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FINITE TIME BLOW-UP AND GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR PSEUDO-PARABOLIC EQUATION WITH EXPONENTIAL NONLINEARITY*

Qunfei Long[1,†], Jianqing Chen[1] and Ganshan Yang[2]

Abstract This paper is concerned with the initial boundary value problem of a class of pseudo-parabolic equation $u_t - \triangle u - \triangle u_t + u = f(u)$ with an exponential nonlinearity. The eigenfunction method and the Galerkin method are used to prove the blow-up, the local existence and the global existence of weak solutions. Moreover, we also obtain other properties of weak solutions by the eigenfunction method.

Keywords Pseudo-parabolic equation, existence, finite time blow-up, exponential nonlinearity.

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1. Introduction

This work was intended as an attempt to study the initial boundary value problem

$$u_t - \triangle u - \triangle u_t + u = f(u), \qquad (x,t) \in \Omega \times (0,T), \tag{1.1}$$
$$u = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, f is a nonlinear function, and $T \in \mathbb{R}^+$ is the maximum existence time of $u(x,t)$.

The equation (1.1) has appeared in a lot of physical phenomena, for example: the unidirectional propagation of nonlinear dispersive long waves, see e.g. [3] and the aggregation of population, see e.g. [18].

If both $-\Delta u_t$ and u vanish, then (1.1) becomes the heat equation

$$u_t - \triangle u = f(u). \tag{1.4}$$

There are a lot of works on (1.4) when f is a power function, see for instance [24–26]. There are also some works on (1.4) when f is an exponential nonlinearity, see

†the corresponding author. Email address: qflongfjnu@126.com(Q. Long)
1Department of Mathematics & FJKLMAA, FuJian Normal University, FuZhou, 350117, China
2Department of Mathematics, Yunnan Nationalities University and Yunnan Normal University, 650031, 650092 , China
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[2,5,7,8,11–13,16,17,20,28–30] and the references therein. In particular, Ruf et al. [20] studied firstly the heat equation with an exponential nonlinearity. For the study of the hyperbolic and pseudo-hyperbolic equations with exponential nonlinearity, one may also find several kinds of results in [1,9,10,21,27]. Especially, the research results of Ibrahim et al. [10] is essentially important for the study of a hyperbolic equation with an exponential nonlinearity.

If u vanishes, then (1.1) is pseudo-parabolic equation

$$u_t - \triangle u - \eta \triangle u_t = f(u), \tag{1.5}$$

when $f(u) = u^p$ (where $1 < p < \infty$ if $n = 1, 2$ and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$), Xu et al. [23] studied the global existence and the finite time blow-up of weak solutions, and the asymptotic behavior of global weak solutions for the initial boundary value problem of (1.5).

If $n = 2$ and f satisfies the following conditions:

- (f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(u)u > 0$ for all $u \neq 0$, and possesses a subcritical exponential growth, that is, for each $\beta > 0$, there exists a positive constant C_β such that
$$|f'(u)|, |f(u)| \leq C_\beta e^{\beta u^2}, u \in \mathbb{R};$$
- (f_2) $f(u) = o(|u|)$ as $u \to 0$;
- (f_3) there exists some $\theta > 1$ such that $\frac{f(u)}{|u|^\theta}$ is strictly increasing $(-\infty, 0)$ and $(0, +\infty)$.

For the problem (1.1)-(1.3), Zhu et al. [31] achieved the following main conclusions:

(1) There exists a local in time weak solution u in $C^1([0, T); H_0^1)$;

(2) If $I(u_0) = \|u_0\|_2^2 + \|\nabla u_0\|_2^2 - \int_\Omega f(u_0)u_0 dx > 0$, then the weak solution u is global;

(3) If $I(u_0) < 0$ or $\|u_0\|_2^2 + \|\nabla u_0\|_2^2 - \mu \int_\Omega F(u_0)dx \leq 0$ (where $\mu = \theta + 1, \theta > 1$, $F(u) = \int_0^u f(s)ds$), then the weak solution u blows up at finite time $t = T_1$, that is,
$$\lim_{t \to T_1} \left(\|u\|_2^2 + \|\nabla u\|_2^2\right) = \infty;$$

(4) If $u_0 \in H_0^1 \backslash \{0\}$ and $u = (x, t; u_0)$ is a global solution of the problem (1.1)-(1.3), then $u \in L^\infty([[0, \infty); H_0^1)$ and there exist $\{t_n\}$ with $t_n \to \infty, c \in [0, +\infty)$ and K_c such that
$$\lim_{n \to \infty} \left(\|u(t_n) - u^*\|_2^2 + \|\nabla(u(t_n) - u^*)\|_2^2\right) = 0,$$
where $K_c = \left\{u \in H_0^1 : J'(u) = 0, J(u) = c\right\}$.

It is obvious that we cannot study the global existence and finite time blow-up of weak solutions on the problem (1.1)-(1.3) by the potential well theory if f does not satisfy (f_3). Therefore, an interesting problem is that: if $n = 2$ and f only satisfies (f_1) and (f_2), what happen?

When $n = 2$ and f only satisfies (f_1) and (f_2), we obtain the following main conclusions:

(1) If $u_0 \in H_0^1$ with $\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \le \frac{\pi}{4\gamma\beta}$, then there exits a local in time weak solution u in $C([0,T); H_0^1)$. Moreover, if β such that $1 - C_\beta C^{\frac{1}{2}} S_4^{-1} \ge 0$, then thus weak solution is global and if β such that $1 - C_\beta C^{\frac{1}{2}} S_4^{-1} < 0$, then thus weak solution u blows up in finite time;

(2) If $|\int_\Omega u_0 w_1 dx| > (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ and $u_0 > (<)0$, then there exists a finite time $T_1 > 0$ such that

$$\lim_{t \to T_1} \int_\Omega u w_1 dx = +(-)\infty;$$

(3) If $|\int_\Omega u_0 w_1 dx| = (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ and $u_0 > (<)0$, then $\int_\Omega u w_1 dx \ge (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ $(\le -(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$) on $[0, +\infty)$;

(4) If $|\int_\Omega u_0 w_1 dx| < (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ and $u_0 > (<)0$, then $\int_\Omega u w_1 dx \ge (\le) e^{-t} \int_\Omega u_0 w_1 dx$ on $[0, +\infty)$,

where $\lambda_1,\ w_1,\ \gamma$ and C will be given later.

Remark 1.1. Our results (1) and (2) show that if we just discuss the global existence and the finite time blow-up of weak solutions on the problem (1.1)-(1.3), then (f_3) is unnecessary.

This paper is organized as follows. In Section 2, we introduce preliminaries, main results, and we also discuss the smoothness of some functionals. In Section 3, we prove the local and the global existence and the criterion for blow-up of weak solutions. In Section 4 and 5, by the eigenfunction method (see [6, 22] and the references therein), we find the integral $\int_\Omega u w_1 dx$ of the positive (or negative) solution $u(x,t)$ which possesses one of the following properties: (1) it blows up in finite time (that is, the solution u blows up in finite time); (2) it has the minimum (or maximum) value; (3) it has the lower (or upper) bound function.

2. Preliminaries and main results

2.1. Preliminaries

Throughout this paper, $L^p(\Omega)$ is simply denoted by L^p with the norm $\|\cdot\|_{L^p}$ and $H^s(\Omega)$ is simply denoted by H^s with the norm $\|\cdot\|_{H^s}$. $L^p([0,T); X)$ is endowed with the norm

$$\|\cdot\|_{L^p([0,T);X)} := \begin{cases} \left(\displaystyle\int_0^T \|\cdot\|_X^p dt \right)^{\frac{1}{p}} & \text{as} \quad 1 \le p < \infty, \\ \operatorname*{ess\ sup}_{0 \le t \le T} \|\cdot\|_X & \text{as} \quad p = \infty. \end{cases}$$

For H_0^1, we use the norm

$$\|u\|_{H_0^1} = \left(\|u\|_2^2 + \|\nabla u\|_2^2 \right)^{\frac{1}{2}}.$$

S_q is the best embedding constant from H_0^1 to L^q, where $2 \le q < +\infty$. $C_i, i = 1, 2, 3, 4, 5, 6, 7$, denote some positive constants.

$u = u(t) = u(x,t)$.

By [14, 19, 31], we can obtain the following key lemma.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then there exists a constant $\hat{C}$ dependent on Ω such that*

$$\int_\Omega e^{\beta u^2}dx = \int_\Omega e^{\beta\|u\|^2_{H_0^1}\frac{u^2}{\|u\|^2_{H_0^1}}}dx \le \hat{C} \quad \text{for } \beta\|u\|^2_{H_0^1} \le 4\pi, \tag{2.1}$$

where $\|u\|^2_{H_0^1} = \int_\Omega |u|^2 + |\nabla u|^2 dx$.

By the mean value theorem, the Hölder's inequality, Lemma 2.1, the Sobolev imbedding theorem and taking $\|u\|^2_{H^1} \le \frac{\pi}{4\gamma\beta}$, we obtain easily the following continuity Lemma.

Lemma 2.2. *Define a mapping $f : H_0^1 \to L^2$. Suppose further that f satisfy (f_1). Then f is Lipschitz continuous.*

Lemma 2.3. *Suppose that $u \in L^2([0,T); H_0^1)$ with $(I-\Delta)u_t \in L^2([0,T); H^{-1})$. Then*

(i) $u \in C([0,T); H_0^1)$ *(after possibly being redefined on a set measure zero);*

(ii) the mapping $t \to \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2$ is absolutely continuous, and

$$\frac{d}{dt}\big(\|u(t)\|_2^2+\|\nabla u(t)\|_2^2\big)=2(u_t(t),u(t))+2(\nabla u_t(t),\nabla u(t))=2((I-\Delta)u_t(t),u(t))$$

for a.e. $0 \le t < T$, *where* $(\cdot,\cdot)$ *denotes the pairing of* H^{-1} *and* H_0^1.

Proof. Similar to the proof of Theorem 3 in §5.9.2 in [6]. We extend u to the larger interval $[-\sigma, T+\sigma]$ for $\sigma > 0$, and define the regularization $u^\varepsilon = \eta_\varepsilon * u$, as in the proof of Theorem 1 in §5.3.1 in [6]. Then for $\varepsilon, \delta > 0$,

$$\frac{d}{dt}\big(\|u^\varepsilon(t)-u^\delta(t)\|_2^2+\|\nabla u^\varepsilon(t)-\nabla u^\delta(t)\|_2^2\big)=2\langle(I-\Delta)u_t^\epsilon(t)-(I-\Delta)u_t^\delta(t),u^\epsilon(t)-u^\delta(t)\rangle.$$

Thus

$$\begin{aligned}&\|u^\varepsilon(t)-u^\delta(t)\|_2^2+\|\nabla u^\varepsilon(t)-\nabla u^\delta(t)\|_2^2\\ &=\|u^\varepsilon(s)-u^\delta(s)\|_2^2+\|\nabla u^\varepsilon(s)-\nabla u^\delta(s)\|_2^2\\ &\quad+2\int_s^t\langle(I-\Delta)u_t^\epsilon(t)-(I-\Delta)u_t^\delta(t),u^\epsilon(t)-u^\delta(t)\rangle d\tau\end{aligned} \tag{2.2}$$

for all $0 \le s,t < T$. For any point $s \in [0,T)$ for which

$$u^\varepsilon(s) \to u(s) \quad \text{in } H_0^1 \quad \text{as } \varepsilon \to 0.$$

Consequently (2.2) implies

$$\begin{aligned}&\limsup_{\varepsilon,\delta\to 0}\sup_{0\le t<T}\big(\|u^\varepsilon(t)-u^\delta(t)\|_2^2+\|\nabla u^\varepsilon(t)-\nabla u^\delta(t)\|_2^2\big)\\ &\le \lim_{\varepsilon,\delta\to 0}\int_0^T \|(I-\Delta)u_\tau^\epsilon(\tau)-(I-\Delta)u_\tau^\delta(\tau)\|^2_{H^{-1}}\\ &\qquad+\big(\|u^\varepsilon(t)-u^\delta(t)\|_2^2+\|\nabla u^\varepsilon(t)-\nabla u^\delta(t)\|_2^2\big)d\tau=0.\end{aligned}$$

Thus the smoothed functions $\{u^\varepsilon\}_{0\le\varepsilon\le 1}$ converge to a limit $v \in C([0,T); H_0^1)$ in $C([0,T); H_0^1)$. Since $u^\varepsilon \to u$ for a.e. t as $\varepsilon \to 0$, it follows that $u = v$ a.e.

Similarly, we obtain

$$\|u^\varepsilon(t)\|_2^2 + \|\nabla u^\varepsilon(t)\|_2^2 = \|u^\varepsilon(s)\|_2^2 + \|\nabla u^\varepsilon(s)\|_2^2 + 2\int_s^t \langle (I-\Delta)u_\tau^\epsilon(\tau), u^\epsilon(\tau)\rangle d\tau,$$

and so, identifying u with v above,

$$\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 = \|u(s)\|_2^2 + \|\nabla u(s)\|_2^2 + 2\int_s^t \langle (I-\Delta)u_\tau(\tau), u(\tau)\rangle d\tau$$

for all $0 \le s < t < T$, which proves Lemma 2.3. □

Definition 2.1. A function $u \in L^2([0,T]; H_0^1)$ with $(I-\Delta)u_t \in L^2([0,T]; H^{-1})$ is called a weak solution of the initial boundary value problem (1.1)-(1.3) on $\Omega \times [0,T)$ if

(i) for each $v \in H_0^1$ and a.e. $t \in (0,T)$,

$$\langle (I-\Delta)u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle = \langle f(u), v\rangle. \tag{2.3}$$

(ii) $u(x,0) = u_0$ in H_0^1.

Definition 2.2 (Maximal existence time)**.** Suppose that u is a weak solution of the problem (1.1)-(1.3). Then the maximal existence time T of weak solution u is defined as follows:

(i) if u exists for any $t \in [0, +\infty)$, then $T = +\infty$;

(ii) if there exists a $t_0 \in (0, +\infty)$ such that u exists for any $0 \le t < t_0$, but u does not exist at $t = t_0$, then $T = t_0$.

2.2. Main results

Theorem 2.1. *Let f satisfy (f_1) and (f_2), and $u_0 \in H_0^1$.*

(i) (Local existence) If there exists a constant $\gamma > 1$ such that

$$\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \le \frac{\pi}{4\gamma\beta},$$

then the problem (1.1)-(1.3) admits a local in time weak solution u in $L^2([0,T];H_0^1)$, with $(I-\Delta)u_t \in L^2([0,T]; H^{-1})$.

(ii) (Global existence) If β such that $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} \ge 0$, then the weak solution u is global for u_0 satisfying

$$\|u_0\|_2^2 + \|\nabla u_0\|_2^2 \le \frac{\pi}{4\beta}.$$

Further, if β such that $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} > 0$, then thus global weak solution u decays exponentially.

(iii) (Criterion for blow-up) If β such that $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} < 0$, then the weak solution u blows up in finite time.

Remark 2.1. By the regularity theory, Lemma 2.3 and Definition 2.1, we obtain that if u is a weak solution of the problem (1.1)-(1.3), then $u \in C([0,T); H_0^1)$ with $u_t \in L^2([0,T); H_0^1)$.

Remark 2.2. The conditions $f(s)\to o(|s|)$ as $s\to 0$ in (f_2) and $f(s)s>0$ for any $s\neq 0$ in (f_1) imply that there exist $l(s)>0$ for any $s\in\mathbb{R}$ and constants $\alpha>1$ and $C>0$ such that $l(s)\geq C$ and $f(s)=s|s|^{\alpha-1}l(s)$ for any $s\in\mathbb{R}$. Therefore, we have

$$Cs^{\alpha}\leq f(s)\quad\text{for any } s\in[0,+\infty) \tag{2.4}$$

and

$$Cs|s|^{\alpha-1}\geq f(s)\quad\text{for any } s\in(-\infty,0]. \tag{2.5}$$

Let λ_1 is a principal eigenvalue of $-\Delta$ with homogenous Dirichlet boundary condition, w_1 is an eigenfunction corresponding to λ_1. By the theory of eigenvalues on symmetric elliptic operators, it is well known that w_1 is smooth and we may furthermore assume that $w_1>0$ in Ω and $\int_\Omega w_1dx=1$.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, suppose further that $u_0>0$. Then the weak solution u of the problem* (1.1)-(1.3) *is positive and possesses one of the following properties:*

(*i*) *If $\int_\Omega u_0w_1dx>(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then u blows up in finite time. That is, there exists a finite time $T_1>0$ such that*

$$\lim_{t\to T_1}\int_\Omega uw_1dx=+\infty,$$

where

$$T_1=-\frac{1}{\alpha-1}\ln\frac{C\left(\int_\Omega u_0w_1dx\right)^{\alpha-1}-(1+\lambda_1)}{C\left(\int_\Omega u_0w_1dx\right)^{\alpha-1}}>0.$$

(*ii*) *If $\int_\Omega u_0w_1dx=(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then $\int_\Omega uw_1dx\geq(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ on $[0,+\infty)$;*

(*iii*) *If $\int_\Omega u_0w_1dx<(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then $\int_\Omega uw_1dx\geq e^{-t}\int_\Omega u_0w_1dx$ on $[0,+\infty)$.*

Similarly, we also obtain the following result.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, suppose further that $u_0<0$. Then the weak solution u of the problem* (1.1)-(1.3) *is negative and possesses one of the following properties:*

(*i*) *If $\int_\Omega u_0w_1dx<-(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then u blows up in finite time. That is, there exists a finite time $T_1>0$ such that*

$$\lim_{t\to T_1}\int_\Omega uw_1dx=-\infty,$$

where

$$T_1=-\frac{1}{\alpha-1}\ln\frac{C\left(-\int_\Omega u_0w_1dx\right)^{\alpha-1}-(1+\lambda_1)}{C\left(-\int_\Omega u_0w_1dx\right)^{\alpha-1}}>0.$$

(*ii*) *If $\int_\Omega u_0w_1dx=-(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then $\int_\Omega uw_1dx\leq-(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$ on $[0,+\infty)$;*

(*iii*) *If $\int_\Omega u_0w_1dx>-(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then $\int_\Omega uw_1dx\leq e^{-t}\int_\Omega u_0w_1dx$ on $[0,+\infty)$.*

3. Proof of Theorem 2.1

To prove Theorem 2.1, we first prove the following Lemma.

Lemma 3.1. *For $\gamma > 1$, define*

$$W := \left\{ u \in C\left([0,T); H_0^1\right) \middle| \quad \|u\|_{H_0^1}^2 \le \frac{\pi}{4\beta} \quad and \quad \|u_0\|_{H_0^1}^2 \le \frac{\pi}{4\gamma\beta} \right\}.$$

For any $u \in ([0,T); H_0^1)$ with satisfying

$$\|u\|_{H_0^1}^2 \le e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t} \|u_0\|_{H_0^1}^2$$

and Lemmas 2.1-2.2, if $u_0 \in H_0^1$ satisfies $\|u_0\|_{H_0^1}^2 \le \frac{\pi}{4\gamma\beta}$, then there exists a finite time $T > 0$ such that $u \in W$ for any $t \in [0,T)$.

Proof. (a) If $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} \ge 0$, then, for $\|u_0\|_{H_0^1}^2 \le \frac{\pi}{4\gamma\beta}$, we obtain

$$16\beta\|u\|_{H_0^1}^2 \le 16\beta e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t} \|u_0\|_{H_0^1}^2 \le 4e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t} \frac{\pi}{\gamma} \le 4\pi,$$

which implies $t \ge 0$ and $\|u\|_{H_0^1}^2 \le \frac{\pi}{4\beta}$;

(b) If $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} < 0$, then, for $\|u_0\|_{H_0^1}^2 \le \frac{\pi}{4\gamma\beta}$, we obtain

$$16\beta\|u\|_{H_0^1}^2 \le 16\beta e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t} \|u_0\|_{H_0^1}^2 \le 4e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t} \frac{\pi}{\gamma} \le 4\pi,$$

which implies $t \le \frac{1}{2\left(C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} - 1\right)} \ln^\gamma$ and $\|u\|_{H_0^1}^2 \le \frac{\pi}{4\beta}$.

Combining (a) with (b), we can assert that there exists some $T > 0$ such that $u \in W$ for any $t \in [0,T)$. □

Proof of Theorem 2.3. (i) Let $\{w_j\}_{j=1}^\infty$ be a group of orthogonal basis in H_0^1 and a group of orthonormal basis in L^2. We construct the approximate weak solutions of the initial value problem (1.1)-(1.3)

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad (m = 1, 2, \cdots, \ j = 1, 2, \cdots, m) \tag{3.1}$$

satisfying

$$\begin{aligned} &\langle u_{mt}, w_j \rangle + \langle \nabla u_{mt}, \nabla w_j \rangle + \langle u_m, w_j \rangle + \langle \nabla u_m, \nabla w_j \rangle \\ &= \langle f(u_m), w_j \rangle, \quad (0 \le t < T, \ j = 1, 2, \cdots, m) \end{aligned} \tag{3.2}$$

$$g_{jm}(0) = \langle u_0, w_j(x) \rangle \quad j = 1, 2, \cdots, m \quad \text{ in } H_0^1. \tag{3.3}$$

Multiplying (3.2) by $g_{jm}(t)$, summing for $j = 1, 2, \cdots, m$, integrating the resulting equation over Ω and applying integration by parts, it follows that

$$\frac{d}{dt}\|u_m\|_{H_0^1}^2 + 2\|u_m\|_{H_0^1}^2 = 2\int_\Omega f(u_m) u_m dx \tag{3.4}$$

for a.e. $0 \le t < T \le T'$.

By (f_1), (f_2), Lemma 2.1 (here we take $\|u_m\|^2_{H^1_0} \le \frac{2\pi}{\beta}$), the Hölder's inequality and the embedding theorem, it follows that

$$\begin{aligned}\int_\Omega |f(u_m)||u_m|dx &= \int_\Omega |\int_0^1 f'(su_m)u_m ds||u_m|dx \\ &\le C_\beta \int_\Omega e^{\beta u_m}|u_m|^2 dx \\ &\le C_\beta \left(\int_\Omega e^{2\beta u_m}dx\right)^{\frac{1}{2}} \left(\int_\Omega |u_m|^4 dx\right)^{\frac{1}{2}} \\ &\le C_\beta \hat{\mathrm{C}}^{\frac{1}{2}} S_4^{-1}\|u_m\|^2_{H^1_0}\end{aligned} \tag{3.5}$$

for each $0 \le t < T$. Therefore, we conclude from (3.4) and (3.5) that

$$\frac{d}{dt}\|u_m\|^2_{H^1_0} + 2\|u_m\|^2_{H^1_0} \le 2\int_\Omega |f(u_m)||u_m|dx \le 2C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\|u_m\|^2_{H^1_0}$$

for each $0 \le t < T$. We further deduce from the above inequality that

$$\frac{d}{dt}\|u_m\|^2_{H^1_0} + 2\left(1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)\|u_m\|^2_{H^1_0} \le 0 \tag{3.6}$$

for a.e. $0 \le t < T$. Multiplying (3.6) by $e^{2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t}$ and integrating on [0, t], we obtain

$$\|u_m\|^2_{H^1_0} \le e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t}\|u_{0m}\|^2_{H^1_0} \tag{3.7}$$

for a.e. $0 \le t < T$. Since $\|u_{0m}\|^2_{H^1_0} \le \|u_0\|^2_{H^1_0}$ by (3.3), we obtain from (3.7) the estimate

$$\|u_m\|^2_{H^1_0} \le e^{-2\left(1-C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\right)t}\|u_0\|^2_{H^1_0}. \tag{3.8}$$

By Lemma 3.1, we deduce that there exists a finite time $T > 0$ such that Lemmas 2.1-2.2, (3.5), (3.7) and the following (3.13) hold for $\|u_0\|^2_{H^1_0} \le \frac{\pi}{4\gamma\beta}$ and any $t \in [0, T)$. Thus, (3.8) implies that exists a $C_0 > 0$ such that

$$\int_0^T \|u_m\|^2_{H^1_0} dt \le C_0\|u_0\|^2_{H^1_0}. \tag{3.9}$$

We conclude from (3.5) and (3.9) that there exists a $C_1 > 0$ such that

$$\int_0^T \int_\Omega |f(u_m)||u_m|dxdt \le C_1\|u_0\|^2_{H^1_0}. \tag{3.10}$$

For any $v \in H^1_0$ with $\left(\|v\|_2^2 + \|\nabla v\|_2^2\right) \le 1$ and write $v = v^1 + v^2$, where $v^1 \in \text{span}\{w_j\}_{j=1}^\infty$ and $(v^2, w_j) = 0, j = 1, 2, \cdots$. Since the functions $\{w_j\}_{j=1}^\infty$ are orthogonal in H^1_0, $\|v^1\|_2^2 + \|\nabla v^1\|_2^2 \le \|v\|_2^2 + \|\nabla v\|_2^2 \le 1$. Using (3.2), we obtain

$$\langle u_{mt}, v^1\rangle + \langle \nabla u_{mt}, \nabla v^1\rangle + \langle u_m, v^1\rangle + \langle \nabla u_m, \nabla v^1\rangle = \langle f(u_m), v^1\rangle, \tag{3.11}$$

which together with (3.1), we get

$$\begin{aligned}|((I-\Delta)u_{mt}, v)| &= |(u_{mt}, v) + (\nabla u_{mt}, \nabla v)| \\ &= |\langle u_{mt}, v^1\rangle + \langle \nabla u_{mt}, \nabla v^1\rangle| \\ &= \langle f(u_m), v^1\rangle - \langle u_m, v^1\rangle - \langle \nabla u_m, \nabla v^1\rangle,\end{aligned} \tag{3.12}$$

where $(\cdot,\cdot)$ denotes the pairing of H^{-1} and H_0^1. Since $\|v^1\|_{H_0^1} \le 1$, we deduce from (3.12) that there exists a $C_3 > 0$ such that

$$\|(I-\Delta)u_{mt}\|_{H^{-1}} \le C_3 \left(\int_\Omega |f(u_m)|^2 dx \right)^{\frac{1}{2}} + C_3 \left(\|u_m\|_{H_0^1}^2 \right)^{\frac{1}{2}}. \tag{3.13}$$

By (f_1), (f_2), Lemma 2.1 (here we take $\|u_m\|_{H_0^1}^2 \le \frac{\pi}{\beta}$), the Hölder's inequality, the embedding theorem and (3.9), it follows that there exists a $C_4 > 0$ such that

$$\begin{aligned} \int_0^T \int_\Omega |f(u_m)|^2 dxdt &= \int_0^T \int_\Omega |\int_0^1 f'(su_m)u_m ds|^2 dxdt \\ &\le \int_0^T \int_\Omega C_\beta^2 e^{2\beta u_m^2} u_m^2 dxdt \\ &\le C_\beta^2 \int_0^T \left(\int_\Omega e^{4\beta u_m^2} dx \right)^{\frac{1}{2}} \left(\int_\Omega |u_m|^4 dx \right)^{\frac{1}{2}} dt \\ &\le C_\beta^2 \hat{C}^{\frac{1}{2}} S_4^{-1} \int_0^T \|u_m\|_2^2 + \|\nabla u_m\|_2^2 dt \\ &\le C_4 \|u_0\|_{H_0^1}^2. \end{aligned} \tag{3.14}$$

Combining (3.9), (3.13) and (3.14), it follows that there exists a $C_5 > 0$ such that

$$\int_0^T \|(I-\Delta)u_{mt}\|_{H^{-1}}^2 dt \le C_5 \|u_0\|_{H_0^1}^2. \tag{3.15}$$

It is concluded from (3.9), (3.10), (3.14) and (3.15) that $\{u_m\}_{m=1}^\infty$ is bounded in $L^2([0,T];H_0^1)$, $\{(I-\Delta)u_{mt}\}_{m=1}^\infty$ is bounded in $L^2([0,T];H^{-1})$, $f(u_m)$ is bounded in $L^2([0,T];L^2)$ and $f(u_m)u_m$ is bounded in $L^1([0,T];L^1)$ for $t \in [0,T)$.

Consequently, there exist a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty \subset L^2([0,T];H_0^1)$ and a function $u \in L^2([0,T];H_0^1)$ such that

$$u_{m_l} \rightharpoonup u \quad \text{weakly in } L^2([0,T];H_0^1), \tag{3.16}$$

$$(I-\Delta)u_{m_l t} \rightharpoonup (I-\Delta)u_t \quad \text{weakly in } L^2([0,T];H^{-1}), \tag{3.17}$$

$$f(u_{m_l}) \rightharpoonup f(u) \quad \text{weakly in } L^2([0,T];L^2). \tag{3.18}$$

We now prove

$$f(u_m)u_m \rightharpoonup f(u)u \quad \text{weakly in } L^1([0,T];L^1). \tag{3.19}$$

By (3.16), (3.17) and Remark 2.2, we obtain $u_t \in L^2([0,T];H_0^1)$. And since $H_0^1 \hookrightarrow L^l$ is compact, we have, thanks to Aubin-Lions-lemma(or theorem) [4, 15] that

$$u_m \to u \text{ in } L^2([0,T];L^l) \text{ strongly as } m \to +\infty,$$

which implies

$$\lim_{m\to+\infty} \int_0^t \|u_m - u\|_{L^l}^2 ds = 0, \tag{3.20}$$

where $l \in [2,+\infty)$ since $n = 2$.

By the mean value theorem, the Hölder's inequality and (f_1), it follows that

$$\begin{aligned}
&\int_0^t \int_\Omega |f(u_m)u_m - f(u)u| dxds \\
=& \int_0^t \int_\Omega |f(u_m)u_m - f(u)u_m + f(u)u_m - f(u)u| dxds \\
\leq& \int_0^t \int_\Omega |f(u_m) - f(u)|\,|u_m| dxds + \int_0^t \int_\Omega |f(u_m)|\,|u_m - u| dxds \\
\leq& \int_0^t \int_\Omega |f'(u_m + \theta(u - u_m))|\,|u_m - u||u_m| dxds + \int_0^t \int_\Omega |f(u_m)|\,|u_m - u| dxds \\
\leq& \left(\int_0^t \left(\int_\Omega |f'(u_m + \theta(u - u_m))|^4 dx\right)^{\frac{1}{2}} \|u_m\|_{L^4}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|u_m - u\|_{L^2}^2 ds\right)^{\frac{1}{2}} \\
&+ \left(\int_0^t \int_\Omega |f(u_m)|^2 dxds\right)^{\frac{1}{2}} \left(\int_0^t \|u_m - u\|_{L^2}^2 ds\right)^{\frac{1}{2}} \\
\leq& C_\beta \left(\int_0^t \left(\int_\Omega e^{4\beta(u_m + \theta(u - u_m))^2} dx\right)^{\frac{1}{2}} \|u_m\|_{L^4}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|u_m - u\|_{L^2}^2 ds\right)^{\frac{1}{2}} \\
&+ \left(\int_0^t \int_\Omega |f(u_m)|^2 dxds\right)^{\frac{1}{2}} \left(\int_0^t \|u_m - u\|_{L^2}^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $f(u_m)$ is bounded in $L^2([0,T);L^2)$ and u_m is bounded in $L^2([0,T);H_0^1)$, we deduce from the obtained formula, (3.16), (3.17) and Lemma 2.1 that there exists a $C_6 > 0$ such that

$$\int_0^t \int_\Omega |f(u_m)u_m - f(u)u| dxds \leq C_6 \int_0^t \|u_m - u\|_{L^2} ds. \tag{3.21}$$

Hence, (3.19) is obtained by (3.20) and (3.21).

Next, fix an integer N and choose a function $v \in C^1([0,T);H_0^1)$ with the form

$$v(t) = \Sigma_{k=1}^N d^k(t) w_k, \tag{3.22}$$

where $\{d^k\}_{k=1}^N$ are given smooth functions. Taking $m \geq N$, multiplying (3.2) by $d^k(t)$, summing for $k = 1, 2, \cdots, N$, and then integrating it on $[0,T)$, it follows that

$$\int_0^T \langle (I - \Delta)u_{mt}, v\rangle + \langle u_m, v\rangle + \langle \nabla u_m, \nabla v\rangle dt = \int_0^T \langle f(u_m), v\rangle dt. \tag{3.23}$$

Using (3.16)-(3.19), taking the limit for (3.23) with respect to m, it follows that

$$\int_0^T \langle (I - \Delta)u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle dt = \int_0^T \langle f(u), v\rangle dt. \tag{3.24}$$

Hence (3.24) holds for all functions $v \in L^2([0,T);H_0^1)$ as functions of the form (3.22) are dense in $L^2([0,T);H_0^1)$. Further, it follows that

$$\langle (I - \Delta)u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle = \langle f(u), v\rangle \tag{3.25}$$

for each $v \in H_0^1$ and a.e. $0 \leq t < T$. Using Lemma 2.3, it follows that $u \in C([0,T);H_0^1)$.

We next prove $u(0) = u_0$ in H_0^1 as $m \to \infty$. Integrating by parts with respect to time t, we deduce from (3.24) that

$$\begin{aligned}&\int_0^T -\langle v_t, (I-\Delta)u\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle dt \\ =&\int_0^T \langle f(u), v\rangle dt + ((I-\Delta)u(0), v(0)) \\ =&\int_0^T \langle f(u), v\rangle dt + (u(0), v(0)) + (\nabla u(0), \nabla v(0))\end{aligned} \tag{3.26}$$

for each $v \in C^1([0,T); H_0^1)$ with $v(T) = 0$. Similarly, we conclude from (3.23) that

$$\begin{aligned}&\int_0^T -\langle v_t, (I-\Delta)u_m\rangle + \langle u_m, v\rangle + \langle \nabla u_m, \nabla v\rangle dt \\ =&\int_0^T \langle f(u_m), v\rangle dt + ((I-\Delta)u_m(0), v(0)) \\ =&\int_0^T \langle f(u_m), v\rangle dt + (u_m(0), v(0)) + (\nabla u_m(0), \nabla v(0)).\end{aligned} \tag{3.27}$$

Let

$$\lim_{m\to\infty} u_m(0) = u_0 \quad \text{in} \quad H_0^1.$$

Once again employing (3.16)-(3.19), taking the limit for (3.27) with respect to m, it follows that

$$\begin{aligned}&\int_0^T -\langle v_t, (I-\Delta)u\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle dt \\ =&\int_0^T \langle f(u), v\rangle dt + (u_0, v(0)) + (\nabla u_0, \nabla v(0)).\end{aligned} \tag{3.28}$$

Comparing (3.26) and (3.28), it follows that $u(0) = u_0$ in H_0^1 since $v(0)$ is arbitrary, which proves Theorem 2.1.

(ii) From the proving process of (i), we see that if β such that $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} \geq 0$, then the weak solution u of the problem (1.1)-(1.3) is global for $\|u_0\|_{H_0^1}^2 \leq \frac{\pi}{4\beta}$. Further, if $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} > 0$, then thus global weak solution u decay exponentially.

(iii) By the proving process of (i), we know that if β such that $1 - C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1} < 0$, then the conclusion of (iii) is easily gotten. □

4. Proof of Theorem 2.2

To prove Theorem 2.2, we first prove the following key Lemma.

Lemma 4.1. *Let $T > 0$, $u_0 \in H_0^1$, u be a solution of the problem* (1.1)-(1.3) *and f satisfy* (f_1) *and* (f_2). *If $u_0 \geq 0$, then $u \geq 0$ for any $(x,t) \in \Omega \times [0,T)$.*

Proof. For any $t \in [0,T)$, let

$$l(t) = \int_\Omega (u^- - \Delta u^-) u^- dx.$$

Now, we prove $l(t) \equiv 0$ as $u_0 > 0$. By arguing by contradiction, suppose that $l(t) \neq 0$. By the definitions of $l(t)$, u^- and ∇u^-, Lemma 2.1 (here taking $\|u\|_2^2 + \|\nabla u\|_2^2 \leq \frac{2\pi}{\beta}$), (f_1), (f_2) and the Hölder's inequality, it follows that

$$\begin{aligned} l'(t) &= -2\int_\Omega (u_t - \Delta u_t)u^- dx \\ &= -2\int_\Omega (\Delta u - u + f(u))u^- dx \\ &\leq 2\int_\Omega |f(u)||-u^-|dx = 2\int_\Omega |\int_0^1 f'(su)uds||-u^-|dx \\ &\leq 2C_\beta \int_\Omega e^{\beta u^2}|-u^-|^2 dx \\ &\leq 2C_\beta \left(\int_\Omega e^{2\beta u^2}dx\right)^{\frac{1}{2}}\left(\int_\Omega |u^-|^4 dx\right)^{\frac{1}{2}} \leq 2C_\beta \hat{C}^{\frac{1}{2}} S_4^{-1}\|u^-\|_{H_0^1}^2, \end{aligned}$$

which together with

$$l(t) = \int_\Omega (u^- - \Delta u^-)u^- dx = \int_\Omega |u^-|^2 + |\nabla u^-|^2 dx = \|u^-\|_{H_0^1}^2,$$

it follows that there exists a $C_7 > 0$ such that

$$l'(t) \leq C_7\|u^-\|_{H_0^1}^2 = C_7 l(t).$$

Multiplying the above inequality by $e^{-C_7 t}$, it deduces that

$$l(t) \leq l(0)e^{C_7 t} = 0$$

for $u_0 \geq 0$. Hence $u \geq 0$ for any $t \in [0, T)$, which proves Lemma 4.1. □

Proof of Theorem 2.2. We first prove that $u > 0$. By $u_0 > 0$ and Lemma 4.1, it follows that $u > 0$. Thus $f(u) > 0$ since $uf(u) > 0$ for any $u \in \mathbb{R}\backslash\{0\}$.

Define

$$h(t) := \int_\Omega (I - \Delta)uw_1 dx. \tag{4.1}$$

Then

$$\frac{d}{dt}h(t) = \int_\Omega (I - \Delta)\frac{du}{dt}w_1 dx = \int_\Omega (u_t - \Delta u_t)w_1 dx. \tag{4.2}$$

Therefore, we deduce from (1.1) and (4.2) that

$$\begin{aligned} \frac{d}{dt}h(t) &= \int_\Omega (\Delta u - u + f(u))w_1 dx \\ &= -\int_\Omega (I - \Delta)uw_1 dx + \int_\Omega f(u)w_1 dx = -h(t) + \int_\Omega f(u)w_1 dx. \end{aligned} \tag{4.3}$$

Combining (4.3) and (2.4), we obtain

$$\frac{d}{dt}h(t) \geq -h(t) + C\int_\Omega u^\alpha w_1 dx. \tag{4.4}$$

On the other hand, using $-\Delta w_1 = \lambda_1 w_1$, $\int_\Omega w_1 dx = 1, \alpha > 1$ and the Hölder's inequality, it follows from (4.1) that

$$\begin{aligned} h(t) &= (1+\lambda_1)\int_\Omega u w_1 dx \\ &\leq (1+\lambda_1)\left(\int_\Omega u^\alpha w_1 dx\right)^{\frac{1}{\alpha}}\left(\int_\Omega w_1 dx\right)^{\frac{\alpha-1}{\alpha}} \\ &\leq (1+\lambda_1)\left(\int_\Omega u^\alpha w_1 dx\right)^{\frac{1}{\alpha}}, \end{aligned} \tag{4.5}$$

which implies

$$\frac{1}{(1+\lambda_1)^\alpha} h^\alpha(t) \leq \int_\Omega u^\alpha w_1 dx. \tag{4.6}$$

Substituting (4.6) into (4.4), it follows that

$$\frac{d}{dt}h(t) \geq -h(t) + \frac{C}{(1+\lambda_1)^\alpha} h^\alpha(t). \tag{4.7}$$

Let

$$\eta(t) := e^t h(t). \tag{4.8}$$

Then

$$\frac{d}{dt}\eta(t) = e^t h(t) + e^t \frac{d}{dt}h(t). \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we conclude

$$\frac{d}{dt}\eta(t) \geq \frac{C}{(1+\lambda_1)^\alpha} e^{-(\alpha-1)t}\eta^\alpha(t). \tag{4.10}$$

Since $u, w_1, \lambda_1 > 0$, (4.5) implies $h(t) > 0$. Combining this with the definition of $\eta(t)$, it follows that $\eta(t) > 0$. Thus, (4.10) is equivalent to

$$-\frac{1}{\alpha-1}\frac{d\frac{1}{\eta^{\alpha-1}(t)}}{dt} \geq \frac{C}{(1+\lambda_1)^\alpha} e^{-(\alpha-1)t}. \tag{4.11}$$

Integrating (4.11) on [0, t], it follows that

$$-\frac{1}{\alpha-1}\left(\frac{1}{\eta^{\alpha-1}(t)} - \frac{1}{\eta^{\alpha-1}(0)}\right) \geq \frac{C}{(1+\lambda_1)^\alpha}\int_0^t e^{-(\alpha-1)s} ds. \tag{4.12}$$

We now calculate $\int_0^t e^{-(\alpha-1)s} ds$. Write $y = -(\alpha-1)s$. It follows that $y = 0$ if $s = 0$, $y = -(\alpha-1)t$ if $s = t$ and $ds = -\frac{dy}{\alpha-1}$. Hence, we obtain

$$\int_0^t e^{-(\alpha-1)s} ds = -\frac{1}{\alpha-1}\int_0^{-(\alpha-1)t} e^s ds = \frac{1}{\alpha-1}\frac{e^{(\alpha-1)t}-1}{e^{(\alpha-1)t}}.$$

Substituting the above formula into (4.12), it follows that

$$-\frac{1}{\alpha-1}\left(\frac{1}{\eta^{\alpha-1}(t)} - \frac{1}{\eta^{\alpha-1}(0)}\right) \geq \frac{C(e^{(\alpha-1)t}-1)}{(\alpha-1)(1+\lambda_1)^\alpha e^{(\alpha-1)t}}.$$

Using $\alpha > 1$, we conclude

$$\eta(t) \geq \frac{\eta(0)(1+\lambda_1)^{\frac{\alpha}{\alpha-1}} e^t}{\left(\left((1+\lambda_1)^\alpha - C\eta^{\alpha-1}(0)\right) e^{(\alpha-1)t} + C\eta^{\alpha-1}(0)\right)^{\frac{1}{\alpha-1}}}. \tag{4.13}$$

Since

$$\eta(t) = e^t h(t) = e^t \int_\Omega (1-\Delta) u w_1 dx = (1+\lambda_1) e^t \int_\Omega u w_1 dx,$$

we deduce from (4.13) that

$$\int_\Omega u w_1 dx \geq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{\left(\left((1+\lambda_1)^\alpha - C\eta^{\alpha-1}(0)\right) e^{(\alpha-1)t} + C\eta^{\alpha-1}(0)\right)^{\frac{1}{\alpha-1}}}. \tag{4.14}$$

We next discuss the properties of (4.14) according to the size of the relationship between the initial data $\int_\Omega u(x,0) w_1 dx$ and $(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$.

(i) If $\int_\Omega u(x,0) w_1 dx > (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, by $\eta(0) = h(0) = (1+\lambda_1)\int_\Omega u(x,0) w_1 dx > 0$, it follows that $-C\eta^{\alpha-1}(0) < (1+\lambda_1)^\alpha - C\eta^{\alpha-1}(0) < 0$ and $C\eta^{\alpha-1}(0) > (1+\lambda_1)^\alpha$. From this, we know that (4.14) makes sense and the right side of (4.14) closes to the positive infinity as $t \to T_1$. Hence, it is deduced from (4.14) that

$$\lim_{t\to T_1} \int_\Omega u w_1 dx = +\infty, \tag{4.15}$$

where

$$T_1 = -\frac{1}{\alpha-1} \ln \frac{C\left(\int_\Omega u_0 w_1 dx\right)^{\alpha-1} - (1+\lambda_1)}{C\left(\int_\Omega u_0 w_1 dx\right)^{\alpha-1}} > 0.$$

In this case, we say u blowing up at finite time T_1.

(ii) If $\int_\Omega u(x,0) w_1 dx = (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, it is concluded from (4.14) that

$$\int_\Omega u w_1 dx \geq (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}} \quad \text{for any } t \geq 0.$$

(iii) If $\int_\Omega u(x,0) w_1 dx < (\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, it is deduced from (4.14) that

$$\begin{aligned}\int_\Omega u w_1 dx &\geq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{\left((1+\lambda_1)^\alpha e^{(\alpha-1)t} - C\eta^{\alpha-1}(0)\left(e^{(\alpha-1)t}-1\right)\right)^{\frac{1}{\alpha-1}}} \\ &\geq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{(1+\lambda_1)^{\frac{\alpha}{\alpha-1}} e^t} = \eta(0)(1+\lambda_1)^{-1} e^{-t} = e^{-t}\int_\Omega u_0 w_1 dx\end{aligned}$$

for any $t \geq 0$, which proves Theorem 2.2. □

5. Proof of Theorem 2.3

Similar to Lemma (4.1), we obtain the following Lemma.

Lemma 5.1. *Under the hypotheses of Lemma 4.1, if $u_0 \le 0$, then $u \le 0$ for any $(x,t) \in \Omega \times [0,T)$.*

Proof of Theorem 2.3. By Lemma 5.1 and $u_0 < 0$, it follows that $u < 0$. Thus, $f(u) < 0$ since $uf(u) > 0$ for any $u \in \mathbb{R}\backslash\{0\}$.

Inserting (2.5) into (4.3), it follows that

$$\frac{d}{dt}h(t) \le -h(t) + C\int_\Omega u|u|^{\alpha-1}w_1 dx = -h(t) - C\int_\Omega |u|^\alpha w_1 dx. \tag{5.1}$$

On the other hand, using $-\Delta w_1 = \lambda_1 w_1$, $\int_\Omega w_1 dx = 1, \alpha > 1$ and the Hölder's inequality, it is deduced from (4.1) that

$$\begin{aligned} h(t) &= -(1+\lambda_1)\int_\Omega |u| w_1 dx \\ &\ge -(1+\lambda_1)\left(\int_\Omega |u|^\alpha w_1 dx\right)^{\frac{1}{\alpha}}\left(\int_\Omega w_1 dx\right)^{\frac{\alpha-1}{\alpha}} \\ &\ge -(1+\lambda_1)\left(\int_\Omega |u|^\alpha w_1 dx\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{5.2}$$

Since $w_1, \lambda_1 > 0$ and $u < 0$, (5.2) implies $h(t) < 0$. This is equivalent to $-h(t) > 0$ for any $t \in [0,T)$. Thus, we obtain from (5.2) that

$$\frac{(-h(t))^\alpha}{(1+\lambda_1)^\alpha} \le \int_\Omega |u|^\alpha w_1 dx. \tag{5.3}$$

We conclude from (5.1) and (5.3) that

$$\frac{d}{dt}h(t) \le -h(t) - \frac{C}{(1+\lambda_1)^\alpha}(-h(t))^\alpha. \tag{5.4}$$

Combining (4.8), (4.9) and (5.4), it follows that

$$\frac{d}{dt}\eta(t) \le -\frac{C}{(1+\lambda_1)^\alpha}e^{-(\alpha-1)t}(-\eta(t))^\alpha. \tag{5.5}$$

Using $h(t) < 0$ and the definition of $\eta(t)$, it follows that $\eta(t) < 0$. Thus, (5.5) is equivalent to

$$\frac{1}{\alpha-1}\frac{d\frac{1}{(-\eta(t))^{\alpha-1}}}{dt} \le -\frac{C}{(1+\lambda_1)^\alpha}e^{-(\alpha-1)t}. \tag{5.6}$$

Integrating (5.6) on [0, t] and using $\alpha > 1$, we obtain

$$\frac{1}{(-\eta(t))^{\alpha-1}} - \frac{1}{(-\eta(0))^{\alpha-1}} \le -\frac{(\alpha-1)C}{(1+\lambda_1)^\alpha}\int_0^t e^{-(\alpha-1)s}ds, \tag{5.7}$$

where

$$\eta(0) = \lim_{t\to 0} e^t h(t) = h(0) = -(1+\lambda_1)\int_\Omega |u_0| w_1 dx < 0.$$

By $\int_0^t e^{-(\alpha-1)s}ds = \frac{1}{\alpha-1}\frac{e^{(\alpha-1)t}-1}{e^{(\alpha-1)t}}$, we conclude from (5.7) that

$$\frac{1}{(-\eta(t))^{\alpha-1}} - \frac{1}{(-\eta(0))^{\alpha-1}} \le -\frac{C(e^{(\alpha-1)t}-1)}{(1+\lambda_1)^\alpha e^{(\alpha-1)t}},$$

which is equivalent to

$$\eta(t) \leq \frac{\eta(0)(1+\lambda_1)^{\frac{\alpha}{\alpha-1}} e^t}{\left(\left((1+\lambda_1)^\alpha - C(-\eta(0))^{\alpha-1}\right) e^{(\alpha-1)t} + C(-\eta(0))^{\alpha-1}\right)^{\frac{1}{\alpha-1}}}. \tag{5.8}$$

Since

$$\eta(t) = e^t h(t) = e^t \int_\Omega (1-\Delta) u(x,t) w_1(x) dx = (1+\lambda_1) e^t \int_\Omega u(x,t) w_1(x) dx,$$

it is deduced from (5.8) that

$$\int_\Omega u w_1 dx \leq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{\left(\left((1+\lambda_1)^\alpha - C(-\eta(0))^{\alpha-1}\right) e^{(\alpha-1)t} + C(-\eta(0))^{\alpha-1}\right)^{\frac{1}{\alpha-1}}}. \tag{5.9}$$

We next discuss the properties of (5.9) according to the size of the relationship between the initial data $\int_\Omega u(x,0) w_1 dx$ and $(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$.

(i) If $\int_\Omega u(x,0) w_1 dx < -(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, by $\eta(0) = h(0) = (1+\lambda_1)\int_\Omega u(x,0) w_1 dx < 0$, it follows that

$$-C(-\eta(0))^{\alpha-1} < (1+\lambda_1)^\alpha - C(-\eta(0))^{\alpha-1} < 0 \text{ and } C(-\eta(0))^{\alpha-1} > (1+\lambda_1)^\alpha.$$

From this, we know that (5.8) makes sense and the right side of (5.9) closes to the negative infinity as $t \to T_1$. Hence, it concluded from (5.9) that

$$\lim_{t \to T_1} \int_\Omega u w_1 dx = -\infty,$$

where

$$T_1 = -\frac{1}{\alpha-1} \ln \frac{C\left(-\int_\Omega u_0 w_1 dx\right)^{\alpha-1} - (1+\lambda_1)}{C\left(-\int_\Omega u_0 w_1 dx\right)^{\alpha-1}} > 0.$$

(ii) If $\int_\Omega u(x,0) w_1 dx = -(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, it is deduced from (5.9) that

$$\int_\Omega u w_1 dx \leq -(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}} \quad \text{for any } t \geq 0.$$

(iii) If $\int_\Omega u(x,0) w_1 dx > -(\frac{1+\lambda_1}{C})^{\frac{1}{\alpha-1}}$, then $(1+\lambda_1)^\alpha > C(-\eta(0))^{\alpha-1}$. We further obtain $((1+\lambda_1)^\alpha - C(-\eta(0))^{\alpha-1}) e^{(\alpha-1)t} + C(-\eta(0))^{\alpha-1} > 0$. Thus, it is concluded from (5.9) that

$$\begin{aligned}\int_\Omega u w_1 dx &\leq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{\left((1+\lambda_1)^\alpha e^{(\alpha-1)t} - C(-\eta(0))^{\alpha-1}\left(e^{(\alpha-1)t} - 1\right)\right)^{\frac{1}{\alpha-1}}} \\ &\leq \frac{\eta(0)(1+\lambda_1)^{\frac{1}{\alpha-1}}}{(1+\lambda_1)^{\frac{\alpha}{\alpha-1}} e^t} = e^{-t} \int_\Omega u_0 w_1 dx\end{aligned}$$

for any $t \geq 0$, which proves Theorem 2.3. □

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