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MINIMIZERS FOR THE EMBEDDING OF BESOV SPACES

Mingjuan Chen

Abstract Using the profile decomposition, we will show the relatively compactness of the minimizing sequence to the critical embeddings between Besov spaces, which implies the existence of minimizer of the critical embeddings of Besov spaces $\dot{B}^{s_1}_{p_1,q_1} \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}$ in d dimensions with $s_1 - d/p_1 = s_2 - d/p_2$, $s_1 > s_2$ and $1 \le q_1 < q_2 \le \infty$. Moreover, we establish the nonexistence of the minimizer in the case $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}$.

Keywords Profile decomposition, Besov embedding, minimizer, compactness.

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1. Introduction

Let X and Y be two Banach spaces. We say that $X \hookrightarrow Y$ if X is a subset of Y and there exists a constant $R > 0$ such that $R\|u\|_Y \le \|u\|_X$ holds for all $u \in X$. Obviously, the constant $R > 0$ is not unique, but it has a maximal one, which is denoted by S. It is easy to see that

$$S = \inf\big\{\|u\|_X;\ u \in X,\ \|u\|_Y = 1\big\}.$$

If there exists $u_0 \in X$ and $\|u_0\|_Y = 1$ such that,

$$\|u_0\|_X = \inf\big\{\|u\|_X;\ u \in X,\ \|u\|_Y = 1\big\},$$

then u_0 is said to be a *minimizer* of the embedding $X \hookrightarrow Y$. In view of the definition of S, one easily sees that there exists a sequence of $\{u_n\}$ satisfying the following properties:

$$u_n \in X,\quad \|u_n\|_Y = 1,\quad \|u_n\|_X \ge S \ \text{ and } \lim_{n\to\infty} \|u_n\|_X = S,$$

which is said to be a *minimizing sequence*. If $\{u_n\}$ is compact in X, which means that it has a subsequence converging to $u_0 \in X$, then we see that u_0 is a *minimizer* of the embedding $X \hookrightarrow Y$.

In the present work we are mainly interested in the existence of the minimizer in the embeddings of Besov spaces $\dot{B}^s_{p,q}$ and we consider the following question:

Question. Let $\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d)$. Does the minimizer exist or not?

Email address:mjchenhappy@pku.edu.cn(M. Chen)
School of Mathematical Sciences, Peking University, Beijing, 100871, China

The existence of the minimizer for the classical Sobolev's embeddings was established in Lions [17] in the 1980s using the concentration-compactness principle (see also Struwe [20]). In this paper we will use the profile decomposition to solve this problem in the Besov's embeddings. Similarly, the conclusion in the Sobolev's embeddings can also be obtained by using the profile decomposition, which is much easier than the concentration-compactness principle (see [11]).

It is known that we have the critical embedding (cf. [21])

$$\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d), \quad s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}, \quad s_1 > s_2, \quad q_1 \le q_2.$$

These two spaces have the same scaling properties, which means that for any function f and $\lambda > 0$, we have

$$\|f(\lambda\cdot)\|_{\dot{B}^{s_1}_{p_1,q_1}} = \lambda^r \|f\|_{\dot{B}^{s_1}_{p_1,q_1}}, \quad \|f(\lambda\cdot)\|_{\dot{B}^{s_2}_{p_2,q_2}} = \lambda^r \|f\|_{\dot{B}^{s_2}_{p_2,q_2}}, \quad r = s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}.$$

This inclusion lacks compactness: there exists a bounded sequence in $\dot{B}^{s_1}_{p_1,q_1}$, whose any subsequence is not strongly convergent in $\dot{B}^{s_2}_{p_2,q_2}$. However, one can rewrite a bounded sequence which is known as profile decomposition, and the "profile" is the typical obstacle to compactness [11,12,16]. A profile decomposition shows that the lack of compactness of such embeddings generally comes from linear combinations of the profiles which are norm-invariant transformations of some non-zero elements. Firstly, the profiles can be isolated to recover some aspects of compactness. Then the orthogonal inequalities are established due to the orthogonality of each two profiles. As a result, we can see that at most one profile in the decomposition is different from zero, leading to the relatively compactness of the minimizing sequence.

In 1998, the profile decomposition was established by Gérard in [11] for the embedding of the homogeneous Sobolev space $X = \dot{H}^s$ into the Lebesgue space $Y = L^p$ with $0 < s = d/2 - d/p$. Jaffard [12] used the wavelet expansions to generalize the profile decomposition to the case where X is a Riesz potential space. Until 2010, such a profile decomposition for Lebesgue and Besov space embeddings was obtained by Koch in [16] by use of wavelet bases. Such a profile decomposition has particular applications to the regularity theory of nonlinear partial differential equations (cf. [2,4,5,7–10,13–15]). In our paper, we present that it has another application to recover the existence of the minimizer for critical space embeddings. That is to say, the profile decomposition can be used to recover the existence of the minimizer in the following cases:

(1) $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ where $0 < s < d/2$ and $p = 2d/(d-2s)$;

(2) $\dot{H}^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ where $s > 0$, $s - d/p = -d/q$;

(3) $L^p(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_{p,r}}_{r,q}(\mathbb{R}^d)$ where $2 \le p < q$, $r \le \infty$, $s_{p,r} := d(1/r - 1/p) < 0$;

(4) $\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ where $s_1 - d/p_1 = s_2 - d/p_2$, $1 \le p_1 < p_2 \le \infty$, $1 \le q_1 < q_2 \le \infty$.

In the first case, the orthogonal property and the corresponding result can be obtained easily because $\dot{H}^s$ has Hilbert structure. And in the other three cases, we can get the results in a similar way by using wavelet bases. Therefore, in our paper we only consider the last case.

We list the main results of this paper as follows.

Theorem 1.1. *(i) Let $X = \dot{B}^{s_1}_{p_1,q}$, $Y = \dot{B}^{s_2}_{p_2,q}$ where $1 \le p_1 < p_2 \le \infty$, $1 \le q < \infty$, $s_1 - d/p_1 = s_2 - d/p_2$. Then the minimizer does not exist.*

(ii) Let $X = \dot{B}^{s_1}_{p_1,q_1}$, $Y = \dot{B}^{s_2}_{p_2,q_2}$ where $1 < p_1 < p_2 \le \infty$, $1 < q_1 < q_2 \le \infty$, $s_1 - d/p_1 = s_2 - d/p_2$. Suppose $\max(p_1, q_1) \le \min(p_2, q_2)$, then the minimizer exists.

It is easy to note that these two results in Theorem 1.1 are the corollaries of the main theorems in Section 4 and Section 5. The profile decompositions for the embeddings of Tribel-Lizorkin, Lorentz, Hölder and BMO spaces were established by Bahouri etc [1]. If the relevant orthogonal results can be recovered, we believe that the questions in these spaces' embeddings can also be solved.

In order to more clearly express that why the same sequence index q leads to the nonexistence of minimizer. Let us introduce the relatively compactness of the minimizing sequence of frequency-localized operator. From Bernstein's inequality, we know that for $1 \le p_1 < p_2 \le \infty$, $\|f\|_{p_2} \lesssim \|f\|_{p_1}$ if supp$\hat{f}$ is compact. Therefore, we consider the existence of the minimizer for

$$S_j \|\Delta_j u\|_{p_2} \le \|\Delta_j u\|_{p_1}, \quad 1 \le p_1 < p_2 \le \infty, \quad j \in \mathbb{Z}, \tag{1.1}$$

where S_j is the maximal constant of the frequency-localized operator Δ_j (the definition is in next section). Suppose $\{\Delta_j u_n\}$ is a minimizing sequence, that is

$$\Delta_j u_n \in L^{p_1}, \quad \|\Delta_j u_n\|_{p_2} = 1, \quad \lim_{n\to\infty} \|\Delta_j u_n\|_{p_1} = S_j. \tag{1.2}$$

If the minimizing sequence is compact in L^{p_1}, the minimizer exists. We have the following theorem:

Theorem 1.2. *Let $1 < p_1 < p_2 < \infty$, $S_j\|\Delta_j u\|_{p_2} \le \|\Delta_j u\|_{p_1}$. Then the minimizer exists for any fixed $j \in \mathbb{Z}$. In particular, $S_j = 2^{jd(1/p_2 - 1/p_1)} S_0$.*

The proof of Theorem 1.2 is in Section 3. We find that the difference between p_1 and p_2 ensures the relatively compactness of the minimizing sequence at every frequency torus, but the same q shall lead to the non-compactness.

2. Preliminaries

First, we introduce the dyadic decomposition and homogeneous Besov spaces, see [21]. Let $\psi : \mathbb{R}^d \to [0, 1]$ be a smooth radial cut-off function which is supported in $\{\xi \in \mathbb{R}^d; |\xi| \le 2\}$ and equals 1 on $\{\xi \in \mathbb{R}^d; |\xi| \le 1\}$. Denote $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$, then supp $\varphi \subset \{\xi : 2^{-1} \le |\xi| \le 2\}$. Next we introduce the function sequence $\{\varphi_j\}^{+\infty}_{-\infty}$:

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad j \in \mathbb{Z}. \tag{2.1}$$

It is easy to see that supp $\varphi_j \subset \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{2.2}$$

Define the homogeneous dyadic decomposition operators as following:

$$\Delta_j = \mathscr{F}^{-1}\varphi_j\mathscr{F}, \quad j \in \mathbb{Z}. \tag{2.3}$$

Noticing that in (2.1)-(2.3), there is no restriction on $\xi = 0$, and so one needs to modify the Schwartz space $\mathscr{S}$ and its dual $\mathscr{S}'$. Denote

$$\dot{\mathscr{S}}(\mathbb{R}^d) = \{f \in \mathscr{S}(\mathbb{R}^d) : \ (D^\alpha \hat{f})(0) = 0, \ \forall \alpha\}.$$

As a subspace of $\mathscr{S}$, $\dot{\mathscr{S}}(\mathbb{R}^d)$ is equipped with the same topology as $\mathscr{S}$. we denote by $\dot{\mathscr{S}}'(\mathbb{R}^d)$ the dual space of $\dot{\mathscr{S}}(\mathbb{R}^d)$. We now introduce the homogeneous Besov space $\dot{B}^s_{p,q}$:

$$\dot{B}^s_{p,q} = \{f \in \dot{\mathscr{S}}'(\mathbb{R}^d) : \ \|f\|_{\dot{B}^s_{p,q}} < \infty\}, \quad -\infty < s < \infty, \ 1 \le p, q \le \infty;$$

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p^q \right)^{1/q}.$$

Using the dilation, we have the scaling property

$$\|f(2^l \cdot)\|_{\dot{B}^s_{p,q}} = 2^{l(s-d/p)} \|f\|_{\dot{B}^s_{p,q}}.$$

Next, we introduce the wavelet bases and the equivalent norm of Besov spaces, see [16]. Fix any $m \in \mathbb{N}$, denote $\lambda = (i,j,k) \in \{1, 2, \cdots, 2^d - 1\} \times \mathbb{Z} \times \mathbb{Z}^d := \Lambda$. There exists a real-valued set of functions $\{\varphi^{(i)}\}_{1 \le i \le 2^d - 1} \subset \mathcal{C}^m(\mathbb{R}^d) \cup (\cap_{p>1} L^p(\mathbb{R}^d))$ such that $\varphi_\lambda = \varphi^{(i)}_{j,k}$ defined by $\varphi_\lambda(x) = 2^{jd/2} \varphi^{(i)}(2^j x - k)$ is an orthonormal basis of L^2:

$$f \in L^2(\mathbb{R}^d) \iff f = \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda, \qquad \text{where } \ c_\lambda = c^{(i)}_{j,k} := \int_{\mathbb{R}^d} \varphi_\lambda \cdot f.$$

Moreover, the equivalent norm of Besov spaces is defined by:

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \|f\tilde{\|}_{\dot{B}^s_{p,q}} := \left\| 2^{j(s+d(1/2-1/p))} \|c^{(i)}_{j,k}\|_{\ell^p_{i,k}} \right\|_{\ell^q_j}, \quad 1 \le p, q \le \infty, \quad |s| < m. \tag{2.4}$$

By calculating, one has the scaling property

$$\|f(2^l \cdot)\tilde{\|}_{\dot{B}^s_{p,q}} = 2^{l(s-d/p)} \|f\tilde{\|}_{\dot{B}^s_{p,q}}.$$

Finally, we recall some known lemmas which would be used.

Lemma 2.1 (Compactness in L^p, [19])**.** *Fix $1 \le p < \infty$. A family of functions $\mathcal{F} \subset L^p(\mathbb{R}^d)$ is precompact in this topology if and only if it obeys the following three conditions:*

(i) There exists $A > 0$ so that $\|f\|_p \le A$ for all $f \in \mathcal{F}$.

(ii) For any $\varepsilon > 0$ there exists $\delta > 0$ so that $\int_{\mathbb{R}^d} |f(x) - f(x+y)|^p dx < \varepsilon$ for all $f \in \mathcal{F}$ and all $|y| < \delta$.

(iii) For any $\varepsilon > 0$ there exists R so that $\int_{|x| \ge R} |f|^p dx < \varepsilon$ for all $f \in \mathcal{F}$.

Remark 2.1. It is easy to note that when we consider a compact set $K \subset \mathbb{R}^d$, the third condition of Lemma 2.1 is trivial.

Lemma 2.2 (Bernstein multiplier theorem, [21])**.** *Let $L > d/2$ be an integer, $\partial^\alpha_{x_i} \rho \in L^2$, $i = 1, \cdots, d$ and $0 \le \alpha \le L$. Then we have $\rho \in M_p$, $1 \le p \le \infty$ and*

$$\|\rho\|_{M_p} \lesssim \|\rho\|_2^{1-d/2L} \left(\sum_{i=1}^d \|\partial^L_{x_i} \rho\|_2 \right)^{d/2L}.$$

3. Minimizers for the frequency-localized operator

First, we consider the minimizer of the frequency-localized operator Δ_0. The main result of this section is as follows.

Theorem 3.1. *Let $1 < p_1 < p_2 < \infty$, $S_0\|\Delta_0 u\|_{p_2} \le \|\Delta_0 u\|_{p_1}$. Then the minimizing sequence $\{\Delta_0 u_n\}$ is relatively compact in L^{p_1}. Moreover, $\Delta_0 u_n$ is convergence to $\Delta_0 u$ strongly in L^{p_1}.*

Before proving Theorem 3.1, we introduce the following lemma.

Lemma 3.1. *Let $1 \le p_1 < p_2 < \infty$, K be any compact set in $\mathbb{R}^d$. Then the restriction of the minimizing sequence $\{\Delta_0 u_n\}$ on K, which is denoted by $\Delta_0 u_n|_K$, is relatively compact in $L^{p_i}(K)$, $i \in \{1, 2\}$.*

Proof. In the next proceeding, $i = 1$ or 2. It is easy to see that $\Delta_0 u_n|_K$ is uniformly bounded in $L^{p_i}(K)$. In fact from (1.2), we know that there exists C_i such that

$$\|\Delta_0 u_n\|_{L^{p_i}(K)} \le \|\Delta_0 u_n\|_{L^{p_i}(\mathbb{R}^d)} \le C_i.$$

Next we claim that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(\Delta_0 u_n)(x+h) - (\Delta_0 u_n)(x)\|_{L^{p_i}(K)} < \varepsilon$ for all $|h| < \delta$. Indeed,

$$\|(\Delta_0 u_n)(x+h) - (\Delta_0 u_n)(x)\|_{L^{p_i}(K)} \le \|(\mathscr{F}^{-1}(\mathrm{e}^{ih\xi} - 1)\varphi_0(\xi)\mathscr{F}u_n)(x)\|_{L^{p_i}(\mathbb{R}^d)}.$$

From the construction of φ_j, one has the almost orthogonality $\varphi_j = \sum_{|l|\le 1}\varphi_j\varphi_{j+l}$. Then we have

$$\begin{aligned}
&\|(\Delta_0 u_n)(x+h) - (\Delta_0 u_n)(x)\|_{L^{p_i}(K)}\\
\le&\|(\mathscr{F}^{-1}(\mathrm{e}^{ih\xi} - 1)(\varphi_{-1}(\xi) + \varphi_0(\xi) + \varphi_1(\xi))\mathscr{F}\mathscr{F}^{-1}\varphi_0(\xi)\mathscr{F}u_n)(x)\|_{L^{p_i}(\mathbb{R}^d)}\\
\le&\|(\mathrm{e}^{ih\xi} - 1)(\varphi_{-1}(\xi) + \varphi_0(\xi) + \varphi_1(\xi))\|_{M_{p_i}}\|\Delta_0 u_n\|_{L^{p_i}(\mathbb{R}^d)}\\
\le&C_i\|(\mathrm{e}^{ih\xi} - 1)(\varphi_{-1}(\xi) + \varphi_0(\xi) + \varphi_1(\xi))\|_{M_{p_i}}.
\end{aligned}$$

By using Bernstein multiplier theorem, we can get that there exists $\delta > 0$ so that $\|(\mathrm{e}^{ih\xi} - 1)(\varphi_{-1}(\xi) + \varphi_0(\xi) + \varphi_1(\xi))\|_{M_{p_i}} < \varepsilon/C_i$ for all $|h| < \delta$. Thus from Lemma 2.1, the conclusion is obtained. □

Proof of Theorem 3.1. From Lemma 3.1, we know that there exists $g(x)$ such that $(\Delta_0 u_n)(x) \to g(x)$ in $L^{p_1}(K)$. Then there is a subsequence $\{u_n\}$ such that $(\Delta_0 u_n)(x) \to g(x)$ pointwise almost everywhere on K. We may assume that $\Delta_0 u_n \rightharpoonup \Delta_0 u$ weakly in $L^{p_1}(1 < p_1 < \infty)$ since $\Delta_0 u_n$ is uniformly bounded. By the arbitrariness of compact set K, we obtain $(\Delta_0 u_n)(x) \to \Delta_0 u(x)$ pointwise almost everywhere on $\mathbb{R}^d$. Moreover, if the family $(|\Delta_0 u_n - \theta\Delta_0 u|^{p_1-1}\Delta_0 u)$ is uniformly integrable and tight over $\mathbb{R}^d$ for all $\theta \in [0, 1]$, by Vitali's convergence theorem we can get that

$$\begin{aligned}
&\lim_{n\to\infty}\int|\Delta_0 u_n|^{p_1}dx - \int|\Delta_0 u_n - \Delta_0 u|^{p_1}dx\\
=&\lim_{n\to\infty} -\int\int_0^1 \frac{d}{d\theta}|\Delta_0 u_n - \theta\Delta_0 u|^{p_1}d\theta dx\\
=&\lim_{n\to\infty} p_1\int\int_0^1 |\Delta_0 u_n - \theta\Delta_0 u|^{p_1-2}(\Delta_0 u_n - \theta\Delta_0 u)\Delta_0 u d\theta dx
\end{aligned}$$

$$=p_1\int\int_0^1 |\Delta_0 u-\theta\Delta_0 u|^{p_1-2}(\Delta_0 u-\theta\Delta_0 u)\Delta_0 u d\theta dx=\int|\Delta_0 u|^{p_1}dx. \tag{3.1}$$

Indeed, it is well known that $\forall\ \varepsilon>0$, there exist $\delta>0$ and $E\subset\mathbb{R}^d$ such that

$$\Big(\int_e|\Delta_0 u|^{p_1}dx\Big)^{1/p_1}<\varepsilon,\quad \text{if } m(e)<\delta;$$
$$\Big(\int_{E^c}|\Delta_0 u|^{p_1}dx\Big)^{1/p_1}<\varepsilon,\quad \text{if } m(E)<\infty.$$

Thus from Hölder's inequality, we have

$$\begin{aligned}\int_e|\Delta_0 u_n-\theta\Delta_0 u|^{p_1-1}|\Delta_0 u|dx&\le\|\Delta_0 u\|_{L^{p_1}(e)}\|\Delta_0 u_n-\theta\Delta_0 u\|_{L^{p_1}}^{p_1-1}\\&<\varepsilon(\|\Delta_0 u_n\|_{L^{p_1}}+\|\Delta_0 u\|_{L^{p_1}})^{p_1-1}\le C\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\int_{E^c}|\Delta_0 u_n-\theta\Delta_0 u|^{p_1-1}|\Delta_0 u|dx&\le\|\Delta_0 u\|_{L^{p_1}(E^c)}\|\Delta_0 u_n-\theta\Delta_0 u\|_{L^{p_1}}^{p_1-1}\\&<\varepsilon(\|\Delta_0 u_n\|_{L^{p_1}}+\|\Delta_0 u\|_{L^{p_1}})^{p_1-1}\le C\varepsilon.\end{aligned}$$

Hence we get the conclusion from (3.1) that

$$\int|\Delta_0 u_n|^{p_1}dx-\int|\Delta_0 u_n-\Delta_0 u|^{p_1}dx\to\int|\Delta_0 u|^{p_1}dx. \tag{3.2}$$

It is easy to see that this result is also true if p_1 is replaced by p_2, that is

$$\int|\Delta_0 u_n|^{p_2}dx-\int|\Delta_0 u_n-\Delta_0 u|^{p_2}dx\to\int|\Delta_0 u|^{p_2}dx. \tag{3.3}$$

We denote $\lambda:=\int|\Delta_0 u|^{p_2}dx$. From (1.2) and (3.3), we have $\int|\Delta_0 u_n-\Delta_0 u|^{p_2}dx\to 1-\lambda$. Then from (3.2) we see that

$$\begin{aligned}S_0^{p_1}&=\lim_{n\to\infty}\int|\Delta_0 u_n|^{p_1}dx=\lim_{n\to\infty}\int|\Delta_0 u_n-\Delta_0 u|^{p_1}dx+\int|\Delta_0 u|^{p_1}dx\\&\ge S_0^{p_1}\lim_{n\to\infty}\Big(\int|\Delta_0 u_n-\Delta_0 u|^{p_2}dx\Big)^{\frac{p_1}{p_2}}+\lambda^{\frac{p_1}{p_2}}S_0^{p_1}\\&=S_0^{p_1}\big((1-\lambda)^{\frac{p_1}{p_2}}+\lambda^{\frac{p_1}{p_2}}\big)\ge S_0^{p_1}(1-\lambda+\lambda)^{\frac{p_1}{p_2}}=S_0^{p_1},\end{aligned}$$

where the first inequality is from (1.1), and the second inequality is by strict concavity. The equality holds if and only if $\lambda\in\{0,1\}$ since $p_1<p_2$. By making translation and dilation to $\Delta_0 u_n$, we may assure $\lambda=\int|\Delta_0 u|^{p_2}dx\neq 0$. Thus

$$\lambda=\int|\Delta_0 u|^{p_2}dx=1,\quad \|\Delta_0 u_n-\Delta_0 u\|_{p_2}\to 0.$$

Furthermore,

$$S_0=S_0\|\Delta_0 u\|_{p_2}\le\|\Delta_0 u\|_{p_1}\le\liminf_{n\to\infty}\|\Delta_0 u_n\|_{p_1}=S_0.$$

It implies $\|\Delta_0 u_n\|_{p_1}\to\|\Delta_0 u\|_{p_1}$. Combined with $\Delta_0 u_n\rightharpoonup\Delta_0 u$ in L^{p_1}, it follows that $\Delta_0 u_n\to\Delta_0 u$ in L^{p_1}, as desired. The proof is completed. □

Next, we extend the result of Δ_0 to every frequency-localized operator Δ_j, $j\in\mathbb{Z}$.

Proposition 3.1. *$S_0\|\Delta_0 u\|_{p_2} \le \|\Delta_0 u\|_{p_1}$ holds if and only if*

$$S_0\|\Delta_j u\|_{p_2} \le 2^{jd(1/p_1-1/p_2)}\|\Delta_j u\|_{p_1}$$

holds for any $j \in \mathbb{Z}$.

Proof. On the one hand, $\forall j \in \mathbb{Z}$, let $\tilde{u}(x) := u(2^{-j}x)$, it is easy to calculate that $(\Delta_0\tilde{u})(x) = (\Delta_j u)(2^{-j}x)$. By $S_0\|\Delta_0\tilde{u}\|_{p_2} \le \|\Delta_0\tilde{u}\|_{p_1}$, we can get $S_0\|\Delta_j u\|_{p_2} \le 2^{jd(1/p_1-1/p_2)}\|\Delta_j u\|_{p_1}$.

On the other hand, let $\bar{u}(x) := u(2^j x)$. Then from

$$S_0\|\Delta_j\bar{u}\|_{p_2} \le 2^{jd(1/p_1-1/p_2)}\|\Delta_j\bar{u}\|_{p_1},$$

we have $S_0\|\Delta_0 u\|_{p_2} \le \|\Delta_0 u\|_{p_1}$. □

Proof of Theorem 1.2. We can get from Proposition 3.1 that

$$S_j = 2^{jd(1/p_2-1/p_1)}S_0.$$

Suppose $\{\Delta_0 u_n\}$ is a minimizing sequence of Δ_0, that is

$$\Delta_0 u_n \in L^{p_1}, \quad \|\Delta_0 u_n\|_{p_2} = 1, \quad \lim_{n\to\infty}\|\Delta_0 u_n\|_{p_1} = S_0.$$

Theorem 3.1 implies that there exists u such that $\Delta_0 u_n \to \Delta_0 u$ in L^{p_1}. For any fixed $j \in \mathbb{Z}$, take the sequence $v_n := 2^{jd/p_2}u_n(2^j x)$ and $v := 2^{jd/p_2}u(2^j x)$. We can obtain

$$\|\Delta_j v_n\|_{p_2} = \|\Delta_0 u_n\|_{p_2} = 1,$$
$$\lim_{n\to\infty}\|\Delta_j v_n\|_{p_1} = 2^{jd(1/p_2-1/p_1)}\lim_{n\to\infty}\|\Delta_0 u_n\|_{p_1} = 2^{jd(1/p_2-1/p_1)}S_0 = S_j.$$

Then $\{\Delta_j v_n\}$ is a minimizing sequence of Δ_j, and

$$\|\Delta_j v_n - \Delta_j v\|_{p_1} = 2^{jd(1/p_2-1/p_1)}\|\Delta_0 u_n - \Delta_0 u\|_{p_1} \to 0.$$

The proof is completed. □

Finally, we establish a proposition to supplement.

Proposition 3.2. *Let $1 \le p_1 < p_2 \le \infty$, $s_1 - d/p_1 = s_2 - d/p_2$, and S is the maximal constant for $S\|u\|_{\dot{B}^{s_2}_{p_2,q}} \le \|u\|_{\dot{B}^{s_1}_{p_1,q}}$, then $S = S_0$.*

Proof. On the one hand, by Proposition 3.1,

$$S_0\|\Delta_j u\|_{p_2} \le 2^{jd(1/p_1-1/p_2)}\|\Delta_j u\|_{p_1}, \quad \forall j \in \mathbb{Z}.$$

From the embedding condition $s_1 - d/p_1 = s_2 - d/p_2$, we can get

$$S_0\|\Delta_j u\|_{p_2} \le 2^{j(s_1-s_2)}\|\Delta_j u\|_{p_1}, \quad \forall j \in \mathbb{Z}.$$

Transfering 2^{js_2} to the left side of the inequality and taking the ℓ^q norm about $j \in \mathbb{Z}$, we have

$$\left(\sum_{j\in\mathbb{Z}}\left(2^{js_2}S_0\|\Delta_j u\|_{p_2}\right)^q\right)^{1/q} \le \left(\sum_{j\in\mathbb{Z}}\left(2^{js_1}\|\Delta_j u\|_{p_1}\right)^q\right)^{1/q}.$$

It means that $S_0\|u\|_{\dot{B}^{s_2}_{p_2,q}} \le \|u\|_{\dot{B}^{s_1}_{p_1,q}}$, then $S_0 \le S$.

On the other hand, $S\|u\|_{\dot{B}^{s_2}_{p_2,q}} \le \|u\|_{\dot{B}^{s_1}_{p_1,q}}$ means that

$$\left(\sum_{j\in\mathbb{Z}} \left(2^{js_2} S\|\Delta_j u\|_{p_2}\right)^q\right)^{1/q} \le \left(\sum_{j\in\mathbb{Z}} \left(2^{js_1}\|\Delta_j u\|_{p_1}\right)^q\right)^{1/q}.$$

There exists at least one $j_0 \in \mathbb{Z}$ such that

$$2^{j_0 s_2} S\|\Delta_{j_0} u\|_{p_2} \le 2^{j_0 s_1}\|\Delta_{j_0} u\|_{p_1}.$$

Transfering $2^{j_0 s_2}$ to the right side of the inequality and using $s_1 - d/p_1 = s_2 - d/p_2$, we obtain

$$S\|\Delta_{j_0} u\|_{p_2} \le 2^{j_0 d(1/p_1 - 1/p_2)}\|\Delta_{j_0} u\|_{p_1}.$$

By the proof of Proposition 3.1, we have

$$S\|\Delta_0 u\|_{p_2} \le \|\Delta_0 u\|_{p_1},$$

then $S \le S_0$. The proof is completed. □

4. Nonexistence cases of the minimizer in Besov embeddings

The Besov's embeddings $\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ hold when $s_1 - d/p_1 = s_2 - d/p_2$, $1 \le p_1 < p_2 \le \infty$, $1 \le q_1 \le q_2 \le \infty$. In this section we consider the case $q_1 = q_2$. Let S be the maximal constant for

$$S\|u\|_{\dot{B}^{s_2}_{p_2,q}} \le \|u\|_{\dot{B}^{s_1}_{p_1,q}}, \quad s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}, \quad 1 \le p_1 < p_2 \le \infty.$$

We have the following theorem.

Theorem 4.1. *Let $s_1 - d/p_1 = s_2 - d/p_2$, $1 \le p_1 < p_2 \le \infty$ and $1 \le q < \infty$, then there exists a minimizing sequence $\{u_n\}$ satisfying*

$$u_n \in \dot{B}^{s_1}_{p_1,q}, \quad \|u_n\|_{\dot{B}^{s_2}_{p_2,q}} = 1, \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{\dot{B}^{s_1}_{p_1,q}} = S, \tag{4.1}$$

such that $\{u_n\}$ is not relatively compact in $\dot{B}^{s_1}_{p_1,q}$.

Proof. The strategy of the proof is by contradiction. We assume that the conclusion is false. It is to say that any minimizing sequence $\{u_n\}$ is relatively compact in $\dot{B}^{s_1}_{p_1,q}$.

We choose a fixed sequence $\{u_n\}$ satisfying (4.1). Then there exists $u \in \dot{B}^{s_1}_{p_1,q}$ such that $\|u_n - u\|_{\dot{B}^{s_1}_{p_1,q}} \to 0$, $\|u_n - u\|_{\dot{B}^{s_2}_{p_2,q}} \to 0$ as $n \to \infty$. It means that $\|u\|_{\dot{B}^{s_1}_{p_1,q}} = S$, $\|u\|_{\dot{B}^{s_2}_{p_2,q}} = 1$. It is easy to see that

$$\sum_{|j|>n} \left(2^{js_2}\|\Delta_j u\|_{p_2}\right)^q \le \varepsilon(n), \quad \lim_{n\to\infty} \varepsilon(n) = 0.$$

Next we consider a sequence $\{v_n\}$ which has the following form:

$$v_n(x) = u(x) + (2^{3n})^{\frac{d}{p_2} - s_2} u(2^{3n}x) =: u(x) + w_n(x). \tag{4.2}$$

For any $j \in \mathbb{Z}$, we have

$$\Delta_j v_n = \Delta_j u + \Delta_j w_n = \Delta_j u + (2^{3n})^{\frac{d}{p_2}-s_2}(\Delta_{j-3n}u)(2^{3n}x).$$

Therefore,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\left(2^{js_2}\|\Delta_j v_n\|_{p_2}\right)^q &= \sum_{|j|\le n}\left(2^{js_2}\|\Delta_j u+\Delta_j w_n\|_{p_2}\right)^q + \sum_{|j|>n}\left(2^{js_2}\|\Delta_j u+\Delta_j w_n\|_{p_2}\right)^q\\ &= \sum_{|j|\le n}\left(2^{js_2}\|\Delta_j u\|_{p_2}\right)^q + \sum_{|j|>n}\left(2^{js_2}\|\Delta_j w_n\|_{p_2}\right)^q + \theta(\varepsilon(n))\\ &= \sum_{|j|\le n}\left(2^{js_2}\|\Delta_j u\|_{p_2}\right)^q + \sum_{\substack{j>-2n \text{ or}\\ j<-4n}}\left(2^{js_2}\|\Delta_j u\|_{p_2}\right)^q + \theta(\varepsilon(n)),\end{aligned} \tag{4.3}$$

where $\theta(\varepsilon(n)) \to 0$ as $\varepsilon(n) \to 0$. Let $n \to +\infty$ in (4.3), we obtain

$$\lim_{n\to\infty}\|v_n\|^q_{\dot{B}^{s_2}_{p_2,q}} = 2\|u\|^q_{\dot{B}^{s_2}_{p_2,q}} = 2. \tag{4.4}$$

From $s_1 - d/p_1 = s_2 - d/p_2$, we can get by similar discussion that

$$\lim_{n\to\infty}\|v_n\|^q_{\dot{B}^{s_1}_{p_1,q}} = 2\|u\|^q_{\dot{B}^{s_1}_{p_1,q}} = 2S^q. \tag{4.5}$$

Combining (4.4) and (4.5), we see that $S\lim_{n\to\infty}\|v_n\|_{\dot{B}^{s_2}_{p_2,q}} = \lim_{n\to\infty}\|v_n\|_{\dot{B}^{s_1}_{p_1,q}}$. It implies that the sequence $\{v_n\}$ is a minimizing sequence after normalizing. For any subsequence of $\{v_n\}$ which is still denoted by $\{v_n\}$, we have $v_n \rightharpoonup u$ weakly in $\dot{B}^{s_1}_{p_1,q}$ because of (4.2). However, $\lim_{n\to\infty}\|v_n\|_{\dot{B}^{s_1}_{p_1,q}} = 2^{1/q}\|u\|_{\dot{B}^{s_1}_{p_1,q}} > \|u\|_{\dot{B}^{s_1}_{p_1,q}}$, which is contrary to the assumption. Now we complete the proof of Theorem 4.1. □

5. Existence cases of the minimizer in Besov embeddings

In this section we consider the existence of minimizers for the critical Besov's embeddings

$$\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d), \quad s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}, \quad s_1 > s_2, \quad 1 \le q_1 < q_2 \le \infty.$$

We shall use the equivalent norms of Besov spaces as (2.4). Let S be the maximal constant for $S\|u\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} \le \|u\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}}$. Assume that $\{u_n\}$ is a minimizing sequence for S, that is

$$u_n \in \dot{B}^{s_1}_{p_1,q_1}, \quad \|u_n\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = 1, \quad \text{and} \quad \lim_{n\to\infty}\|u_n\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} = S.$$

We use the profile decomposition to get the following theorem.

Theorem 5.1. *Let $s_1 - d/p_1 = s_2 - d/p_2$, $1 < p_1 < p_2 \le \infty$ and $1 < q_1 < q_2 \le \infty$. Suppose $\max(p_1, q_1) \le \min(p_2, q_2)$, then any minimizing sequence $\{u_n\}$ up to translation and dilation is relatively compact in $\dot{B}^{s_1}_{p_1,q_1}$.*

To prove Theorem 5.1, we introduce the profile decomposition for bounded sequences in Besov spaces which is established by Koch (see [1,16]).

Lemma 5.1. *Let $1 \leq p_1 < p_2 \leq \infty$, $1 \leq q_1 < q_2 \leq \infty$, $\alpha = s_1 - d/p_1 = s_2 - d/p_2$. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $\dot{B}^{s_1}_{p_1,q_1}$. There exists a sequence of profiles $\{\phi_l\}_{l=1}^{\infty} \subset \dot{B}^{s_1}_{p_1,q_1}$ and a set of sequences $\{(j_n^l, k_n^l)\}_{n=1}^{\infty} \subset \mathbb{Z} \times \mathbb{Z}^d$ for $l \in \mathbb{N}$, both depending on $\{u_n\}$, such that, after possibly passing to a subsequence in n,*

$$u_n(x) = \sum_{l=1}^{L} (2^{j_n^l})^{-\alpha} \phi_l\big(2^{j_n^l} x - k_n^l\big) + r_n^L(x) \tag{5.1}$$

for any $L \in \mathbb{N}$ where the following properties hold:

(i) For $l \neq l'$, the sequences $\{(j_n^l, k_n^l)\}$ and $\{(j_n^{l'}, k_n^{l'})\}$ are orthogonal in the following sense:

$$\lim_{n\to\infty} \left| \log\big(2^{(j_n^l - j_n^{l'})}\big) \right| + \left| 2^{(j_n^l - j_n^{l'})} k_n^{l'} - k_n^l \right| = +\infty. \tag{5.2}$$

(ii) The remainder r_n^L satisfies the following smallness condition:

$$\lim_{L\to\infty} \limsup_{n\to\infty} \|r_n^L\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = 0. \tag{5.3}$$

(iii) (Stability) For each $n \in \mathbb{N}$,

$$\left| \Big(\|(2^{j_n^l})^{-\alpha} \phi_l\big(2^{j_n^l} x - k_n^l\big)\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} \Big)_{l=1}^{\infty} \right\|_{\ell^\tau} \leq \liminf_{n'\to\infty} \|u_{n'}\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}}, \tag{5.4}$$

where $\tau := \max(p_1, q_1)$, and for any $L \in \mathbb{N}$,

$$\|r_n^L\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} \leq \|u_n\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} + \circ(1) \quad \text{as} \quad \mathrm{n} \to \infty. \tag{5.5}$$

Remark 5.1. In [16], (5.3) needs the condition $q_2/q_1 \geq p_2/p_1$. However, the authors [1] put this condition off.

The proof of Lemma 5.1 can be found in [1,16]. It used the wavelet bases. Let me sketch the proceeding to get the profile decomposition.

We consider the embeddings $\dot{B}^{s_1}_{p_1,q_1} \hookrightarrow \dot{B}^{s_2}_{p_2,q_2} \hookrightarrow \dot{B}^{\alpha}_{\infty,\infty}$, where $\alpha = s_1 - d/p_1 = s_2 - d/p_2$. From the equivalent norm (2.4), we have

$$\|f\tilde{\|}_{\dot{B}^{\alpha}_{\infty,\infty}} = \sup_{\lambda\in\Lambda} 2^{j(\alpha+d/2)} |c_{j,k}^{(i)}|.$$

Thus we define

$$a_{j,k}^{(i)} := 2^{j(\alpha+d/2)} c_{j,k}^{(i)}, \quad \psi_\lambda(x) := 2^{-j(\alpha+d/2)} \varphi_\lambda(x).$$

Therefore, for a given f one may write

$$f = \sum_\lambda c_\lambda \varphi_\lambda = \sum_\lambda a_\lambda \psi_\lambda = \sum_\lambda a_\lambda 2^{-j\alpha} \varphi^{(i)}(2^j x - k), \tag{5.6}$$

and we have the equivalence norm:

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \|f\tilde{\|}_{\dot{B}^s_{p,q}} := \left\| 2^{j(s-\alpha-d/p)} \|a_{j,k}^{(i)}\|_{\ell^p_{i,k}} \right\|_{\ell^q_j}.$$

In particular,

$$\|f\|_{\tilde{\dot{B}}^{\alpha}_{\infty,\infty}} = \sup_{\lambda\in\Lambda} |a^{(i)}_{j,k}|, \quad \|f\|_{\tilde{\dot{B}}^{s_1}_{p_1,q_1}} = \big\| \|a^{(i)}_{j,k}\|_{\ell^{p_1}_{i,k}} \big\|_{\ell^{q_1}_j}, \quad \|f\|_{\tilde{\dot{B}}^{s_2}_{p_2,q_2}} = \big\| \|a^{(i)}_{j,k}\|_{\ell^{p_2}_{i,k}} \big\|_{\ell^{q_2}_j}. \tag{5.7}$$

For convenience, for any $\lambda = \{i, j, k\} \in \Lambda$, let

$$\tau(x) := 2^j x - k, \quad (\tau\varphi^{(i)})(x) := 2^{-j\alpha}\varphi^{(i)}(\tau(x)), \quad |\tau| := |\log 2^j| + \left|\frac{k}{2^j}\right|.$$

Then for a given f one may write from (5.6) that

$$f = \sum_{\lambda} a_\lambda(\tau\varphi^{(i)})(x).$$

Moreover, note that $\tau_1(\tau_2\varphi^{(i)}) = (\tau_2 \circ \tau_1)\varphi^{(i)}$ for any such $\tau_1,\ \tau_2$. For sequences $\tau_{1,n}(x) := 2^{j^1_n}x - k^1_n$ and $\tau_{2,n}(x) := 2^{j^2_n}x - k^2_n$, let $\tau^{(2,1)}_n := \tau_{2,n} \circ \tau^{-1}_{1,n}$, then

$$\tau^{(2,1)}_n(x) = 2^{j^2_n - j^1_n}x - (k^2_n - 2^{j^2_n - j^1_n}k^1_n), \quad |\tau^{(2,1)}_n| = \left|\log 2^{(j^2_n - j^1_n)}\right| + \left|2^{(j^1_n - j^2_n)}k^2_n - k^1_n\right|.$$

We say "$\tau_{1,n}$ and $\tau_{2,n}$ are orthogonal" if $|\tau^{(2,1)}_n| \to \infty$ as $n \to \infty$, which is in conformity with (5.2).

The strategy of getting the profile decomposition is by method of iteration. The wavelet decomposition is

$$u_n = \sum_{\lambda} a_{\lambda,n}\psi_\lambda = \sum_{\lambda} a_{\lambda,n}(\tau\varphi^{(i)}).$$

At every iteration l, we sort out a wavelet component $a_{\lambda^l_n}\psi_{\lambda^l_n} = a_{\lambda^l_n}\tau_{l,n}\varphi^{(i_l)}$ whose coefficient has the largest possible modulus. Here $\lambda^l_n = (i_l, j^l_n, k^l_n)$, $\tau_{l,n} = 2^{j^l_n}x - k^l_n$, $a_{\lambda^l_n} \to a_l \neq 0$ as $n \to \infty$ (passing to a subsequence if necessary). If for any $\bar{l} < l$, $\tau_{l,n}$ and $\tau_{\bar{l},n}$ are orthogonal, we shall build a new profile; else if there is some $\bar{l} < l$ such that $\tau_{l,n}$ and $\tau_{\bar{l},n}$ are not orthogonal, we shall use $a_l\psi_{\lambda^l_n}$ to modify the profile contains $\tau_{\bar{l},n}$. We show out the general form which can be found in [16],

$$\begin{aligned}
u_n &= \sum_{l=1}^{L_N}\sum_{\mu=1}^{M_N(l)} a_{\lambda^{m_\mu(l)}_n}\tau_{m_\mu(l),n}\varphi^{i_{m_\mu(l)}} + u^N_n \\
&= \sum_{l=1}^{L_N}\tau_{m_1(l),n}\left(\sum_{\mu=1}^{M_N(l)} a_{\lambda^{m_\mu(l)}_n}\tau^{(m_\mu(l),m_1(l))}\varphi^{i_{m_\mu(l)}}\right) + u^N_n \\
&=: \sum_{l=1}^{L_N}\tau_{m_1(l),n}\left(\sum_{\mu=1}^{M_N(l)} a_{m_\mu(l)}\tau^{(m_\mu(l),m_1(l))}\varphi^{i_{m_\mu(l)}}\right) + r^N_n,
\end{aligned} \tag{5.8}$$

where the second equality is obtained by passing to a subsequence such that $\tau^{(m_\mu(l),m_1(l))}_n \equiv \tau^{(m_\mu(l),m_1(l))} = \text{const}$ (because of working on lattices). Let iteration $N \to \infty$ (if iteration stops at some finite $\bar{N}$, it is similar and easier), and denote $L_\infty :=$

$\lim_{N\to\infty} L_N$, $M_\infty(l) := \lim_{N\to\infty} M_N(l)$ for any $l \in [1, L_\infty]$. Then for any $L \in [1, L_\infty]$, we have from (5.8) that:

$$u_n = \sum_{l=1}^{L} \tau_{m_1(l),n} \left(\sum_{\mu=1}^{M_\infty(l)} a_{m_\mu(l)} \tau^{(m_\mu(l), m_1(l))} \varphi^{i_{m_\mu(l)}} \right) + r_n^L. \tag{5.9}$$

Let

$$\phi_l := \sum_{\mu=1}^{M_\infty(l)} a_{m_\mu(l)} \tau^{(m_\mu(l), m_1(l))} \varphi^{i_{m_\mu(l)}},$$

we know $\phi_l \in \dot{B}^{s_1}_{p_1,q_1}$ in the proof of Lemma 5.1 in [16]. Therefore (5.9) implies (5.1). Replace both L_∞ and $M_\infty(l)$ by ∞. If either is finite, we set the extra components are zero. It is to say that

$$\begin{aligned} u_n &= \sum_{l=1}^{\infty} \tau_{m_1(l),n} \left(\sum_{\mu=1}^{\infty} a_{\lambda_n^{m_\mu(l)}} \tau^{(m_\mu(l), m_1(l))} \varphi^{i_{m_\mu(l)}} \right) \\ &=: \sum_{l=1}^{\infty} \tau_{m_1(l),n} \left(\sum_{\mu=1}^{\infty} a_{m_\mu(l)} \tau^{(m_\mu(l), m_1(l))} \varphi^{i_{m_\mu(l)}} \right) + r_n^\infty \\ &=: \sum_{l=1}^{\infty} \tau_{m_1(l),n} \phi_l + r_n^\infty = \sum_{l=1}^{\infty} \sum_{\mu=1}^{\infty} a_{m_\mu(l)} \psi_{\lambda_n^{m_\mu(l)}} + r_n^\infty. \end{aligned} \tag{5.10}$$

To prove Theorem 5.1, we also need the following orthogonal inequalities.

Lemma 5.2. *$\phi_l \in \dot{B}^{s_1}_{p_1,q_1} \hookrightarrow \dot{B}^{s_2}_{p_2,q_2}$ for any $l \in \mathbb{N}^+$, and the symbols are as the above discussion, then for any $n \in \mathbb{N}$,*

$$\left\| \sum_{l=1}^{\infty} \tau_{m_1(l),n} \phi_l \right\|^{\sigma}_{\dot{B}^{s_2}_{p_2,q_2}} \tilde{\ } \le \sum_{l=1}^{\infty} \| \tau_{m_1(l),n} \phi_l \|\tilde{}^{\sigma}_{\dot{B}^{s_2}_{p_2,q_2}},$$

where $\sigma = \min(p_2, q_2)$.

Proof. Since u_n can be decomposed into the form like (5.10), we may divide Λ into a disjoint union of sets E_l: $\Lambda = \cup_{l=1}^\infty E_l$. The proof is ascribed to proving the following inequality:

$$\left\| \sum_{\lambda\in\Lambda} a_\lambda \psi_\lambda \right\|\tilde{}^{\sigma}_{\dot{B}^{s_2}_{p_2,q_2}} \le \sum_{l=1}^{\infty} \left\| \sum_{\lambda\in E_l} a_\lambda \psi_\lambda \right\|\tilde{}^{\sigma}_{\dot{B}^{s_2}_{p_2,q_2}}.$$

(1) The case $\sigma = p_2 \le q_2$. We have from (5.7) that

$$\begin{aligned} \sum_{l=1}^{\infty} \left\| \sum_{\lambda\in E_l} a_\lambda \psi_\lambda \right\|\tilde{}^{p_2}_{\dot{B}^{s_2}_{p_2,q_2}} &= \sum_{l=1}^{\infty} \left(\sum_j \left(\sum_{\lambda\in E_{l,j}} |a_\lambda|^{p_2} \right)^{\frac{q_2}{p_2}} \right)^{\frac{p_2}{q_2}} \\ &\ge \left(\sum_j \left(\sum_{l=1}^{\infty} \sum_{\lambda\in E_{l,j}} |a_\lambda|^{p_2} \right)^{\frac{q_2}{p_2}} \right)^{\frac{p_2}{q_2}} = \left\| \sum_{\lambda\in\Lambda} a_\lambda \psi_\lambda \right\|\tilde{}^{p_2}_{\dot{B}^{s_2}_{p_2,q_2}}, \end{aligned}$$

where $\lambda \in E_{l,j}$ means $\lambda = \{i, j, k\}$ having the same index j in E_l, and the inequality is by Minkowski's inequality.

(2) The case $\sigma = q_2 \le p_2$.

$$\begin{aligned}\sum_{l=1}^{\infty}\Big\|\sum_{\lambda\in E_l} a_\lambda\psi_\lambda\Big\tilde{\|}^{q_2}_{\dot B^{s_2}_{p_2,q_2}} &= \sum_{l=1}^{\infty}\sum_j\Big(\sum_{\lambda\in E_{l,j}}|a_\lambda|^{p_2}\Big)^{\frac{q_2}{p_2}} = \sum_j\sum_{l=1}^{\infty}\Big(\sum_{\lambda\in E_{l,j}}|a_\lambda|^{p_2}\Big)^{\frac{q_2}{p_2}}\\ &\ge \sum_j\Big(\sum_{l=1}^{\infty}\sum_{\lambda\in E_{l,j}}|a_\lambda|^{p_2}\Big)^{\frac{q_2}{p_2}} = \Big\|\sum_{\lambda\in\Lambda}a_\lambda\psi_\lambda\Big\tilde{\|}^{q_2}_{\dot B^{s_2}_{p_2,q_2}},\end{aligned}$$

where the inequality is from strict concavity. □

Remark 5.2. In Lemma 5.2, when $p_2 \ne q_2$, the equality in the result holds if and only if at most one of the terms $\tau_{m_1(l),n}\phi_l$, $l \in \mathbb{N}^+$, is different from zero.

Proof of Theorem 5.1. From $\|u_n\tilde{\|}_{\dot B^{s_2}_{p_2,q_2}} = 1$ and $\lim_{n\to\infty}\|u_n\tilde{\|}_{\dot B^{s_1}_{p_1,q_1}} = S$, we know that $\|u_n\tilde{\|}_{\dot B^{s_2}_{p_2,q_2}}$ and $\|u_n\tilde{\|}_{\dot B^{s_1}_{p_1,q_1}}$ are uniformly bounded. Thus after possibly passing to a subsequence in n, $\{u_n\}$ can be decomposed into (5.1) in $\dot B^{s_1}_{p_1,q_1}$ and satisfies all the properties in Lemma 5.1. Moreover, since $\dot B^{s_1}_{p_1,q_1} \hookrightarrow \dot B^{s_2}_{p_2,q_2}$, (5.1) is also a decomposition of u_n in $\dot B^{s_2}_{p_2,q_2}$. Combining (5.3) and (5.10), we know after passing a subsequence

$$\lim_{n\to\infty}\|r_n^\infty\tilde{\|}_{\dot B^{s_2}_{p_2,q_2}} = 0.$$

As Lemma 5.1 and Lemma 5.2, let $\tau = \max(p_1, q_1)$ and $\sigma = \min(p_2, q_2)$. From (5.10) we have

$$\begin{aligned}1 &= \|u_n\tilde{\|}^\sigma_{\dot B^{s_2}_{p_2,q_2}} = \|u_n - r_n^\infty\tilde{\|}^\sigma_{\dot B^{s_2}_{p_2,q_2}} + o(1)\\ &= \Big\|\sum_{l=1}^{\infty}\tau_{m_1(l),n}\phi_l\Big\tilde{\|}^\sigma_{\dot B^{s_2}_{p_2,q_2}} + o(1) \le \sum_{l=1}^{\infty}\|\tau_{m_1(l),n}\phi_l\tilde{\|}^\sigma_{\dot B^{s_2}_{p_2,q_2}} + o(1)\\ &\le S^{-\sigma}\sum_{l=1}^{\infty}\|\tau_{m_1(l),n}\phi_l\tilde{\|}^\sigma_{\dot B^{s_1}_{p_1,q_1}} + o(1) \le S^{-\sigma}\Big(\sum_{l=1}^{\infty}\|\tau_{m_1(l),n}\phi_l\tilde{\|}^\tau_{\dot B^{s_1}_{p_1,q_1}}\Big)^{\sigma/\tau} + o(1)\\ &\le S^{-\sigma}\big(\lim_{n'\to\infty}\|u_{n'}\tilde{\|}_{\dot B^{s_1}_{p_1,q_1}}\big)^\sigma + o(1) = 1 + o(1),\end{aligned} \tag{5.11}$$

where $o(1) \to 0$ $(n \to \infty)$, the first inequality is by Lemma 5.2, the second inequality is by $S\|u\tilde{\|}_{\dot B^{s_2}_{p_2,q_2}} \le \|u\tilde{\|}_{\dot B^{s_1}_{p_1,q_1}}$, the third inequality is by strict convexity, and the forth inequality is from (5.4).

Let $n \to \infty$, the inequalities in (5.11) must ensure all equalities hold. If $\max(p_1, q_1) < \min(p_2, q_2)$, we know that the equality in the third inequality holds if and only if at most one of the items ϕ_l, $l \in \mathbb{N}^+$, is not zero. If $\max(p_1, q_1) = \min(p_2, q_2)$, from the conditions $p_1 < p_2$, $q_1 < q_2$, we only have two cases:

$$p_1 < q_1 = p_2 < q_2, \quad q_1 < p_1 = q_2 < p_2.$$

From Remark 5.2, we get that the equality in the first inequality holds if and only if at most one of the terms ϕ_l, $l \in \mathbb{N}^+$, is different from zero. Therefore, there exists $l' \in \mathbb{N}^+$ such that

$$u_n(x) = (2^{j_n^{l'}})^{-\alpha}\phi_{l'}\big(2^{j_n^{l'}}x - k_n^{l'}\big) + r_n. \quad \limsup_{n\to\infty}\|r_n\tilde{\|}_{\dot B^{s_2}_{p_2,q_2}} = 0.$$

Let

$$v_n(x) := (2^{j_n^{l'}})^{\alpha} u_n\Big(\frac{x + k_n^{l'}}{2^{j_n^{l'}}}\Big).$$

Passing to a subsequence and using the norm invariance, we have

$$\lim_{n\to\infty} \|v_n - \phi_{l'}\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = 0, \quad \|\phi_{l'}\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = \lim_{n\to\infty} \|v_n\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = \lim_{n\to\infty} \|u_n\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} = 1.$$

Moreover, we have

$$S = S\|\phi_{l'}\tilde{\|}_{\dot{B}^{s_2}_{p_2,q_2}} \le \|\phi_{l'}\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} \le \lim_{n\to\infty} \|v_n\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} = \lim_{n\to\infty} \|u_n\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} = S.$$

It follows that $\|v_n\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}} \to \|\phi_{l'}\tilde{\|}_{\dot{B}^{s_1}_{p_1,q_1}}$ as $n \to \infty$. Combining with $v_n \rightharpoonup \phi_{l'}$ in $\dot{B}^{s_1}_{p_1,q_1}$, we know that $v_n \to \phi_{l'}$ in $\dot{B}^{s_1}_{p_1,q_1}$(here the condition $1 < p_1,\ q_1 < \infty$ is necessary). The proof is completed. □

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