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BIFURCATIONS AND CHAOS CONTROL IN A DISCRETE-TIME PREDATOR-PREY SYSTEM OF LESLIE TYPE

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Abstract We investigate the dynamics of a discrete-time predator-prey system of Leslie type. We show algebraically that the system passes through a flip bifurcation and a Neimark-Sacker bifurcation in the interior of $\mathbb{R}^2_+$ using center manifold theorem and bifurcation theory. Numerical simulations are implimented not only to validate theoretical analysis but also exhibits chaotic behaviors, including phase portraits, period-11 orbits, invariant closed circle, and attracting chaotic sets. Furthermore, we compute Lyapunov exponents and fractal dimension numerically to justify the chaotic behaviors of the system. Finally, a state feedback control method is applied to stabilize the chaotic orbits at an unstable fixed point.

Keywords Discrete-time predator-prey system, bifurcations, chaos, Lyapunov exponents, feedback control.

MSC(2010) 92D25, 37D45, 39A28, 39A33.

1. Introduction

The population models in ecology and mathematical ecology which describe predator-prey interaction geverned by differential equations studied extensively by many researchers [6,8] and the reference therein. Qualitative analyses of these works found many rich dynamics which include global stability, stable limit cycle, bifurcatioins and persistence analysis. But in recent years, there is a growing evidence that the discretization of predator-prey models governed by difference equations are more appropriate than the continuous ones, especially when the populations have non-overlapping generations [4, 5, 7, 12, 13, 15, 16, 18, 19]. The main studied subjects in discrete-time models were the posibility of bifurcations and chaos phenomenon those had been performed either by using numerical simulations or by using the center manifold theorem and bifurcation theory.

In this paper, we consider the following predator-prey system of Leslie type [6]:

$$\begin{aligned} \dot{x} &= rx\left(1 - \tfrac{x}{K}\right) - mxy, \\ \dot{y} &= sy\left(1 - \tfrac{hy}{x}\right), \end{aligned} \tag{1.1}$$

where x and y stand for density of prey and predator, respectively; r, K, m, s and h are all positive parameters, and mx is Holling type I functional response. In [6],

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it is shown that the positive equilibrium of system (1.1) is globally asymptotically stable. For the sake of simplicity, we make the following scaling

$$\frac{x}{K} \to x, \frac{my}{r} \to y, rt \to t \text{ and } \alpha = \frac{s}{r}, \beta = \frac{sh}{mK}.$$

Then the system (1.1) takes the form

$$\begin{aligned} \dot{x} &= x\left(1-x\right) - xy, \\ \dot{y} &= y\left(\alpha - \frac{\beta y}{x}\right). \end{aligned} \tag{1.2}$$

Applying the forward Euler scheme to system (1.2), we obtain the discrete-time predator-prey system as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \delta\left[x\left(1-x\right) - xy\right] \\ y + \delta\left[\alpha y - \frac{\beta y^2}{x}\right] \end{pmatrix}, \tag{1.3}$$

where δ is the step size. The aim is to study systematically the existence of a flip bifurcation and a NS bifurcation in the interior of $\mathbb{R}^2_+$ using bifurcation theory and center manifold theory (see section 4, [10]).

This paper is organized as follows. In Section 2, we present the existence and stability of positive fixed point for system (1.3) in $\mathbb{R}^2_+$. In Section 3, we prove that under certain parametric condition system (1.3) admits a flip bifurcation and a NS bifurcation in the interior of $\mathbb{R}^2_+$. In Section 4, we perform numerical simulations which include the bifurcation diagrams, the phase portraits, maximum Lyapunov exponents and Fractal dimensions to characterize the chaotic behaviors of the system. In Section 5, chaos is controlled to an unstable fixed point using the feedback control method. Finally a short discussion is carried out in Section 6.

2. Existence and stability of fixed points

The fixed points of (1.3) satisfy the following equations:

$$\begin{aligned} x + \delta\left[x\left(1-x\right) - xy\right] &= x, \\ y + \delta\left[\alpha y - \frac{\beta y^2}{x}\right] &= y. \end{aligned} \tag{2.1}$$

By a simple algebraic computation we obtain the following result:

Lemma 2.1. *System* (1.3) *always has two fixed points* $E_1(1,0)$ *and* $E_2(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$ *for all permissible parameter values.*

Now we investigate the local stability of the system (1.3) around each fixed points. Note that for each fixed point $E(x,y)$ the local stability is determined by the modules of eigenvalues of Jacobian matrix evaluated at the fixed point. The Jacobian matrix of system (1.3) at a fixed point $E(x,y)$ is

$$J(x,y) = \begin{pmatrix} 1+\delta(1-2x-y) & -\delta x \\ \frac{\delta\beta y^2}{x^2} & 1+\delta(\alpha - \frac{2\beta y}{x}) \end{pmatrix}. \tag{2.2}$$

The characteristic equation associated with (2.2) is

$$\lambda^2 - (trJ)\lambda + detJ = 0, \tag{2.3}$$

where trJ and $detJ$ are the trace and determinant of J.

Using Jury's criterion [2], a simple calculation shows the following propositions.

Proposition 2.1. *The fixed point $E_1(1,0)$ is a saddle if $0 < \delta < 2$, it is a source if $\delta > 2$ and it is non-hyperbolic if $\delta = 2$.*

We see that when $\delta = 2$, the two eigenvalues of $J(E_1)$ are $\lambda_1 = 1 - \delta$ and $\lambda_2 = 1 + \alpha\delta$. Thus, a flip bifurcation may occur when parameters change in the small neighborhood of

$$FB_{E_1} = \left\{(\alpha, \beta, \delta) \in (0, +\infty) : \delta = 2\right\}.$$

In this case, the predator becomes extinction and the prey undergoes the period-doubling bifurcation to chaos in the sense of Li-Yorke by choosing bifurcation parameter δ.

The characteristic equation of the jacobian matrix (2.2) at the fixed point $E_2(x^*, y^*)$ is written as

$$F(\lambda) := \lambda^2 + p(x^*, y^*)\lambda + q(x^*, y^*) = 0,$$

where

$$p(x^*, y^*) = 2 - \alpha\delta - \frac{\beta\delta}{\alpha + \beta},\ q(x^*, y^*) = \frac{\beta - \beta\delta + \alpha^2(-1+\delta)\delta + \alpha(1 + \beta(-1+\delta)\delta)}{\alpha + \beta}.$$

Then $F(1) = \alpha\delta^2 > 0$ and $F(-1) = \frac{-2\beta(-2+\delta)+\alpha^2(-2+\delta)\delta+\alpha(4+\beta(-2+\delta)\delta)}{\alpha+\beta}$.

We state the local dynamics of fixed point E_2 in the following Proposition.

Proposition 2.2. *Let E_2 be a positive fixed point of* (1.3)*. Then*

(i) it is a sink if one of the following conditions holds

(i.1) $\Delta \geq 0$ *and* $\delta < \frac{N-\sqrt{\Delta}}{M}$;

(i.2) $\Delta < 0$ *and* $\delta < \frac{N}{M}$;

(ii) it is a source if one of the following conditions holds

(ii.1) $\Delta \geq 0$ *and* $\delta > \frac{N+\sqrt{\Delta}}{M}$;

(ii.2) $\Delta < 0$ *and* $\delta > \frac{N}{M}$;

(iii) it is non-hyperbolic if one of the following conditions holds

(iii.1) $\Delta \geq 0$ *and* $\delta = \frac{N\pm\sqrt{\Delta}}{M}$;

(iii.2) $\Delta < 0$ *and* $\delta = \frac{N}{M}$;

(iv) it is a saddle except for the parameter values those lie in (i)–(iii),

where

$$M = \alpha^2 + \beta + \alpha\beta,$$
$$N = \alpha^2 + \alpha\beta,$$
$$\Delta = N^2 - 4(\alpha+\beta)M.$$

If the term (iii.1) of Proposition 2.2 holds, then one of the eigenvalues of $J(E_2)$ is -1 and the other is neither 1 nor -1. Therefore, there may be a flip bifurcation of the fixed point E_2 if δ varies in the small neighborhood of $FB^1_{E_2}$ or $FB^2_{E_2}$ where

$$FB^1_{E_2} = \left\{ (\alpha, \beta, \delta) \in (0, +\infty) : \delta = \frac{N - \sqrt{\Delta}}{M},\ \Delta \geq 0 \right\},$$

or

$$FB^2_{E_2} = \left\{ (\alpha, \beta, \delta) \in (0, +\infty) : \delta = \frac{N + \sqrt{\Delta}}{M},\ \Delta \geq 0 \right\}.$$

Also when the term (iii.2) of Proposition 2.2 holds, then the eigenvalues of $J(E_2)$ are a pair of conjugate complex numbers with module one. The conditions in the term (iii.2) of Proposition 2.2 can be written as the following set:

$$NSB_{E_2} = \left\{ (\alpha, \beta, \delta) \in (0, +\infty) : \delta = \frac{N}{M},\ \Delta < 0 \right\},$$

and if the parameter δ changes in the small neighborhood of NSB_{E_2}, then the NS bifurcation will appear.

The next result is obtained from the above analysis to study the bifurcation of map (1.3) at E_2.

Proposition 2.3. *The positive fixed point E_2 loses its stability:*

(i) via a flip point when $\Delta \geq 0$ and $\delta = \frac{N \pm \sqrt{\Delta}}{M}$;

(ii) via a Neimark-Sacker point when $\Delta < 0$ and $\delta = \frac{N}{M}$.

In the following section, we shall use bifurcation theory in (see Section 4 in [10]; see also [3,14,17]) to study the flip bifurcation and the Neimark-Sacker bifurcation of system (1.3) around E_2, respectively where the parameter δ is chosen as bifurcation parameter.

3. Bifurcation analysis

We first discuss the flip bifurcation of (1.3) at E_2. Suppose that $\Delta > 0$.
If

$$\delta = \delta_F = \frac{N - \sqrt{\Delta}}{M},$$

or

$$\delta = \delta_F = \frac{N + \sqrt{\Delta}}{M},$$

then the eigenvalues of the positive fixed point (x^*, y^*) are

$$\lambda_1(\delta_F) = -1 \text{ and } \lambda_2(\delta_F) = 3 - \alpha\delta_F - \frac{\beta\delta_F}{\alpha + \beta}.$$

The condition $|\lambda_2(\delta_F)| \neq 1$ leads to

$$\alpha\delta_F + \frac{\beta\delta_F}{\alpha + \beta} \neq 2, 4. \tag{3.1}$$

Let $\tilde{x} = x - x^*$, $\tilde{y} = y - y^*$ and $A(\delta) = J(x^*, y^*)$, we transform the fixed point (x^*, y^*) of system (1.3) into the origin, then system (1.3) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to A(\delta) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} F_1(\tilde{x}, \tilde{y}, \delta) \\ F_2(\tilde{x}, \tilde{y}, \delta) \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} F_1(\tilde{x}, \tilde{y}, \delta) &= -\left(\tilde{x}^2 + \tilde{x}\tilde{y}\right)\delta + O(\|X\|^4), \\ F_2(\tilde{x}, \tilde{y}, \delta) &= \frac{\beta\delta}{{x^*}^4}\tilde{x}\left(x^*\tilde{y} - y^*\tilde{x}\right)^2 - \frac{\beta\delta}{{x^*}^3}\tilde{x}\left(x^*\tilde{y} - y^*\tilde{x}\right)^2 + O(\|X\|^4), \end{aligned} \tag{3.3}$$

and $X = (\tilde{x}, \tilde{y})^T$. It follows that

$$\begin{aligned} B_1(x, y) &= \sum_{j,k=1}^{2} \frac{\delta^2 F_1(\xi, \delta)}{\delta\xi_j \delta\xi_k}\bigg|_{\xi=0} x_j y_k = (-2x_1 y_1 - x_1 y_2 - x_2 y_1)\delta, \\ B_2(x, y) &= \sum_{j,k=1}^{2} \frac{\delta^2 F_2(\xi, \delta)}{\delta\xi_j \delta\xi_k}\bigg|_{\xi=0} x_j y_k = -\frac{2\beta\delta(x^* x_2 - y^* x_1)(x^* y_2 - y^* y_1)}{{x^*}^3}, \\ C_1(x, y, u) &= \sum_{j,k,l=1}^{2} \frac{\delta^2 F_1(\xi, \delta)}{\delta\xi_j \delta\xi_k \delta\xi_l}\bigg|_{\xi=0} x_j y_k u_l = 0, \\ C_2(x, y, u) &= \sum_{j,k,l=1}^{2} \frac{\delta^2 F_2(\xi, \delta)}{\delta\xi_j \delta\xi_k \delta\xi_l}\bigg|_{\xi=0} x_j y_k u_l = \frac{2\beta\delta}{{x^*}^2}(x_1 y_2 u_2 + x_2 y_1 u_2 + x_2 y_2 u_1) \\ &\quad - \frac{4\beta\delta y^*}{{x^*}^3}(x_1 y_1 u_2 + x_1 y_2 u_1 + x_2 y_1 u_1) + \frac{6\beta\delta {y^*}^2}{{x^*}^4} x_1 y_1 u_1, \end{aligned} \tag{3.4}$$

and $\delta = \delta_F$.

Therefore, $B(x, y) = \begin{pmatrix} B_1(x, y) \\ B_2(x, y) \end{pmatrix}$ and $C(x, y, u) = \begin{pmatrix} C_1(x, y, u) \\ C_2(x, y, u) \end{pmatrix}$ are symmetric multilinear vector functions of $x, y, u \in \mathbb{R}^2$.

Let $p, q \in \mathbb{R}^2$ be left and right eigenvectors of A for the eigenvalue $\lambda_1(\delta_F) = -1$ respectively. Then $A(\delta_F)q = -q$ and $A^T(\delta_F)p = -p$. By direct calculation we get

$$\begin{aligned} q &\sim \left(2 - \alpha\delta_F, -\frac{\alpha^2\delta_F}{\beta}\right)^T, \\ p &\sim \left(2 - \alpha\delta_F, \frac{\beta\delta_F}{\alpha + \beta}\right)^T. \end{aligned}$$

We set

$$p = \gamma_1 \left(2 - \alpha\delta_F, \frac{\beta\delta_F}{\alpha + \beta}\right)^T$$

to normalize p with respect to q, where

$$\gamma_1 = \frac{1}{(2 - \alpha\delta_F)^2 - \frac{\alpha^2\delta_F^2}{\alpha + \beta}}.$$

We see that $\langle p, q\rangle = 1$, where $\langle ., .\rangle$ is the standard scalar product in $\mathbb{R}^2$ defined by $\langle p, q\rangle = p_1q_1 + p_2q_2$.

The direction of the flip bifurcation is obtained by the sign of the critical normal form coefficient $c(\delta_F)$ as in [10]. It is given by the formula:

$$c(\delta_F) = \frac{1}{6}\langle p, C(q, q, q)\rangle - \frac{1}{2}\langle p, B(q, (A - I)^{-1}B(q, q))\rangle. \tag{3.5}$$

We state the following result.

Theorem 3.1. *The system* (1.3) *undergoes a flip bifurcation at positive fixed point* $E_2(x^*, y^*)$ *when the parameter* δ *changes in a small neighborhood of* δ_F *and if the condition* (3.1) *holds and* $c(\delta_F) \neq 0$. *Moreover, the period-2 orbits that bifurcate from* $E_2(x^*, y^*)$ *are stable (resp., unstable) if* $c(\delta_F) > 0$ *(resp.,* $c(\delta_F) < 0$*).*

We next use the NS theorem in [3,10,14,17] to study the existence of a Neimark-Sacker bifurcation.

The eigenvalues of the matrix associated with the linearization of the map (3.2) at $(\tilde{x}, \tilde{y}) = (0, 0)$ are given as

$$\lambda, \bar{\lambda} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Since $p^2 - 4q = \frac{\delta^2}{(\alpha+\beta)^2}\Delta$, so the eigenvalues are complex conjugate if $\Delta < 0$. Let

$$\delta = \delta_{NS} = \frac{N}{M}. \tag{3.6}$$

Then $detJ(\delta_{NS}) = 1$ and $\lambda, \bar{\lambda} = -\frac{p}{2} \pm \frac{i\delta}{2(\alpha+\beta)}\sqrt{-\Delta}$. Under (3.6), we have

$$|\lambda(\delta_{NS})| = 1, \quad \left.\frac{d|\lambda(\delta)|}{d\delta}\right|_{\delta=\delta_{NS}} = \frac{\alpha^2 + \beta + \alpha\beta}{2(\alpha + \beta)} \neq 0. \tag{3.7}$$

In addition, if $p(\delta_{NS}) \neq 0, 1$,

$$\alpha\delta_{NS} + \frac{\beta\delta_{NS}}{\alpha + \beta} \neq 2, 3,$$

which obviously satisfies

$$\lambda^k(\delta_{NS}) \neq 1 \;\text{ for } k = 1, 2, 3, 4. \tag{3.8}$$

Let $q \in \mathbb{C}^2$ be an eigenvector of $A(\delta_{NS})$ corresponding to the eigenvalue $\lambda(\delta_{NS})$ such that

$$A(\delta_{NS})q = \lambda(\delta_{NS})q, \quad A(\delta_{NS})\bar{q} = \bar{\lambda}(\delta_{NS})\bar{q}.$$

Also let $p \in \mathbb{C}^2$ be an eigenvector of the transposed matrix $A^T(\delta_{NS})$ corresponding to its eigenvalue, that is, $\bar{\lambda}(\delta_{NS})$,

$$A^T(\delta_{NS})p = \bar{\lambda}(\delta_{NS})p, \quad A^T(\delta_{NS})\bar{p} = \lambda(\delta_{NS})\bar{p}.$$

By direct calculation we obtain

$$q \sim \left(1 - \alpha\delta_{NS} - \lambda, -\frac{\alpha^2\delta_{NS}}{\beta}\right)^T,$$

$$p \sim \left(1 - \alpha\delta_{NS} - \bar{\lambda}, \frac{\beta\delta_{NS}}{\alpha + \beta}\right)^T$$

We set

$$p = \gamma_2 \left(1 - \alpha\delta_{NS}, \frac{\beta\delta_{NS}}{\alpha+\beta}\right)^T$$

to normalize p with respect to q, where

$$\gamma_2 = \frac{1}{(1+\alpha\delta_{NS}-\bar{\lambda})^2 - \frac{\alpha^2\delta_{NS}^2}{\alpha+\beta}}.$$

It is clear that $\langle p, q\rangle = 1$, where $\langle ., .\rangle$ is the standard scalar product in $\mathbb{C}^2$ defined by $\langle p, q\rangle = \bar{p_1}q_2 + \bar{p_2}q_1$.

Any vector $X \in \mathbb{R}^2$ can be represented for δ near δ_{NS} as $X = zq + \bar{z}\bar{q}$, for some complex z. Indeed, the explicit formula to determine z is $z = \langle p, X\rangle$. Thus, system (3.2) can be transformed for sufficiently small $|\delta|$ (near δ_{NS}) into the following form:

$$z \mapsto \lambda(\delta)z + g(z, \bar{z}, \delta),$$

where $\lambda(\delta)$ can be written as $\lambda(\delta) = (1+\varphi(\delta))e^{i\theta(\delta)}$ (where $\varphi(\delta)$ is a smooth function with $\varphi(\delta_{NS}) = 0$) and g is a complex-valued smooth function of $z, \bar{z}$, and δ, whose Taylor expression with respect to $(z, \bar{z})$ contains quadratic and higher-order terms:

$$g(z, \bar{z}, \delta) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(\delta) z^k \bar{z}^l, \quad \text{with} \quad g_{kl} \in \mathbb{C}, \ k, l = 0, 1, \cdots .$$

By symmetric multilinear vector functions, the Taylor coefficients g_{kl} can be expressed by the formulas

$$g_{20}(\delta_{NS}) = \langle p, B(q, q)\rangle, \quad g_{11}(\delta_{NS}) = \langle p, B(q, \bar{q})\rangle$$
$$g_{02}(\delta_{NS}) = \langle p, B(\bar{q}, \bar{q})\rangle, \quad g_{21}(\delta_{NS}) = \langle p, C(q, q, \bar{q})\rangle,$$

and the coefficient $a(\delta_{NS})$, which determines the direction of the appearance of the invariant closed curve, can be computed via

$$a(\delta_{NS}) = \mathrm{Re}\left(\frac{e^{-i\theta(\delta_{NS})}g_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1-2e^{i\theta(\delta_{NS})})e^{-2i\theta(\delta_{NS})}}{2(1-e^{i\theta(\delta_{NS})})} g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2,$$

where $e^{i\theta(\delta_{NS})} = \lambda(\delta_{NS})$.

Clearly, (3.7) and (3.8) demonstrate that the transversal condition and the non-degenerate condition of system (1.3) hold well. We obtain the following result.

Theorem 3.2. *If $a(\delta_{NS}) \neq 0$, then system* (1.3) *undergoes a Neimark-Sacker bifurcation at the positive fixed point E_2 when the parameter δ changes in the small neighborhood of NSB_{E_2}. Moreover, if $a(\delta_{NS}) < 0$ (resp., > 0), then the NS bifurcation of system* (1.3) *at $\delta = \delta_{NS}$ is supercritical (resp., subcritical) and there exists a unique invariant closed curve bifurcates from E_2 for $\delta = \delta_{NS}$, which is attracting (resp., repelling).*

Next we give an example, which illustrates Theorem 3.1.

Example 3.1. Consider system (1.3) with $\alpha = 0.35, \beta = 2.5$, and $\delta = \delta_F = 2.57604$. Then $(\alpha, \beta, \delta) \in FB1_{E_2}$ and there is a unique positive fixed point $(0.877193, 0.122807)$ with multipliers $\lambda_1 = -1, \lambda_2 = -0.161293$, and $c(\delta_F) = 0.952787$. Hence, a flip bifurcation emerges from the fixed point $(0.877193, 0.122807)$ at $\delta = \delta_F$. This verifies Theorem 3.1.

4. Numerical simulations

In this section, numerical simulation works have been performed to present bifurcation diagrams, phase portraits, Lyapunov exponents and fractal dimension of system (1.3) to confirm our theoretical results and to show some new interesting complex dynamical behaviors existing in system (1.3). We consider the bifurcation parameters in the following cases:

case (i) varying δ in range $1.7 \leq \delta \leq 1.9556$, and fixing $\alpha = 0.75, \beta = 1.0$;
case (ii) varying δ in range $1.7 \leq \delta \leq 1.9556$, β in range $1.0 \leq \beta \leq 1.3$, and fixing $\alpha = 0.75$.

For case (i): The bifurcation diagrams of map (1.3) in $(\delta - x)$ plane and in $(\delta - y)$ plane are given in Figure 1(a-b). We observe that at the fixed point $(0.571429, 0.428571)$ of map (1.3), a NS bifurcation emerges at $\delta = \delta_{NS} \sim 1.7619$ and $(\alpha, \beta, \delta) \in NSB_{E_2}$. It shows the correctness of proposition 2.2. And for $\delta = \delta_{NS}$, we have $\lambda, \bar{\lambda} = -0.164116 \pm 0.986441i, |\lambda| = 1, |\bar{\lambda}| = 1, \frac{d|\lambda(\delta)|}{d\delta}|_{\delta=\delta_{NS}} = 0.660714 > 0, \delta_{NS}(\alpha + \frac{\beta}{\alpha+\beta}) = 2.32823 \neq 2, 3,\ g_{20} = 3.61014 - 2.49718i, g_{11} = 4.0744 - 1.41092i, g_{02} = -2.2728 + 5.54114i, g_{21} = -6.39354 + 11.0869i$ and $a(\delta_{NS}) = -11.8679$. Therefore, the NS bifurcation is supercritical and it verifies Theorem 3.2.

The maximum Lyapunov exponents corresponding to Figure 1(a-b) are computed and plotted in Figure 1(c). From Figure 1(c), we see that some Lyapunov exponents are positive, some are negative, so there exist stable fixed points or stable period windows in the chaotic region. Figure 1(d) is the local amplification corresponding to Figure 1(a) for $\delta \in [1.8702, 1.9142]$. The diagrams in Figure 1(a-b) show that the fixed point E_2 of map (1.3) is stable for $\delta < 1.7619$ and loses its stability as δ increases. NS bifurcation occurs at $\delta \sim 1.7619$ and an invariant circle appears when δ exceeds 1.7619. As δ grows, the circle disappears suddenly and a period-11 orbits appear at $\delta \sim 1.88$. We also see that for $\delta \sim 1.9556$, a fully developed chaos in system (1.3) occurs. The phase portraits associated with Figure 1(a-b) for various values of δ are disposed in Figure 2, which clearly depicts the process of how a smooth invariant circle bifurcates from the stable fixed point.

For case (ii): The dynamics of map (1.3) can change when more parameters vary. The 3-dimensional bifurcation diagrams of map (1.3) for control parameters $\delta \in [1.7, 1.9556],\quad \beta \in [1.0, 1.3]$, respectively, and fixing other parameters as in case (ii), are plotted in Figure 3 (a-b). Since the measure of Lyapunov exponents quantify the chaotic dynamics of the discrete system, so we will compute the maximum Lyapunov exponents of system (1.3) and will study the dependence of these Lyapunov exponents on two real parameters δ and a. The 3-dimensional maximum Lyapunov exponents is plotted in Figure 3(c) and its 2-dimensional projection onto (δ, a) is shown in Figure 3(d). It is clear to see for which choice of parameters map (1.3) is showing chaotic motion, and for which one is map (1.3) exhibiting periodic or quasi-periodic movement. For instance, the chaotic dynamics is on Figure 2 for values of parameters $\delta = 1.9556,\quad \beta = 1.0$, and the non-chaotic dynamics is for values of parameters $\delta = 1.75,\quad \beta = 1.0$, which are consistent with the signs in Figure 3.

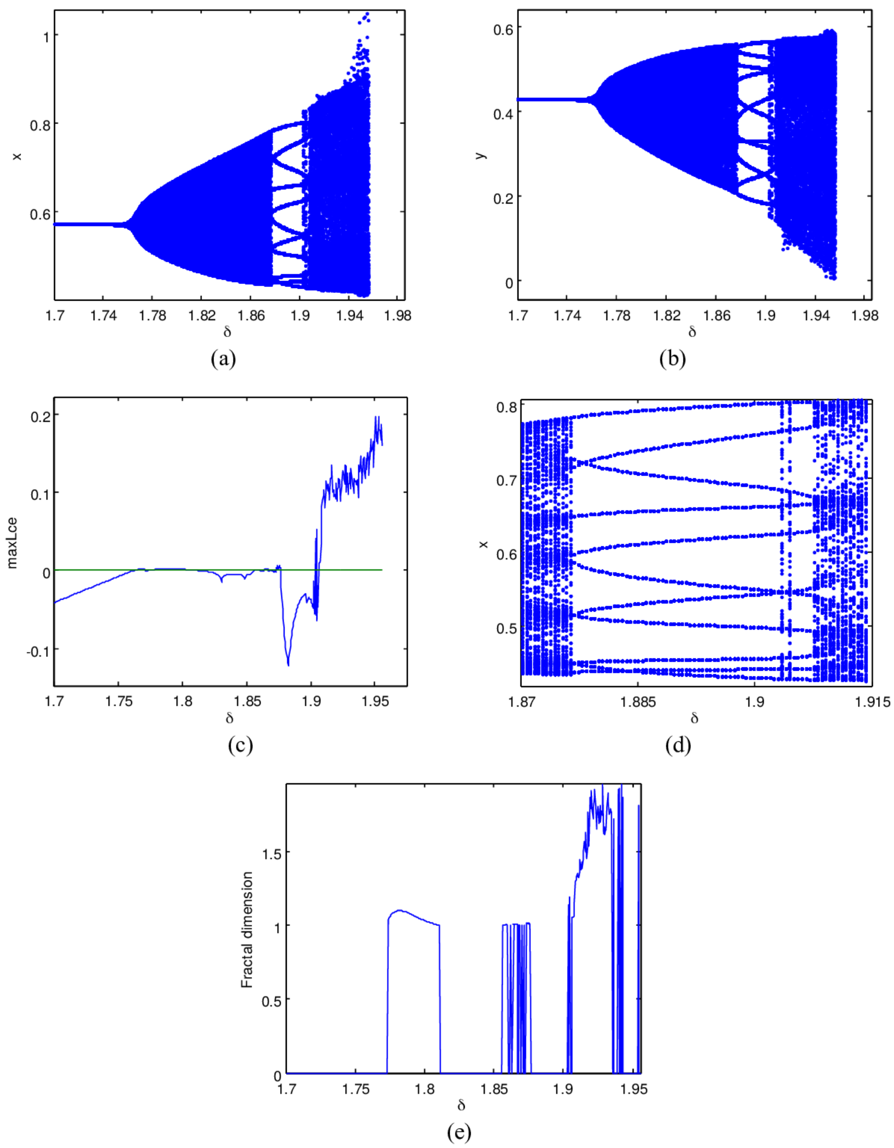

Figure 1. Bifurcation diagrams and maximum Lyapunov exponent of map (1.3) around E_2. (a) Neimark-Sacker bifurcation diagram of map (1.3) in $(\delta - x)$ plane, (b) NS bifurcation diagram in $(\delta - y)$ plane, (c) maximum Lyapunov exponents corresponding to (a-b), (d) local amplification corresponding to (a) for $\delta \in [1.8702, 1.9142]$ (e) Fractal dimension corresponding to (a). The initial value is $(x_0, y_0) = (0.57, 0.42)$.

4.1. Fractal dimension of the map

We compute fractal dimensions to characterize strange attractors of a map. Lyapunov exponents quantify the separation of neighboring chaotic orbits to observe how fast they separate each other. These exponents also indicate a dynamic measure of chaos which average the separation of the orbits of nearby initial conditions as system runs forward in time. The fractal dimension [1, 9] is defined by using

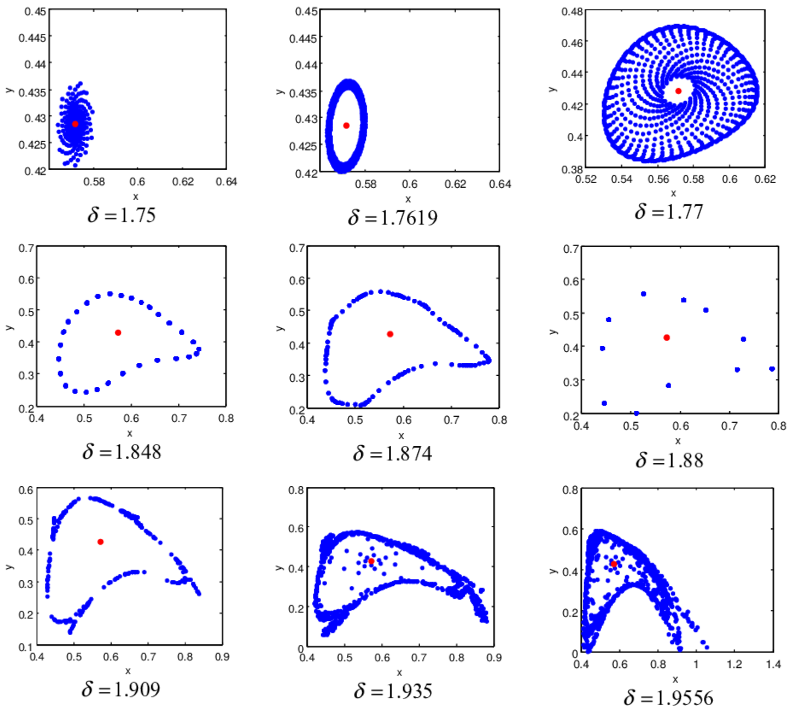

Figure 2. Phase portraits for various values of δ corresponding to Figure 1(a-b).

Lyapunov exponents as follows:

$$d_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_j|}$$

with $\lambda_1, \lambda_2, ..., \lambda_n$, where j is the largest integer such that $\sum_{i=1}^{j} \lambda_i \geq 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$.

The fractal dimension of our two-dimensional map (1.3) is of the form

$$d_L = 1 + \frac{\lambda_1}{|\lambda_2|}, \quad \lambda_1 > 0 > \lambda_2.$$

With given set of parameter values, two Lyapunov exponents are estimated by computer simulation and are found to be $\lambda_1 \approx 0.0901$ and $\lambda_2 \approx -0.2924$ for $\delta = 1.9091$, and $\lambda_1 \approx 0.1157$ and $\lambda_2 \approx -0.1505$ for $\delta = 1.9352$. So the fractal dimensions of the strange attractor of map (1.3) are

$$d_L \approx 1 + \frac{0.0901}{0.2924} = 1.3082 \quad \text{for} \quad \delta = 1.9091 \quad \text{and}$$
$$d_L \approx 1 + \frac{0.1157}{0.1505} = 1.7689 \quad \text{for} \quad \delta = 1.9352.$$

The strange attractors given in Figure 2 and its corresponding fractal dimension illustrate that the predator-prey system of Leslie type (1.3) has a very complex dynamic behaviors as the parameter δ increases.

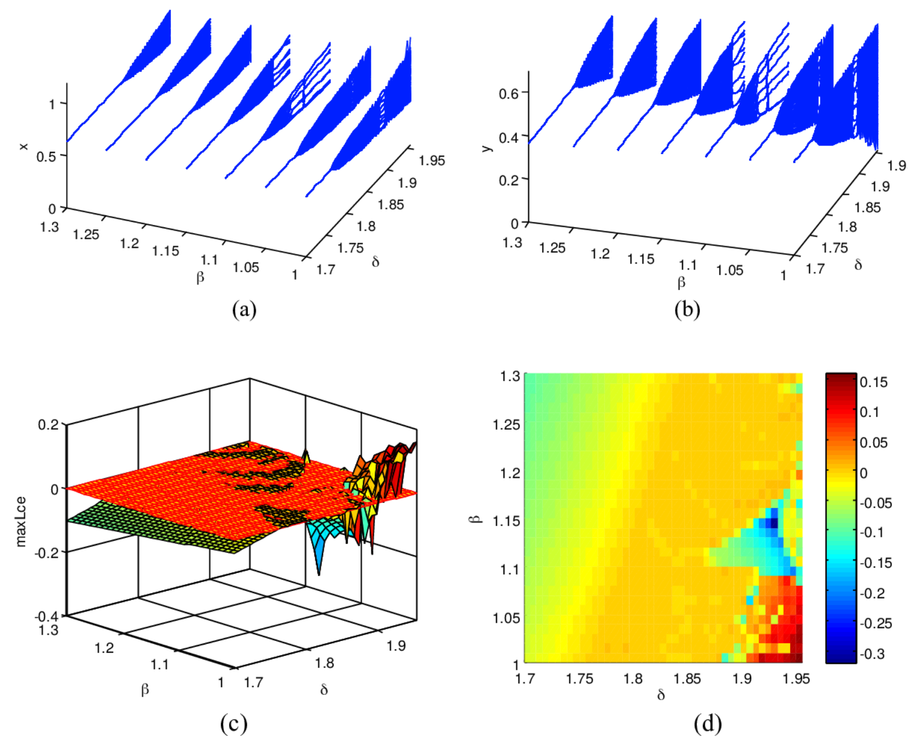

Figure 3. The 3-dimensional bifurcation diagram and maximum Lyapunov exponents of map (1.3) around E_2. (a-b) The 3-dimensional bifurcation diagrams of map (1.3) covering $\delta \in [1.7, 1.9556]$, $\beta \in [1.0, 1.3]$, and $\alpha = 0.75$ in $(\delta - a - x)$ space and $(\delta - a - y)$ space (c) The 3-dimensional maximum Lyapunov exponents corresponding to (a-b) (d) The 2-dimensional projection onto (δ, a) plane. The initial value is $(x_0, y_0) = (0.57, 0.42)$.

5. Chaos control

In order to stabilize chaotic orbits at an unstable fixed point of system (1.3), a state feedback control method [2,11] is applied. By adding a feedback control law as the control force u_n to system (1.3), the controlled form of (1.3) becomes

$$\begin{aligned} x_{n+1} &= x_n + \delta \left[x_n (1 - x_n) - x_n y_n \right] + u_n, \\ y_{n+1} &= y_n + \delta \left[\alpha y_n - \frac{\beta y_n^2}{x_n} \right] \end{aligned} \tag{5.1}$$

and

$$u_n = -k_1(x_n - x^*) - k_2(y_n - y^*),$$

where $k_1$and k_2 are the feedback gains, and (x^*, y^*) is the positive fixed point of system (1.3).

The Jacobian matrix J_c of the controlled system(5.1) is given by

$$J_c(x^*, y^*) = \begin{pmatrix} a_{11} - k_1 & a_{12} - k_2 \\ a_{21} & a_{22} \end{pmatrix}, \tag{5.2}$$

where $a_{11} = \frac{\alpha+\beta-\beta\delta}{\alpha+\beta}, a_{12} = -\frac{\beta\delta}{\alpha+\beta}, a_{21} = \frac{\alpha^2\delta}{\beta}, a_{22} = 1 - \alpha\delta$. The characteristic equation of (5.2) is

$$\lambda^2 - (a_{11} + a_{22} - k_1)\lambda + a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2) = 0. \tag{5.3}$$

Let λ_1 and λ_2 be the eigenvalues. Then

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} - k_1 \tag{5.4}$$

and

$$\lambda_1\lambda_2 = a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2). \tag{5.5}$$

The lines of marginal stability are determined by solving the equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These conditions guarantee that the eigenvalues λ_1 and λ_2 have modulus less than 1.

Assume that $\lambda_1\lambda_2 = 1$, then from (5.5) we have

$$l_1 : a_{22}k_1 - a_{21}k_2 = a_{11}a_{22} - a_{12}a_{21} - 1.$$

Assume that $\lambda_1 = 1$, then from (5.4) and (5.5) we get

$$l_2 : (1 - a_{22})k_1 + a_{21}k_2 = a_{11} + a_{22} - 1 - a_{11}a_{22} + a_{12}a_{21}.$$

Assume that $\lambda_1 = -1$, then from (5.4) and (5.5) we obtain

$$l_3 : (1 + a_{22})k_1 - a_{21}k_2 = a_{11} + a_{22} + 1 + a_{11}a_{22} - a_{12}a_{21}.$$

The stable eigenvalues lie within a triangular region by lines $l_1, l_2,$ and l_3 (see Figure 4(a)) in the (k_1, k_2) plane.

Some numerical simulations have been performed to see how the state feedback method controls the unstable fixed point. Parameter values are fixed as $\delta = 1.8$ and rest as in case(i). The initial value is $(x_0, y_0) = (0.57, 0.42)$, and the feedback gains are $k_1 = 1.0$ and $k_2 = -0.55$. Figures 4(b) and 4(c) show that a chaotic trajectory is stabilized at the fixed point $(0.571429, 0.428571)$.

6. Discussions

We investigated the dynamic behaviors of the discrete-time predator-prey system of Leslie type (1.3) in details and showed that it has a complex dynamics in the closed first quadrant $\mathbb{R}^2_+$. We established the conditions for the existence of a flip bifurcation and a NS bifurcation of map (1.3) at unique positive fixed point by using center manifold theorem and bifurcation theory. Some other dynamical features of system (1.3) have been analyzed by means of bifurcation diagrams, phase portraits, Lyapunov exponents, and fractal dimension. Specifically, as the parameters vary, system (1.3) exhibits a variety of dynamical behaviors, which include orbits of period-11, an invariant cycle, and chaotic sets. These all mean that the predator can coexist with prey in the stable period-n orbits and smooth invariant cycle in case of NS bifurcation. We computed Lyapunov exponents and fractal dimension to confirm the chaotic dynamics. We observed that when the prey shows chaotic dynamic, the predator can tend to extinction or to a stable fixed point. Finally, the chaotic orbits at an unstable fixed point are stabilized by using the feedback control method.

Competing interests. The author declares that there is no competing interest.

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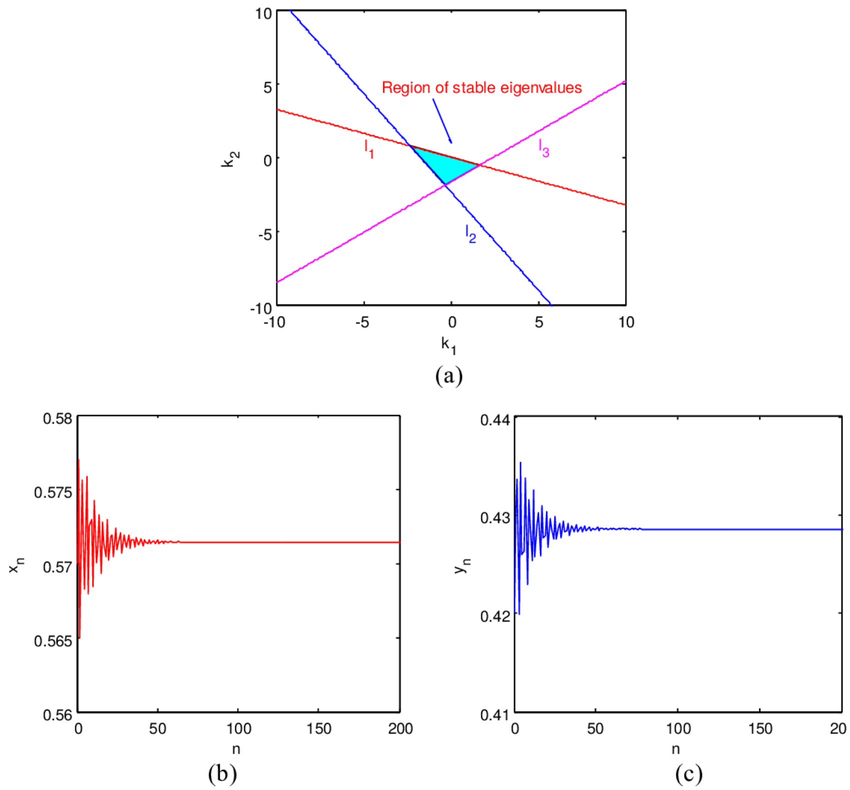

Figure 4. Control of chaotic orbits of system (5.1). (a) The bounded region for the stable eigenvalues of the controlled system (5.1) in the (k_1, k_2) plane (b-c) The time responses for the states x and y of the controlled system (5.1) in the (n, x) and (n, y) planes respectively.

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