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SOLVABILITY OF HYPERBOLIC FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract The main purpose of this paper is to study the existence and uniqueness of solutions for the hyperbolic fractional differential equations with integral conditions. Under suitable assumptions, the results are established by using an energy integral method which is based on constructing an appropriate multiplier. Further we find the solution of the hyperbolic fractional differential equations using Adomian decomposition method. Examples are provided to illustrate the theory.

Keywords Existence and uniqueness, fractional derivatives and integrals, hyperbolic equations, a priori estimates, Adomian decomposition method.

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1. Introduction

Since fractional derivatives and integrals are more suitable than integer order derivatives and integrals in the real world phenomena, many researchers concentrate on fractional calculus in the previous decades. Due to an enthusiastic efforts of researchers, there has been a rapid development on fractional calculus and its applications (see [2, 13, 18]). Because of its framework and novel surprising insights, the continuous growth of the fractional differential equations to cope with engineering and the applied sciences is reflected in the huge number of books and papers involving fractional derivatives in the last few decades. For example, several authors have studied the existence and uniqueness results of fractional differential equations [5, 6, 16]. Observability, controllability and stability of fractional dynamical systems are also discussed in the literature. For further details of qualitative behaviors of fractional dynamical systems, see the papers [3, 4, 17].

Another interesting area of research is the investigation of fractional partial differential equations. In recent years, fractional partial differential equations are more and more helpful in modeling fluid flow and biological systems. Hence many of the authors have investigated fractional partial differential equations. Here we propose some of the works concerning fractional partial differential equations. Guo et al. [12] described the theoretical and numerical aspects of fractional partial differential e-

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quations arising in plasma physics and atmosphere ocean dynamics. Sakamoto and Yamamoto [31] proved the existence and uniqueness of fractional diffusion wave equations and analyzed the asymptotic behavior of the same equation by using the eigenfunction expansions. For fractional diffusion system which appears in biological populations of cells, the existence and uniqueness of solution in the Sobolev space is presented in [14] by the analytic method of fixed point theory. Javidi and Nyamoradi [15] considered the phytoplankton model and analyzed the dynamic behavior of the model through fractional Routh-Hurwits conditions and also examined local stability of the equilibrium points of the corresponding model. In [27], Oussaeif and Bouziani studied about the existence and uniqueness of solution for parabolic fractional differential equations

$$Lv = \frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} v(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right) = F(x,t), \tag{1.1}$$

in a functional weighted Sobolev space with integral conditions by using a priori estimates. Mesloub [23] discussed the existence and uniqueness of a strong solution for a fractional two times evolution equation with initial and boundary integral conditions.

There are a very few articles related to existence of unique solutions of fractional partial differential equations by means of energy inequality method except [27] and [23]. But for partial differential equations, many authors have applied the energy integral method. For instance, the solvability of a class of singular hyperbolic differential equation

$$Lv = v_{tt} - \frac{1}{x}v_x - v_{xx} = f(x,t), \quad (x,t) \in (0,R) \times (0,T) \tag{1.2}$$

with weighted integral conditions has been studied by Mesloub and Bouziani [20]. In [7], Bouziani established the existence and uniqueness results for parabolic and hyperbolic equations with boundary integral conditions by applying the technique based on priori estimates. He also studied the well-posedness of a second order hyperbolic equation with Bessel operator [8]. Mesloub [21] established the well-posedness of a nonlinear pseudoparabolic equation which is a model of heat conduction. By applying an iterative process, he proved the existence and uniqueness of solution for the two dimensional parabolic equation [22]. In addition, see [9, 28] and references therein. Motivated by these results, we extend the results of [27] to hyperbolic fractional partial differential equations and also obtain the approximate solutions by Adomian decomposition method.

The rest of this paper is organized as follows: In Section 2, we state some basic definitions and properties that are inherently tied to fractional calculus. Reformulation of the problem is given in Section 3. In Section 4, the existence and uniqueness of solution of the hyperbolic fractional differential equation with the boundary integral conditions is proved by energy estimate arguments. Section 5 investigates the approximate solution of the problem by using Adomian decomposition method. In Section 6, two examples are proved to illustrate the unique existence result. Last Section contains conclusion of this work.

2. Preliminaries

We begin this section by briefly introducing the basic definitions of fractional calculus. Let $\Gamma(\cdot)$ denote the gamma function. For any $n-1<\alpha<n$, $n\in\mathbb{N}$, the commonly used fractional definitions are defined as follows:

Definition 2.1 ([16]). The partial Riemann-Liouville fractional integral operator of order α with respect to t of a function $f(x,t)$ is defined by

$$I^\alpha f(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x,s)}{(t-s)^{n-\alpha}}\,\mathrm{d}s.$$

Definition 2.2 ([16]). The partial Riemann-Liouville fractional derivative of order α of a function $f(x,t)$ with respect to t is of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}f(x,t)=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t \frac{f(x,s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where the function $f(x,t)$ has absolutely continuous derivatives upto order $(n-1)$.

Definition 2.3 ([16]). The Caputo partial fractional derivative of order α with respect to t of a function $f(x,t)$ is defined as

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}f(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^n f(x,s)}{\partial s^n}\,\mathrm{d}s,$$

where the function $f(x,t)$ has absolutely continuous derivatives upto order $(n-1)$.

To know the properties of these operators see the books [26, 32] and for more facts on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types, see [30]. In this paper we adopt Caputo fractional derivative, since it permits conventional initial conditions which is involved in our problem. Next we give some well known inequalities which are needed throughout this paper.

Let $\Im_x u=\int_0^x u(\xi,t)\,\mathrm{d}\xi$ and $\Im_t u=\int_0^t u(x,\tau)\,\mathrm{d}\tau$.

(i) Cauchy's inequality with ϵ [10]

$$ab\le \epsilon a^2+\frac{1}{4\epsilon}b^2,\qquad a,b>0 \text{ and } \epsilon>0.$$

(ii) Poincare-type inequalities [23]

$$\text{(a)}\quad \|\Im_x u\|^2_{L^2(\Omega)}\le \frac{1}{2}\|u\|^2_{L^2(\Omega)}. \tag{2.1}$$

$$\text{(b)}\quad \|\Im_x^2 u\|^2_{L^2(\Omega)}\le \frac{1}{2}\|\Im_x u\|^2_{L^2(\Omega)}. \tag{2.2}$$

(iii) For any α, we have the inequality [23]

$$\int_0^1 \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x u)^2\,\mathrm{d}x\le 2\int_0^1 \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x u)(\Im_x u)\,\mathrm{d}x. \tag{2.3}$$

3. Formulation of the Problem

Consider the domain $Q = \Omega \times J = (0,1) \times (0,T)$, $0 < T < \infty$ and the equation

$$\mathcal{L}w = g(x,t), \quad 0 < x < 1, \ \ 0 < t < T, \tag{3.1}$$

where

$$\mathcal{L}w = \frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} w(x,t) - a(x,t)\frac{\partial w}{\partial x} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w}{\partial x}\right), \quad 1 < \alpha < 2$$

with the initial conditions

$$l_1 w = w(x,0) = \phi(x), \qquad x \in \Omega, \tag{3.2}$$

$$l_2 w = \frac{\partial w(x,0)}{\partial t} = \psi(x), \quad x \in \Omega, \tag{3.3}$$

and the integral conditions

$$\int_{\Omega} x^i w(x,t)\ \mathrm{d}x = m_i(t), \quad t \in J, \quad (i = 0,1), \tag{3.4}$$

where a, b, g, ϕ, ψ and $m_i's$ are known functions and $\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}}$ denotes the Caputo fractional derivative. The compatibility conditions are

$$\int_{\Omega} x^i \phi(x)\ \mathrm{d}x = m_i(0), \quad (i = 0,1).$$

For our convenience, we change the non-homogeneous boundary conditions to homogeneous ones. So let

$$v(x,t) = m_0(t) + 10\big(2m_1(t) - m_0(t)\big)(4x^3 - 3x^2).$$

Now by introducing a new function $u(x,t) = w(x,t) - v(x,t)$, the problem (3.1)-(3.4) can be written as

$$\begin{aligned}
\mathcal{L}u &= \frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) - a(x,t)\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right), \quad 1 < \alpha < 2 \\
&= f(x,t), & (3.5)\\
l_1 u &= u(x,0) \quad = \varphi(x), \quad x \in \Omega, & (3.6)\\
l_2 u &= \frac{\partial u(x,0)}{\partial t} = \Psi(x), \quad x \in \Omega, & (3.7)\\
&\int_{\Omega} x^i u(x,t)\ \mathrm{d}x = 0, \quad t \in J, \quad (i = 0,1). & (3.8)
\end{aligned}$$

Rather than the function w, it is enough to prove the existence and uniqueness of solution for the function u. Now it is sufficient to show the existence and uniqueness of solution of the problem (3.5)-(3.8).

4. Existence and Uniqueness of Solution

We start this section by introducing the function space $L^2(\Omega)$, the space of square integrable functions on Ω. The solution of the problem (3.5)-(3.8) can be considered as a solution of the operator equation $Lu = F$, where $L = (\mathcal{L}, l_1, l_2)$ and $F = (f, \varphi, \Psi)$. The operator $L : B \to H$ is considered with the domain

$$D(L) = \begin{cases} u \in L^2(Q) : \dfrac{{}^C\partial^\alpha u}{\partial t^\alpha}, u_x, u_{xx} \in L^2(Q) \\ \quad \text{and } u \text{ satisfies the condition (3.8),} \end{cases} \tag{4.1}$$

where B is a Banach space of functions with respect to the norm

$$\|u\|_B^2 = \|\Im_x u\|_{L^2(Q)}^2 \tag{4.2}$$

and H is the Hilbert space endowed with the norm

$$\|F\|_H^2 = \int_J \Big(\|f\|_{L^2(\Omega)}^2 + I^\alpha \|f\|_{L^2(\Omega)}^2 + \|\Im_x \varphi\|_{L^2(\Omega)}^2 + \|\Im_x \Psi\|_{L^2(\Omega)}^2 \Big) \, \mathrm{d}t. \tag{4.3}$$

The central task of this paper is to show the following result.

Theorem 4.1. *Let* $\dfrac{\partial^2 a}{\partial x^2} - \dfrac{\partial a}{\partial x} - \dfrac{\partial^2 b}{\partial x^2} + 4 \inf_{x \in \Omega} b - \epsilon \geq M$ *for sufficiently small* ϵ *and* $M > 0$. *Then there exists a positive constant* C *such that*

$$\|u\|_B \leq C \|Lu\|_H, \qquad u \in D(L), \tag{4.4}$$

where C *is independent of* u.

Proof. Multiplying equation (3.5) by $Mu = -\Im_x^2 u$, then integrating over Ω we obtain

$$\begin{aligned} \int_\Omega Lu \cdot Mu \, \mathrm{d}x &= -\int_\Omega \frac{{}^C\partial^\alpha u}{\partial t^\alpha} (\Im_x^2 u) \, \mathrm{d}x + \int_\Omega a(x,t) \frac{\partial u}{\partial x} (\Im_x^2 u) \, \mathrm{d}x \\ &\quad + \int_\Omega \frac{\partial}{\partial x} \Big(b(x,t) \frac{\partial u}{\partial x} \Big) (\Im_x^2 u) \, \mathrm{d}x \\ &= -\int_\Omega f(x,t) (\Im_x^2 u) \, \mathrm{d}x. \end{aligned} \tag{4.5}$$

Using the conditions (3.6)-(3.8) and integration by parts, we attain

$$-\int_\Omega \frac{{}^C\partial^\alpha u}{\partial t^\alpha} (\Im_x^2 u) \, \mathrm{d}x = \int_\Omega \frac{{}^C\partial^\alpha}{\partial t^\alpha} (\Im_x u)(\Im_x u) \, \mathrm{d}x, \tag{4.6}$$

$$\int_\Omega a(x,t) \frac{\partial u}{\partial x} (\Im_x^2 u) \, \mathrm{d}x = \frac{1}{2} \int_\Omega \frac{\partial^2 a}{\partial x^2} (\Im_x u)^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega \frac{\partial a}{\partial x} (\Im_x u)^2 \, \mathrm{d}x, \tag{4.7}$$

$$\int_\Omega \frac{\partial}{\partial x} \Big(b(x,t) \frac{\partial u}{\partial x} \Big) (\Im_x^2 u) \, \mathrm{d}x = \int_\Omega b(x,t)(u(x,t))^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega \frac{\partial^2 b}{\partial x^2} (\Im_x u)^2 \, \mathrm{d}x. \tag{4.8}$$

With the help of Cauchy's inequality with ϵ, the right hand side of (4.5) becomes

$$-\int_\Omega f(x,t)(\Im_x^2 u) \, \mathrm{d}x \leq \epsilon \int_\Omega (\Im_x^2 u)^2 \, \mathrm{d}x + \frac{1}{4\epsilon} \int_\Omega (f(x,t))^2 \, \mathrm{d}x. \tag{4.9}$$

Substituting (4.6)-(4.9) in (4.5) and applying the inequalities (2.2) and (2.3), we have

$$\begin{aligned}\int_\Omega \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x u)^2\ \mathrm{d}x + \int_\Omega \frac{\partial^2 a}{\partial x^2}(\Im_x u)^2\ \mathrm{d}x - \int_\Omega \frac{\partial a}{\partial x}(\Im_x u)^2\ \mathrm{d}x + 2\int_\Omega b(x,t)(u(x,t))^2\ \mathrm{d}x \\ - \int_\Omega \frac{\partial^2 b}{\partial x^2}(\Im_x u)^2\ \mathrm{d}x \le 2\epsilon \int_\Omega (\Im_x^2 u)^2\ \mathrm{d}x + \frac{1}{2\epsilon}\int_\Omega (f(x,t))^2\ \mathrm{d}x \\ \le \epsilon \int_\Omega (\Im_x u)^2\ \mathrm{d}x + \frac{1}{2\epsilon}\int_\Omega (f(x,t))^2\ \mathrm{d}x. \qquad (4.10)\end{aligned}$$

Simplifying the above inequality, we get

$$\begin{aligned}\int_\Omega \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x u)^2\ \mathrm{d}x + 2\inf_{x\in Q} b\int_\Omega (u(x,t))^2\ \mathrm{d}x + \int_\Omega \left(\frac{\partial^2 a}{\partial x^2} - \frac{\partial a}{\partial x} - \frac{\partial^2 b}{\partial x^2} - \epsilon\right)(\Im_x u)^2\ \mathrm{d}x \\ \le \frac{1}{2\epsilon}\int_\Omega (f(x,t))^2\ \mathrm{d}x. \qquad (4.11)\end{aligned}$$

To estimate the second integral on the left hand side of (4.11), make use of the inequality (2.1) and observe that

$$\begin{aligned}\int_\Omega \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x u)^2\ \mathrm{d}x + \int_\Omega \left(\frac{\partial^2 a}{\partial x^2} - \frac{\partial a}{\partial x} - \frac{\partial^2 b}{\partial x^2} + 4\inf_{x\in Q} b - \epsilon\right)(\Im_x u)^2\ \mathrm{d}x \\ \le \frac{1}{2\epsilon}\int_\Omega (f(x,t))^2\ \mathrm{d}x. \qquad (4.12)\end{aligned}$$

By the assumption $\left(\frac{\partial^2 a}{\partial x^2} - \frac{\partial a}{\partial x} - \frac{\partial^2 b}{\partial x^2} + 4\inf_{x\in Q} b - \epsilon\right) \ge M > 0$, we can write the above inequality as

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}\|\Im_x u\|^2_{L^2(\Omega)} + M\|\Im_x u\|^2_{L^2(\Omega)} \le \frac{1}{2\epsilon}\|f\|^2_{L^2(\Omega)}. \qquad (4.13)$$

This inequality also takes the form

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}\|\Im_x u\|^2_{L^2(\Omega)} + \|\Im_x u\|^2_{L^2(\Omega)} \le \frac{(2\epsilon)^{-1}}{\min\{1,M\}}\|f\|^2_{L^2(\Omega)}.$$

Since $u(x,0)=\varphi(x)$, $\dfrac{\partial u(x,0)}{\partial t} = \Psi(x)$ and the property

$$I^\alpha \frac{{}^C\partial^\alpha}{\partial t^\alpha}u(x,t) = u(x,t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial}{\partial t}u(x,0), \quad n-1<\alpha<n, \qquad (4.14)$$

we deduce from (4.13) that

$$\|\Im_x u\|^2_{L^2(\Omega)} - \|\Im_x \varphi\|^2_{L^2(\Omega)} - t\|\Im_x \Psi\|^2_{L^2(\Omega)} \le \frac{(2\epsilon)^{-1}}{\min\{1,M\}} I^\alpha \|f\|^2_{L^2(\Omega)}, \qquad (4.15)$$

$$\|\Im_x u\|^2_{L^2(\Omega)} \le \frac{(2\epsilon)^{-1}}{\min\{1,M\}}\|f\|^2_{L^2(\Omega)}. \qquad (4.16)$$

Then combining (4.15) and (4.16),

$$\|\Im_x u\|^2_{L^2(\Omega)} \leq C\Big(\|f\|^2_{L^2(\Omega)} + I^\alpha\|f\|^2_{L^2(\Omega)} + \|\Im_x\varphi\|^2_{L^2(\Omega)} + \|\Im_x\Psi\|^2_{L^2(\Omega)}\Big), \tag{4.17}$$

where $C = \dfrac{\max\{1, T, (2\epsilon)^{-1}\}}{\min\{1, M\}}$. Now integrating (4.17) over J, we conclude that

$$\|\Im_x u\|^2_{L^2(Q)} \leq C\int_J \Big(\|f\|^2_{L^2(\Omega)} + I^\alpha\|f\|^2_{L^2(\Omega)} + \|\Im_x\varphi\|^2_{L^2(\Omega)} + \|\Im_x\Psi\|^2_{L^2(\Omega)}\Big)\, \mathrm{d}t.$$

This completes the proof. □

From (4.4), there exists a bounded inverse L^{-1} on the range $R(L)$ of L. Since we don't have any information about $R(L)$ except that $R(L) \subset H$, we should extend the operator L such that (4.4) holds for the extension and range of that extension is the space H.

Corollary 4.1. *The operator L from B into H has a closure $\bar{L}$.*

Proof. The proof is similar to that in [7]. □

Definition 4.1. A solution of the equation $\bar{L}u = (f, \varphi, \Psi)$ is called a strong solution of problem (3.5)-(3.8).

Since $\bar{L}$ is the closure of L, we can extend the inequality (4.4) to the operator $\bar{L}$ by applying limits on both the sides of (4.4). Thus under the conditions of Theorem 4.1, we can say that

$$\|u\|_B \leq C\|\bar{L}u\|_H. \tag{4.18}$$

holds for any $u \in D(\bar{L})$. Hence we establish that under the assumptions of Theorem 4.1, a strong solution of the problem (3.5)-(3.8) is unique if it exists, and depends continuously on $F \in H$. To prove the existence of the solution, it is sufficient to show that the range $R(\bar{L}) = H$. But $R(\bar{L})$ equals to the closure $\overline{R(L)}$ of $R(L)$. So we have to show $R(L)$ is dense in H for all $u \in B$.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 holds. If for all $u \in D_0(L)$, where $D_0(L) = \{u : u \in D(L),\ l_1u = l_2u = 0\}$ and for some $w \in L^2(Q)$ we consider*

$$(Lu, \omega)_{L^2(Q)} = 0, \tag{4.19}$$

then ω vanishes almost everywhere in Q.

Proof. From the equation (3.5), we can see that

$$\left(\frac{{}^C\partial^\alpha}{\partial t^\alpha}u(x,t) - a(x,t)\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial u}{\partial x}\right), \omega\right)_{L^2(Q)} = 0. \tag{4.20}$$

Since (4.19) holds for all functions $u \in D_0(L)$, it can be expressed in a form as

$$u(x,t) = \Im_t z(x,\tau),$$

where $z(x,t)$ satisfies the conditions $(3.6) - (3.8)$ such that

$$z, z_x, \frac{{}^C\partial^\alpha z}{\partial t^\alpha}, \frac{\partial}{\partial x}\left(b\frac{\partial \Im_t z}{\partial x}\right) \in L^2(Q).$$

Therefore the equation (4.20) becomes,

$$\int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}\Im_t z(x,\tau)\cdot\omega \,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(a(x,t)\frac{\partial \Im_t z(x,\tau)}{\partial x}\right)\cdot\omega\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial \Im_t z(x,\tau)}{\partial x}\right)\cdot\omega\,\mathrm{d}x\,\mathrm{d}t. \tag{4.21}$$

Now we express ω in terms of z as

$$\omega = -\Im_x^2\Im_t z.$$

Substituting ω in the equation (4.21) and applying integration by parts, each term can be estimated as

$$\int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_t z)\cdot\omega\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)(\Im_x\Im_t z)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.22}$$

$$\int_Q a(x,t)\frac{(\partial\Im_t z)}{\partial x}\cdot\omega\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_Q \frac{\partial a}{\partial x}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_Q \frac{\partial^2 a}{\partial x^2}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t, \tag{4.23}$$

$$\int_Q \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial\Im_t z}{\partial x}\right)\cdot\omega\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_Q \frac{\partial^2 b}{\partial x^2}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t - \int_Q b(x,t)(\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.24}$$

Now (4.21) can be viewed as,

$$\int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)(\Im_x\Im_t z)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q \frac{\partial^2 a}{\partial x^2}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t + \int_Q b(x,t)(\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_Q \frac{\partial a}{\partial x}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q \frac{\partial^2 b}{\partial x^2}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.25}$$

Using Poincare type inequality and the inequality

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)^2 \le 2\frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)(\Im_x\Im_t z),$$

we arrive at

$$\int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t \le \int_Q \left(\frac{\partial a}{\partial x} + \frac{\partial^2 b}{\partial x^2} - \frac{\partial^2 a}{\partial x^2} - 4\inf_Q b\right)(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t \le -(M+\epsilon)\int_Q (\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.26}$$

Hence the right hand side of (4.26) is nonpositive. That is,

$$\int_Q \frac{{}^C\partial^\alpha}{\partial t^\alpha}(\Im_x\Im_t z)^2\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Thus we obtain $z \equiv 0$ in Q. Therefore $\omega \equiv 0$ in Q. □

As discussed in [7], this implies that the problem (3.5)-(3.8) has a solution $u = L^{-1}F$.

5. Adomian Decomposition Method

In this section, we apply the decomposition method suggested by Adomian to find the approximate solution of the problem (3.1)-(3.4). Adomian decomposition method is the most effective iterative method for finding the solutions of fractional partial differential equations in the form of infinite series. This method has the advantage that we need not linearize the given problem and it has been discussed by many authors [1, 19, 24, 33]. Gepreel [11] has used Adomian decomposition method to find the approximate solutions of time and space fractional partial differential equations. Finally he justified that Adomian decomposition method is very efficient and a powerful tool to fractional partial differential equations. The solution of system of fractional partial differential equations has been found by Parthiban and Balachandran [29] by using Adomian decomposition method. Joice Nirmala and Balachandran [25] determined the solution of time fractional telegraph equation by means of Adomian decomposition method and also analysed the efficiency of this method.

To implement the technique to our problem, we first operate by I^α on both the sides of (3.1) and get

$$I^\alpha\left[\frac{{}^C\partial^\alpha w(x,t)}{\partial t^\alpha}\right] = I^\alpha\left[a(x,t)\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w}{\partial x}\right) + g(x,t)\right].$$

By using the property (4.14) we attain that

$$\begin{aligned} w(x,t) &= w(x,0) + t\frac{\partial w(x,0)}{\partial t} + I^\alpha\left[a(x,t)\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w}{\partial x}\right) + g(x,t)\right] \\ &= \phi(x) + t\psi(x) + I^\alpha\left[a(x,t)\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w}{\partial x}\right) + g(x,t)\right]. \end{aligned}$$

Let

$$w_0 = \phi(x) + t\psi(x) + I^\alpha g(x,t), \tag{5.1}$$

$$w_{n+1} = I^\alpha\left[a(x,t)\frac{\partial w_n}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w_n}{\partial x}\right)\right]. \tag{5.2}$$

Then iteration process leads to

$$\begin{aligned} w_1(x,t) &= I^\alpha\left[a(x,t)\frac{\partial w_0}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w_0}{\partial x}\right)\right], \\ w_2(x,t) &= I^\alpha\left[a(x,t)\frac{\partial w_1}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w_1}{\partial x}\right)\right], \cdots . \end{aligned}$$

By proceeding in this manner, the approximate solution of (3.1) takes the form

$$\begin{aligned} w(x,t) =& \sum_{n=0}^{\infty} w_n(x,t) \\ =& \phi(x) + t\psi(x) + I^\alpha g(x,t) + I^\alpha\left[(x,t)\frac{\partial w_0}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w_0}{\partial x}\right)\right] \end{aligned}$$

$$+ I^{\alpha}\left[a(x,t)\frac{\partial w_1}{\partial x} + \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial w_1}{\partial x}\right)\right] + \cdots . \tag{5.3}$$

In the next section we present two examples that illustrate the existence of solutions of hyperbolic fractional differential equation (3.1) using Adomian decomposition method.

6. Examples

Example 6.1. Consider the equation (3.1) with $a(x,t) = 1$, $b(x,t) = 1$, $\phi(x) = 0$, $\psi(x) = 0$ and $g(x,t) = \frac{3t^2}{2!}\sinh x$. Then the equation (3.1) becomes

$$\frac{{}^C\partial^{\alpha}}{\partial t^{\alpha}}w(x,t) - \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} = \frac{3t^2}{2!}\sinh x, \tag{6.1}$$

with the initial conditions

$$l_1 w = w(x,0) = 0$$

and

$$l_2 w = \frac{\partial w(x,0)}{\partial t} = 0.$$

Then Adomian decomposition method leads to

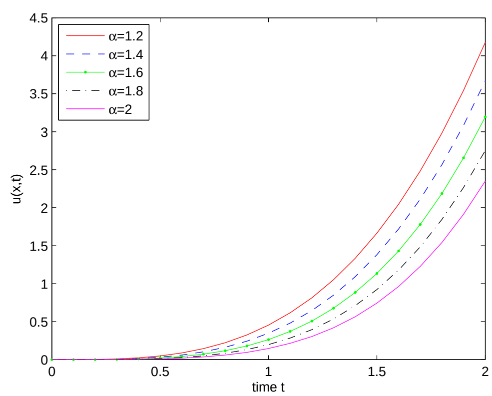

Figure 1. The approximate solution when $1 < \alpha \leq 2$ and $0 < t \leq 2$.

$$w_0(x,t) = \frac{3\sinh x}{\Gamma(3+\alpha)}t^{2+\alpha},$$

$$w_1(x,t) = \frac{3(\cosh x + \sinh x)}{\Gamma(3+2\alpha)}t^{2+2\alpha},$$

$$w_2(x,t) = \frac{6(\cosh x + \sinh x)}{\Gamma(3+3\alpha)}t^{2+3\alpha},$$

$$w_3(x,t) = \frac{12(\cosh x + \sinh x)}{\Gamma(3+4\alpha)}t^{2+4\alpha}$$

Table 1. The approximate solution for the different values of x, t and α.

t	x	α = 1.2	1.4	1.6	1.8	2.0
0.4	0.4	0.00846	0.00539	0.00340	0.00212	0.00131
	0.8	0.01830	0.01166	0.00735	0.00459	0.00284
	1.2	0.03111	0.01982	0.01250	0.00781	0.00483
	1.6	0.04896	0.03119	0.01967	0.01228	0.00760
	2.0	0.07474	0.04762	0.03003	0.01876	0.01161
0.8	0.4	0.07779	0.05693	0.04124	0.02959	0.02103
	0.8	0.16819	0.12309	0.08917	0.06397	0.04547
	1.2	0.28586	0.20921	0.15155	0.10873	0.07728
	1.6	0.44988	0.32925	0.23851	0.17111	0.12163
	2.0	0.68685	0.50267	0.36415	0.26125	0.18570
1.2	0.4	0.28471	0.22597	0.17752	0.13812	0.10647
	0.8	0.61559	0.48858	0.38383	0.29863	0.23020
	1.2	1.04628	0.83040	0.65238	0.50756	0.39125
	1.6	1.64662	1.30687	1.02670	0.79879	0.61575
	2.0	2.51394	1.99525	1.56750	1.21955	0.94008
1.6	0.4	0.71484	0.60095	0.30008	0.41211	0.33649
	0.8	1.54559	1.29934	1.08124	0.89104	0.72754
	1.2	2.62694	2.20841	1.83771	1.51446	1.23655
	1.6	4.13425	3.47557	2.89216	2.38343	1.94607
	2.0	6.31189	5.30627	4.41556	3.63886	2.97112
2.0	0.4	1.45989	1.28331	1.11664	0.96222	0.82150
	0.8	3.15650	2.77471	2.41433	2.08045	1.77621
	1.2	5.36491	4.71601	4.10349	3.53602	3.01892
	1.6	8.44322	7.42198	6.45801	5.56493	4.75114
	2.0	12.89056	11.33139	9.85967	8.49617	7.25372

and so on. Thus the solution of (6.1) is given by

$$w(x,t) = \frac{3\sinh x}{\Gamma(3+\alpha)} t^{2+\alpha} + \sum_{m=1}^{\infty} \frac{3 \cdot 2^{m-1}(\cosh x + \sinh x)}{\Gamma\Big(3+(m+1)\alpha\Big)} t^{2+(m+1)\alpha}.$$

The solution $u(x,t)$ is evaluated and plotted in Figure 1 for different values of α. From Figure 1, we can discern that as time increases the solution $u(x,t)$ grows exponentially with higher power in integer order case when compared to fractional order case. Figure 2 shows that the surface plot of solution of the equation (6.1) at $\alpha = 1.2$ and $\alpha = 2$. In Table 1, we give the values of the approximate solution with $\alpha = 1.2,\ 1.4,\ 1.6,\ 1.8$ and 2 for different values of x and t. The numerical solutions obtained by Adomian decomposition method has the same behavior as those obtained using the exact solution. In this method we need not find large size of computations and sometimes it displays a fast convergence of the solutions. Additionally, the numerical results found by this method signify high degree of accuracy and adaptability to the preferred results.

Example 6.2. Consider the equation (3.1) with $a(x,t) = x$, $b(x,t) = x^2 + 3$, $\phi(x) = 1 + x, \psi(x) = -2x$ and $g(x,t) = t^\alpha + xe^t + 1$. Then the equation (3.1) becomes

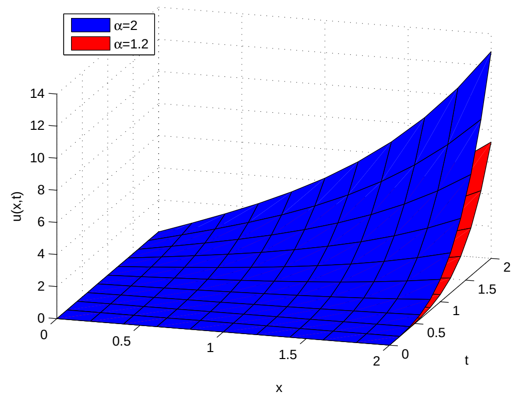

Figure 2. Surface plot when $\alpha = 1.2$ and $\alpha = 2$.

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} w(x,t) - x\frac{\partial w}{\partial x} - \frac{\partial}{\partial x}\left((x^2+3)\frac{\partial w}{\partial x^2}\right) = t^{\alpha} + xe^{t} + 1, \tag{6.2}$$

with the initial conditions

$$l_1 w = w(x,0) = 1 + x$$

and

$$l_2 w = \frac{\partial w(x,0)}{\partial t} = -2x.$$

Then Adomian decomposition method leads to

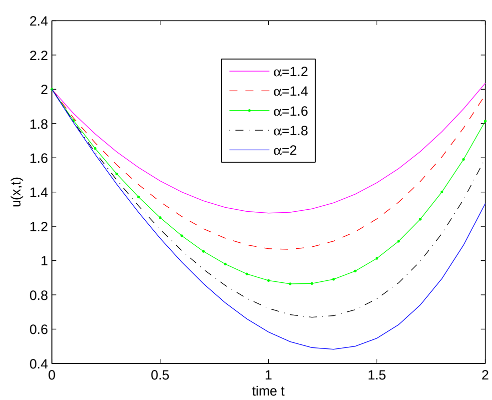

Figure 3. The approximate solution when $1 < \alpha \leq 2$ and $0 < t \leq 2$.

$$w_0(x,t) = (1+x) - 2tx + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} t^2\alpha + x\sum_{k=0}^{\infty} \frac{t^{k+\alpha}}{\Gamma(k+1+\alpha)} + \frac{t^{\alpha}}{\Gamma(1+\alpha)},$$

$$w_1(x,t) = \frac{3xt^{\alpha}}{\Gamma(1+\alpha)} - \frac{6xt^{1+\alpha}}{\Gamma(2+\alpha)} + 3x\sum_{k=0}^{\infty} \frac{t^{k+2\alpha}}{\Gamma(1+k+2\alpha)},$$

Table 2. The approximate solution for the different values of x, t and α.

		α				
t	x	1.2	1.4	1.6	1.8	2.0
0.4	0.2	1.38324	1.28355	1.21128	1.15926	1.12213
	0.4	1.42324	1.32355	1.25128	1.19926	1.16213
	0.6	1.46324	1.36355	1.29128	1.23926	1.20213
	0.8	1.50324	1.40355	1.33128	1.27926	1.24213
	1.0	1.54324	1.44355	1.37128	1.31926	1.28213
0.8	0.2	1.79073	1.61071	1.45971	1.33528	1.23413
	0.4	1.67073	1.49071	1.33971	1.21528	1.11413
	0.6	1.55073	1.37071	1.21971	1.09528	0.99413
	0.8	1.43073	1.25071	1.09971	0.97528	0.87413
	1.0	1.31073	1.13071	0.97971	0.85528	0.75413
1.2	0.2	2.42203	2.20003	1.98673	1.78970	1.61280
	0.4	2.14203	1.92003	1.70673	1.50970	1.33280
	0.6	1.86203	1.64003	1.42673	1.22970	1.05280
	0.8	1.58203	1.36003	1.14673	0.94970	0.77280
	1.0	1.30203	1.08003	0.86673	0.66970	0.49280
1.6	0.2	3.29712	3.10112	2.87312	2.63035	2.38613
	0.4	2.85712	2.66112	2.43312	2.19035	1.94613
	0.6	2.41712	2.22112	1.99312	1.75035	1.50613
	0.8	1.97712	1.78112	1.55312	1.31035	1.06613
	1.0	1.53712	1.34112	1.11312	0.87035	0.62613
2.0	0.2	4.43580	4.36744	4.21416	3.99627	3.73333
	0.4	3.83580	3.76744	3.61416	3.39627	3.13333
	0.6	3.23580	3.16744	3.01416	2.79627	2.53333
	0.8	2.63580	2.56744	2.41416	2.19627	1.93333
	1.0	2.03580	1.96744	1.81416	1.59627	1.33333

$$
\begin{aligned}
w_2(x,t) &= \frac{9xt^2\alpha}{\Gamma(1+2\alpha)} - \frac{18xt^{1+2\alpha}}{\Gamma(2+2\alpha)} + 9x\sum_{k=0}^{\infty}\frac{t^{k+3\alpha}}{\Gamma(1+k+3\alpha)},\\
w_3(x,t) &= \frac{27xt^3\alpha}{\Gamma(1+3\alpha)} - \frac{54xt^{1+3\alpha}}{\Gamma(2+3\alpha)} + 27x\sum_{k=0}^{\infty}\frac{t^{k+4\alpha}}{\Gamma(1+k+4\alpha)},\cdots.
\end{aligned}
$$

Thus the solution of (6.2) is given by

$$
\begin{aligned}
w(x,t) =& (1+x) - 2tx + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}t^{2\alpha} + x\sum_{k=0}^{\infty}\frac{t^{k+\alpha}}{\Gamma(k+1+\alpha)} + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\\
&+ \sum_{i=1}^{\infty}\frac{3^i xt^{i\alpha}}{\Gamma(1+i\alpha)} - \frac{2\cdot 3^i x}{\Gamma(2+i\alpha)}t^{1+i\alpha} + 27x\sum_{k=0}^{\infty}\frac{t^{k+(i+1)\alpha}}{\Gamma(1+k+(i+1)\alpha)}.
\end{aligned}
$$

The evaluated solution $u(x,t)$ is plotted in Figure 3 and we can discern that as time increases the solution $u(x,t)$ decreases and then increases at a faster rate in integer order case when compared to fractional order case. In Table 2, we give the values of the approximate solution with $\alpha = 1.2, 1.4, 1.6, 1.8$ and 2 for different values of x and t.

7. Conclusion

This paper demonstrates the existence and uniqueness of solution of the problem (3.1)-(3.4). The proof is based on the energy inequality method. The main difficulty in this method is to choose a suitable multiplier for the problem so that it provides an estimate from which we can establish the existence and uniqueness of solution of the problem. Also we have applied Adomian decomposition method to investigate the approximate solution of our problem. Finally, two examples were presented to show the existence of solution of our problem by Adomian decomposition method. For various fractional orders the solution of those two examples are analyzed and the values are given in tables.

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