Journal of Applied Analysis and Computation
Volume 8, Number 1, February 2018, 66–80
Website:http://jaac-online.com/
DOI:10.11948/2018.66

HOMOCLINIC SOLUTIONS FOR FOURTH ORDER DIFFERENTIAL EQUATIONS WITH SUPERLINEAR NONLINEARITIES*

Ziheng Zhang[1,†] and Zhisu Liu[2]

Abstract In this paper we investigate the existence of homoclinic solutions for a class of fourth order differential equations with superlinear nonlinearities. Under some superlinear conditions weaker than the well-known (AR) condition, by using the variant fountain theorem, we establish one new criterion to guarantee the existence of infinitely many homoclinic solutions.

Keywords Homoclinic solutions, critical point, variational methods, fountain theorem.

MSC(2010) 34C37, 35A15, 35B38.

1. Introduction

The purpose of this paper is to deal with the existence of homoclinic solutions for the following nonperiodic fourth order nonautonomous differential equations

$$u^{(4)} + wu'' + a(x)u = f(x,u), \tag{FDE}$$

where w is a constant, $a \in C(\mathbb{R},\mathbb{R})$ and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. In (FDE), let $f(x,u)$ be of the form

$$f(x,u) = b(x)u^2 + c(x)u^3,$$

then (FDE) reduces to the following equation

$$u^{(4)} + wu'' + a(x)u - b(x)u^2 - c(x)u^3 = 0, \tag{1.1}$$

which has been put forward as a mathematical model for the study of patten formation in physics and mechanics. For example, the well-known Extended Fisher-Kolmogorov (EFK) equation proposed by Coullet et al. [7] in study of phase transitions, and also by Dee and van Saarlos [8], as well as the Swift-Hohenberg (SH) equation [19] which is a general model for pattern-forming process derived to describe random thermal fluctuations in the Boussinesque equation and in the propagation of lasers [9]. With appropriate changes of variables, stationary solutions of these equations lead to the following fourth order equation

$$u^{(4)} + wu'' - u + u^3 = 0, \tag{1.2}$$

†the corresponding author. Email address:zhzh@mail.bnu.edu.cn(Z. Zhang)
[1]Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China
[2]School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, China
*The authors were supported by National Natural Science Foundation of China (11771044, 11701267).

where $w > 0$ corresponds to (EFK) equation and $w < 0$ to (SH) equation. In addition, in the description of water waves driven by gravity and capillarity [4], the following differential equation can be reduced by means of an argument based on the center manifold theorem

$$u^{(4)} + wu'' - u + u^2 = 0, \tag{1.3}$$

where $w < 0$ is a constant. Meanwhile, in study of weak interactions of dispersive waves, Bretherton [2] gave the following partial differential equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial t^4} + v - v^3 = 0.$$

To obtain traveling wave solutions $v(t,x) = u(x - ct)$ with $c > 0$, one can deduce that

$$u^{(4)} + c^2 u'' - u + u^3 = 0. \tag{1.4}$$

Besides, pulse propagation through optical fibers involving fourth order dispersion leads to a generalized nonlinear Schrödinger equation [5]

$$i\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^4 v}{\partial t^4} + |v|^2 v = 0.$$

Assuming that harmonic spatial dependence $v(t,x)$ be of the form $v(t,x) = u(t)e^{ikx}$ $(k \in \mathbb{N})$, then one obtains

$$u^{(4)} - u'' + ku - u^3 = 0. \tag{1.5}$$

For the problem of finding a homoclinic solution (i.e., a nontrivial solution $u(x)$ such that $u(x) \to 0$ as $|x| \to \infty$) of the fourth order differential equations, we refer the reader to [1, 3, 14] concerned with the autonomous case. If $a(x) = b(x) = 1$, $c(x) = 0$ and $w \le 2$, Amick and Toland [1] proved the existence of homoclinic solutions of Eq.(1.1). Later, their result was extended by Buffoni in [3]. If $a(x) = c(x) = 1$, $b(x) = 0$, Peletier and Troy [14] extensively studied the periodic, homoclinic and heteroclinic solutions of Eq. (1.1). Compared to the autonomous case, the nonautonomous case seems to be more difficult, because of the lack of the translation invariance and the existence of a first integral. Tersian and Chaparova [19] showed that Eq.(1.1) possesses one nontrivial homoclinic solution by using the Mountain Pass Theorem when $a(x)$, $b(x)$ and $c(x)$ are continuous periodic functions and satisfy some other assumptions. Li [12] extended the results to some general nonlinear term, i.e., (FDE), assuming that $a(x)$ and $f(x,u)$ are periodic in x, and $f(x,u)$ satisfies the following Ambrosetti-Rabinowitz condition (shortly denoted by (AR) condition):

(AR) there is a constant $\mu > 2$ such that

$$0 < \mu F(x,u) \le f(x,u)u, \quad \forall (x,u) \in \mathbb{R} \times \mathbb{R},$$

where $F(x,u) = \int_0^u f(x,t)dt$, which implies that $f(x,u)$ is superlinear at infinity. Li [13] dealt with the nonperiodic case of Eq.(1.1) and obtained the existence of nontrivial homoclinic solutions via using a compactness lemma and a mountain pass theorem. Sun and Wu [17] considered the following nonperiodic fourth order differential equations with a perturbation

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R}, \tag{1.6}$$

where w is a constant, $\lambda > 0$ is a parameter, $a \in C(\mathbb{R},\mathbb{R})$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $1 \le p < 2$ and $h \in L^{\frac{2}{2-p}}(\mathbb{R})$, and obtained the existence of at least two homoclinic solutions for the case that $f(x,u)$ is superlinear or asymptotically linear at infinity. More recently, Li et al. [10] studied the existence of infinitely many homoclinic solutions for nonperiodic (FDE) when $f(x,u)$ satisfies the superlinear condition, but does not fulfil the well-known (AR) condition, see its Theorem 1.1. However, we must point out that, for the case that (FDE) is nonperiodic, to obtain the existence of homoclinic solutions, the following coercive condition on a is often needed:

($\mathcal{A}$) $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists some constant $a_1 > 0$ such that

$$0 < a_1 \le a(x) \to \infty \quad \text{as} \quad |x| \to \infty, \tag{1.7}$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see Lemma 2 in [13], Lemma 2.2 in [17] and Lemma 2.3 in [10].

It is obvious that, if a is bounded, then it is not covered by ($\mathcal{A}$). Inspired by the above facts, more recently, the authors [18, 21, 22] investigated the existence of homoclinic solutions of (FDE) for the case that a is nonperiodic and bounded from below. Explicitly, assuming that the following condition hold:

($\mathcal{A}$)′ $a \in C(\mathbb{R},\mathbb{R})$ is continuous and there exists a positive $\tau > 0$ such that

$$a(x) \ge \tau > 0 \quad \text{and} \quad w \le 2\sqrt{\tau};$$

then Yang [21] showed that (FDE) possesses at least one nontrivial homoclinic solution where $f(x,u)$ is of sublinear growth as $|u| \to \infty$. If, in addition, f is odd in u variable, i.e.,

($\mathcal{F}_3$) $f(x,u) = -f(x,-u), \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R}$,

then (FDE) possesses infinitely many homoclinic solutions. In [18], Sun et al. considered the following nonperiodic fourth order differential equations with a parameter:

$$u^{(4)} + wu'' + \lambda a(x)u = f(x,u), \tag{1.8}$$

where w is a constant, $\lambda > 0$ is a parameter and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Assuming that $a(x)$ satisfies the following conditions:

($\mathcal{V}_1$) $a \in C(\mathbb{R},\mathbb{R})$ and $a \ge 0$ on $\mathbb{R}$; there exists $c > 0$ such that the set $\{a < c\} = \{x \in \mathbb{R} \,|\, a(x) < c\}$ is nonempty and $|\{a < c\}| < c_0 S_\infty^{-2}$, where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for the embedding of $H^2(\mathbb{R})$ in $L^\infty(\mathbb{R})$ and c_0 is given in its Lemma 2.1;

($\mathcal{V}_2$) $T = \text{int}a^{-1}(0)$ is nonempty and $\overline{T} = a^{-1}(0)$ such that T is a finite interval;

and $f(x,u)$ is supposed to satisfy some class of sublinear growth conditions, then they showed that there exists $\Lambda_0 > 0$ such that for every $\lambda > \Lambda_0$ Eq.(1.8) has at least one homoclinic solution u_λ, and explored the phenomenon of concentration of homoclinic solutions as $\lambda \to \infty$, which has been improved in recent paper [11] when the nonlinear term $f(x,u)$ satisfies the asymptotically linear condition, and the nonexistence of nontrivial homoclinic solutions is also discussed. In [22], for the case that $a(x)$ is bounded in the following sense

$(\mathcal{A})''$ $a \in C(\mathbb{R},\mathbb{R})$ and there exits two constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$0 < \tau_1 \leq a(x) \leq \tau_2 \quad \text{for all } x \in \mathbb{R},$$

and assuming that $f(x,u)$ satisfies some superlinear condition weaker than (AR) condition, Zhang and Yuan showed that (FDE) has at least one nontrivial homoclinic solution.

Motivated by the above results, in this paper we are interested in the existence of infinitely many homoclinic solutions of (FDE) for the case that $a(x)$ is unnecessarily required to be either nonnegative or coercive, and $f(x,u)$ satisfies some weak superlinear conditions at infinity with respect to u. Now, we make the following assumptions on $a(x)$ and $f(x,u)$:

$(\mathcal{L})_1$ $a \in C(\mathbb{R},\mathbb{R})$ such that $\inf_{x\in\mathbb{R}} a(x) > -\infty$;

$(\mathcal{L})_2$ there exists a constant $r_0 > 0$ such that

$$\lim_{|s|\to\infty} meas(\{x \in (s - r_0, s + r_0) : a(x) \leq M\}) = 0, \quad \forall M > 0,$$

where $meas$ denotes the Lebesgue measure in $\mathbb{R}$;

$(\mathrm{FDE})_1$ $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = \infty$ uniformly with respect to $x \in \mathbb{R}$;

$(\mathrm{FDE})_2$ $F(x,0) = 0$ for all $x \in \mathbb{R}$, and there exist constants $b > 0$ and $\nu > 2$ such that

$$|f(x,u)| \leq b(|u| + |u|^{\nu-1}), \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R};$$

$(\mathrm{FDE})_3$ there exists a constant $\vartheta \geq 1$ such that

$$\vartheta \widetilde{F}(x,u) \geq \widetilde{F}(x,su), \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R} \text{ and } s \in [0,1],$$

where $\widetilde{F}(x,u) = f(x,u)u - 2F(x,u)$;

$(\mathrm{FDE})_4$ $F(x,u) = F(x,-u)$ for all $(x,u) \in \mathbb{R}\times\mathbb{R}$.

Now, we are in the position to state our main result.

Theorem 1.1. *Suppose that* $(\mathcal{L})_1$, $(\mathcal{L})_2$ *and* $(\mathrm{FDE})_1$-$(\mathrm{FDE})_4$ *are satisfied, then* (FDE) *possesses a sequence of homoclinic solutions* $\{u_k\}_{k\in\mathbb{N}}$ *satisfying*

$$\int_{\mathbb{R}} \Big[\frac{1}{2}u_k''(x)^2 - \frac{1}{2}wu_k'(x)^2 + \frac{1}{2}a(x)u_k(x)^2 - F(x,u_k(x))\Big]dx \to \infty \quad as \quad k \to \infty.$$

Remark 1.1. It is easy to see that conditions $(\mathcal{L})_1$ and $(\mathcal{L})_2$ are weaker than $(\mathcal{A})$, $(\mathcal{A})'$, $(\mathcal{A})''$, and the conditions $(\mathcal{V}_1)$-$(\mathcal{V}_2)$. In our Theorem 1.1, $a(x)$ is unnecessarily required to be either nonnegative or coercive, and is allowed to be sign-changing. Besides, the well-known (AR) superlinear condition is not required in our Theorem 1.1. There are functions $a(x)$ and $f(x,u)$ which satisfy all the conditions in our Theorem 1.1 but do not satisfy the corresponding conditions in the aforementioned references for the superlinear case. For example, let

$$a(x) = (|x|\sin^2 x - 1), \quad \forall x \in \mathbb{R}$$

and

$$f(x,u) = \pi(x)u\ln(1+u^2), \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R},$$

where $\pi : \mathbb{R} \to \mathbb{R}$ is a continuously bounded function with positive lower bound, then simple computation shows that $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathrm{FDE})_1$-$(\mathrm{FDE})_4$ are satisfied.

The remaining part of this paper is structured as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

2. Preliminary Results

In order to prove Theorem 1.1 via the critical point theory, we firstly describe some properties of the space E on which the variational framework associated to (FDE) is defined. To do this, we need the following discussion.

Remark 2.1. According to $(\mathrm{FDE})_2$, it is easy to verify that

$$|F(x,u)| = |\int_0^1 f(x,su)uds| \leq \frac{b}{2}|u|^2 + \frac{b}{\nu}|u|^{\nu}, \quad \forall (x,u) \in \mathbb{R}\times\mathbb{R}. \tag{2.1}$$

From $(\mathcal{L})_1$, $(\mathrm{FDE})_1$ and (2.1), we know that there exists a positive constant a_0 such that $\inf_{x\in\mathbb{R}}(a(x)+a_0) > 0$ and $\inf_{(x,u)\in\mathbb{R}\times\mathbb{R}} F(x,u)+a_0u^2 > 0$. Let $\overline{a}(x) = a(x)+a_0$ and $\overline{F}(x,u) = F(x,u) + a_0u^2$ and consider the following fourth order differential equations

$$u^{(4)} + wu'' + \overline{a}(x)u = \overline{f}(x,u), \tag{2.2}$$

then (2.2) is equivalent to (FDE). Moreover, it is obvious that the hypotheses $(\mathcal{L})_1$, $(\mathcal{L})_2$ and $(\mathrm{FDE})_1$-$(\mathrm{FDE})_4$ still hold for $\overline{a}$ and $\overline{F}$. In addition, for the constant w in (FDE), we can choose $a_0 > 0$ large enough such that $w \leq 2\sqrt{\inf_{x\in\mathbb{R}}(a(x)+a_0)}$.

In view of Remark 2.1, in what follows, we investigate the equivalent problem (2.2) and make the following assumption instead of $(\mathcal{L})_1$:

$(\mathcal{L})_1'$ $a \in C(\mathbb{R},\mathbb{R})$ and there exits a positive constant $\tau > 0$ such that $a(x) \geq \tau$.

Meanwhile, due to Remark 2.1, we can also assume that $w \leq 2\sqrt{\tau}$ and $F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}\times\mathbb{R}$.

Now we introduce the functional space on which the variational framework corresponding to (FDE) will be constructed.

Lemma 2.1 ([19, Lemma 8]). *Assume that $a(x) \geq \tau > 0$ and $w \leq 2\sqrt{\tau}$. Then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx \geq c_0\|u\|_{H^2}^2 \quad \textit{for all} \quad u \in H^2(\mathbb{R}), \tag{2.3}$$

where $\|u\|_{H^2} = \Big(\int_{\mathbb{R}}[u''(x)^2 + u'(x)^2 + u(x)^2]dx\Big)^{1/2}$ *is the norm of Sobolev space* $H^2(\mathbb{R})$.

Due to Lemma 2.1, we define

$$E = \left\{u \in H^2(\mathbb{R}) : \int_{\mathbb{R}}[u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx < \infty\right\},$$

with the inner product

$$(u,v) = \int_{\mathbb{R}}[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx$$

and the corresponding norm

$$\|u\| = \Big(\int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx\Big)^{1/2}.$$

Then, it is easy to verify that E is a Hilbert space. By Sobolev embedding theorem,

$$H^2(\mathbb{R}) \subset L^p(\mathbb{R}), \quad 2 \le p \le \infty$$

and the embedding is continuous. That is, there exists a constant $C_p > 0$ such that

$$\|u\|_p \le C_p\|u\|_{H^2}, \quad \forall u \in E, \tag{2.4}$$

for any $p \in [2, \infty]$. Combining (2.4) and (2.3), for any $p \in [2, \infty]$, there is another constant (still denoted by C_p) such that

$$\|u\|_p \le C_p\|u\|, \quad \forall u \in E. \tag{2.5}$$

Here $L^p(\mathbb{R})$ $(2 \le p < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}$ under the norm

$$\|u\|_p := \Big(\int_{\mathbb{R}} |u(x)|^p dx\Big)^{1/p},$$

$L^\infty(\mathbb{R})$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}$ equipped with the norm

$$\|u\|_\infty := \operatorname{ess\,sup}\{|u(x)| : x \in \mathbb{R}\}.$$

Furthermore, we can have the following compact embedding conclusion which plays an essential role in our later argument.

Lemma 2.2. *If $(\mathcal{L})_1'$ and $(\mathcal{L})_2$ are satisfied, then E is compactly embedded into $L^2(\mathbb{R}, \mathbb{R})$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We will show that $u_n \to u$ in $L^2(\mathbb{R}, \mathbb{R})$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in E. The Sobolev embedding theorem implies that $u_n \to 0$ in $L^2_{loc}(\mathbb{R}, \mathbb{R})$. Thus it suffices to show that, for any $\epsilon > 0$, there is $r > 0$ such that

$$\int_{\mathbb{R}\setminus[-r,r]} |u_n(x)|^2 dx < \epsilon, \quad n \in \mathbb{N}. \tag{2.6}$$

To this end, for any $s \in \mathbb{R}$, we denote by $\mathcal{B}_{r_0}(s)$ the interval in $\mathbb{R}$ centered at s with radius r_0, i.e., $\mathcal{B}_{r_0}(s) = (s - r_0, s + r_0)$, where r_0 is constant defined in $(\mathcal{L})_2$. Let $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^\infty \mathcal{B}_{r_0}(s_i)$ and each $x \in \mathbb{R}$ is contained in at most two such intervals. For any $r > 0$ and $M > 0$, let

$$\mathcal{C}(r, M) = \{x \in \mathbb{R} \setminus [-r, r] : a(x) > M\}$$

and

$$\mathcal{D}(r, M) = \{x \in \mathbb{R} \setminus [-r, r] : a(x) \le M\}.$$

Then, one deduces that

$$\int_{\mathcal{C}(r,M)} u_n(x)^2 dx \le \frac{1}{M}\int_{\mathcal{C}(r,M)} a(x)u_n(x)^2 dx \le \frac{1}{M}\int_{\mathbb{R}} a(x)u_n(x)^2 dx$$

and moreover this can be made arbitrarily small by choosing M large enough. In addition, making use of Hölder inequality and (2.5), for a fixed $M>0$, we have

$$\begin{aligned}
\int_{\mathcal{D}(r,M)} u_n(x)^2dx &\leq \sum_{i=1}^{\infty}\int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} u_n(x)^2dx\\
&\leq \sum_{i=1}^{\infty}\Big(\int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} u_n(x)^4dx\Big)^{1/2}\Big(meas(\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i))\Big)^{1/2}\\
&\leq \epsilon_r\sum_{i=1}^{\infty}\Big(\int_{\mathcal{B}_{r_0}(s_i)} u_n(x)^4dx\Big)^{1/2}\\
&\leq \epsilon_r\sum_{i=1}^{\infty}\int_{\mathcal{B}_{r_0}(s_i)} u_n(x)^4dx\\
&\leq 2\epsilon_r\int_{\mathbb{R}} u_n(x)^4dx\\
&\leq 2C_4^4\epsilon_r\int_{\mathbb{R}}[u_n''(x)^2-wu_n'(x)^2+a(x)u_n(x)^2]dx,
\end{aligned}$$

where $\epsilon_r=\sup_{i\in\mathbb{N}}(meas(\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)))^{1/2}$. On account of $(\mathcal{L})_2$, $\epsilon_r\to 0$ as $r\to\infty$ and noting that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E, we can make this term small by choosing r large. This completes the proof. □

Remark 2.2. Due to the fact

$$\int_{\mathbb{R}}|u(x)|^pdx\leq\|u\|_\infty^{p-2}\|u\|_2^2,\quad\forall p\in[2,\infty)$$

and Lemma 2.2, it is easy to check that E is compactly embedded into $L^p(\mathbb{R},\mathbb{R})$ for any $p\in[2,\infty)$.

To obtain the existence of infinitely many homoclinic solutions of (FDE), we need the following variant fountain theorem established in [23].

Let $\mathcal{B}$ be a Banach space with the norm $\|\cdot\|$ and $\mathcal{B}=\overline{\oplus_{j\in\mathbb{N}}X_j}$ with $dimX_j<\infty$ for any $j\in\mathbb{N}$. Set $Y_k=\oplus_{j=1}^kX_j$ and $Z_k=\overline{\oplus_{j=k}^{\infty}X_j}$. Consider the following C^1-functional $\Phi_\lambda:\mathcal{B}\to\mathbb{R}$ defined by

$$\Phi_\lambda(u)=A(u)-\lambda B(u),\quad\lambda\in[1,2].$$

Lemma 2.3 ([23, Theorem 2.1])**.** *Assume that the above functional Φ_λ satisfies*

$(\mathcal{A})_1$ *Φ_λ maps bounded sets to bounded sets for $\lambda\in[1,2]$, and $\Phi_\lambda(-u)=\Phi_\lambda(u)$ for all $(\lambda,u)\in[1,2]\times\mathcal{B}$;*

$(\mathcal{A})_2$ *$B(u)\geq 0$ for all $u\in\mathcal{B}$, and $A(u)\to\infty$ or $B(u)\to\infty$ as $\|u\|\to\infty$;*

$(\mathcal{A})_3$ *there exist $\rho_k>\sigma_k>0$ such that*

$$\alpha_k(\lambda)=\inf_{u\in Z_k,\|u\|=\sigma_k}\Phi_\lambda(u)>\beta_k(\lambda)=\max_{u\in Y_k,\|u\|=\rho_k}\Phi_k(u),\quad\lambda\in[1,2].$$

Then

$$\alpha_k(\lambda)\leq\zeta_k(\lambda)=\inf_{\gamma\in\Gamma_k}\max_{u\in B_k}\Phi_\lambda(\gamma(u)),\quad\forall\lambda\in[1,2],$$

where $B_k = \{u \in Y_k : \|u\| \le \rho_k\}$ *and* $\Gamma_k = \{\gamma \in C(B_k, \mathcal{B}) : \gamma$ *is odd*, $\gamma|_{\partial B_k} = id\}$. *Moreover, for almost every* $\lambda \in [1, 2]$, *there exists a sequence* $\{u^k_m(\lambda)\}^{\infty}_{m=1}$ *such that*

$$\sup_m \|u^k_m(\lambda)\| < \infty, \quad \Phi'_\lambda(u^k_m(\lambda)) \to 0 \quad \textit{and} \quad \Phi_\lambda(u^k_m(\lambda)) \to \zeta_k(\lambda) \quad \textit{as} \quad m \to \infty.$$

3. Proof of Theorem 1.1

The aim of section is to establish the proof of Theorem 1.1. For this purpose, we are going to establish the corresponding variational framework to obtain homoclinic solutions of (FDE). To this end, we choose $\mathcal{B} = E$ as our working functional space. Due to the fact that E is a reflexive and separable Hilbert space, we select an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of E and let $X_j = span\{e_j\}$ for all $j \in \mathbb{N}$. Define the functionals A, B and Φ_λ on E by

$$A(u) = \frac{1}{2}\|u\|^2, \quad B(u) = \int_{\mathbb{R}} F(x, u(x))dx, \tag{3.1}$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}} F(x, u(x))dx \tag{3.2}$$

for all $u \in E$ and $\lambda \in [1, 2]$. Especially, we denote by $\Phi_1 = \Phi$, that is,

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}u''(x)^2 - \frac{1}{2}wu'(x)^2 + \frac{1}{2}a(x)u(x)^2 - F(x, u(x))\Big]dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x, u(x))dx. \end{aligned} \tag{3.3}$$

(2.1) and (2.5) imply that Φ_λ is well defined on E. Furthermore, under the conditions of Theorem 1.1, as usual, we see that $\Phi_\lambda \in C^1(E, \mathbb{R})$, i.e., Φ_λ is a continuously Fréchet-differentiable functional defined on E. Moreover, we have

$$\Phi'_\lambda(u)v = \int_{\mathbb{R}} \Big[u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x) - \lambda f(x, u(x))v(x)\Big]dx \tag{3.4}$$

for all u, $v \in E$, which yields that

$$\Phi'_\lambda(u)u = \|u\|^2 - \lambda \int_{\mathbb{R}} f(x, u(x))u(x)dx. \tag{3.5}$$

Moreover, any nontrivial critical points of Φ are homoclinic solutions of (FDE).

To obtain the existence of infinitely many homoclinic solutions of (FDE) by using the fountain theorem, in the sequel we establish some technical lemmas.

Lemma 3.1. *For any finite dimensional subspace* $\widetilde{E} \subset E$, *there exists a constant* $\varrho > 0$ *such that*

$$meas(\{x \in \mathbb{R} : |u(x)| \ge \varrho\|u\|\}) \ge \varrho, \quad \forall u \in \widetilde{E} \setminus \{0\}.$$

Proof. On the contrary, assume that, for any $n \in \mathbb{N}$, there exists $u_n \in \widetilde{E} \setminus \{0\}$ such that

$$meas(\{x \in \mathbb{R} : |u_n(x)| \ge \frac{\|u_n\|}{n}\}) < \frac{1}{n}.$$

Let $v_n = \frac{u_n}{\|u_n\|} \in \widetilde{E}$ for each $n \in \mathbb{N}$, then we have $\|v_n\| = 1$ and

$$meas(\{x \in \mathbb{R} : |v_n(x)| \geq \frac{1}{n}\}) < \frac{1}{n}. \tag{3.6}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in $\widetilde{E}$ for some $v_0 \in \widetilde{E}$, since $\widetilde{E}$ is of finite dimension. Combining this with (2.5), we have

$$\int_{\mathbb{R}} |v_n(x) - v_0(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Noting that $\|v_0\| = 1$, then there exists a constant $\delta_0 > 0$ such that

$$meas(\{x \in \mathbb{R} : |v_0(x)| \geq \delta_0\}) \geq \delta_0. \tag{3.8}$$

Otherwise, for each fixed $n \in \mathbb{N}$, we have

$$meas(\{x \in \mathbb{R} : |v_0(x)| \geq \frac{1}{n}\}) \leq meas(\{x \in \mathbb{R} : |v_0(x)| \geq \frac{1}{m}\}) \leq \frac{1}{m}, \quad \forall m \geq n.$$

Letting $m \to \infty$, we obtain that

$$meas(\{x \in \mathbb{R} : |v_0(x)| \geq \frac{1}{n}\}) = 0.$$

Consequently, one deduces that

$$\begin{aligned} 0 &\leq meas(\{x \in \mathbb{R} : |v_0(x)| \neq 0\}) \\ &= meas(\cup_{n=1}^{\infty}\{x \in \mathbb{R} : |v_0(x)| \geq \frac{1}{n}\}) \\ &\leq \sum_{n=1}^{\infty} meas(\{x \in \mathbb{R} : |v_0(x)| \geq \frac{1}{n}\}) = 0, \end{aligned}$$

which yields that $v_0 = 0$, a contradiction to $\|v_0\| = 1$. Thus (3.8) holds. In what follows, set $\Omega_0 = \{x \in \mathbb{R} : |v_0(x)| \geq \delta_0\}$, where δ_0 is the constant given in (3.8). For any $n \in \mathbb{N}$, let

$$\Omega_n = \{x \in \mathbb{R} : |v_n(x)| < \frac{1}{n}\} \quad \text{and} \quad \Omega_n^c = \mathbb{R} \setminus \Omega_n = \{x \in \mathbb{R} : |v_n(x)| \geq \frac{1}{n}\}.$$

Then, for n large enough, by (3.6) and (3.8), we have

$$meas(\Omega_n \cap \Omega_0) \geq meas(\Omega_0) - meas(\Omega_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Consequently, for n large enough, it holds that

$$\begin{aligned} \int_{\mathbb{R}} |v_n(x) - v_0(x)|^2 dx &\geq \int_{\Omega_n \cap \Omega_0} |v_n(x) - v_0(x)|^2 dx \\ &\geq \int_{\Omega_n \cap \Omega_0} (|v_0(x)| - |v_n(x)|)^2 dx \\ &\geq \left(\delta_0 - \frac{1}{n}\right)^2 meas(\Omega_n \cap \Omega_0) \\ &\geq \frac{\delta_0^3}{8} > 0, \end{aligned}$$

which contradicts to (3.7). The proof is complete. □

Lemma 3.2. *Assume that* $(\mathcal{L})_1'$, $(\mathcal{L})_2$ *and* $(\text{FDE})_2$ *hold, then there exist a positive integer* k_1 *and a sequence* $\{\sigma_k\}_{k\in\mathbb{N}}$ *satisfying* $\sigma_k \to \infty$ *as* $k \to \infty$ *such that*

$$\alpha_k(\lambda) = \inf_{u\in Z_k, \|u\|=\sigma_k} \Phi_\lambda(u) > 0,$$

where $Z_k = \overline{\oplus_{j=k}^{\infty} X_j} = \overline{span\{e_k, \dots\}}$ *for all* $k \geq k_1$.

Proof. Note that (2.1) and (3.2) imply that

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} F(x, u(x))dx \\ &\geq \frac{1}{2}\|u\|^2 - b\|u\|_2^2 - \frac{2b}{\nu}\|u\|_\nu^\nu, \quad \forall(\lambda, u) \in [1, 2] \times E.\end{aligned} \tag{3.9}$$

For each $k \in \mathbb{N}$, define

$$\ell_2(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_2 \quad \text{and} \quad \ell_\nu(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_\nu. \tag{3.10}$$

Since E is compactly embedded into both $L^2(\mathbb{R}, \mathbb{R})$ and $L^\nu(\mathbb{R}, \mathbb{R})$, then there hold that (see [20, Lemma 3.8])

$$\ell_2(k) \to 0 \quad \text{and} \quad \ell_\nu(k) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.11}$$

Combining (3.9) with (3.10), we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - b\ell_2^2(k)\|u\|^2 - \frac{2b}{\nu}\ell_\nu^\nu\|u\|^\nu, \quad \forall(\lambda, u) \in [1, 2] \times Z_k. \tag{3.12}$$

In view of (3.11), there exists a positive integer k_1 such that

$$b\ell_2^2(k) \leq \frac{1}{4}, \quad \forall k \geq k_1. \tag{3.13}$$

For each $k \geq k_1$, choose

$$\sigma_k = \left(\frac{16b\ell_\nu^\nu(k)}{\nu}\right)^{1/(2-\nu)}. \tag{3.14}$$

Then, it follows from (3.11) that

$$\sigma_k \to \infty \quad \text{as} \quad k \to \infty, \tag{3.15}$$

since $\nu > 2$. Based on (3.12)-(3.14), a direct computation shows that

$$\alpha_k(\lambda) = \inf_{u\in Z_k, \|u\|=\sigma_k} \Phi_\lambda(u) \geq \frac{\sigma_k^2}{8} > 0, \quad \forall k \geq k_1,$$

which implies that the proof is complete. □

Lemma 3.3. *Suppose that* $(\mathcal{L})_1'$, $(\mathcal{L})_2$, $(\text{FDE})_1$ *and* $(\text{FDE})_2$ *are satisfied. Then, for the positive integer* k_1 *and the sequence* $\{\sigma_k\}_{k\in\mathbb{N}}$ *determined in Lemma 3.2, there exists* $\rho_k > \sigma_k$ *for each* $k \geq k_1$ *such that*

$$\beta_k = \max_{u\in Y_k, \|u\|=\rho_k} \Phi_\lambda(u) < 0,$$

where $Y_k = \oplus_{j=1}^{k} X_j = span\{e_1, \dots, e_k\}$.

Proof. Note that Y_k is a finite dimensional subspace for all $k \geq k_1$. Then, by Lemma 3.1, for all $k \geq k_1$, there exists a constant $\varrho_k > 0$ such that

$$meas(\Omega_u^k) \geq \varrho_k, \quad \forall u \in Y_k \setminus \{0\}, \tag{3.16}$$

where $\Omega_u^k = \{x \in \mathbb{R} : |u(x)| \geq \varrho_k \|u\|\}$ for all $k \geq k_1$ and $u \in Y_k \setminus \{0\}$. By (FDE)$_1$, for each $k \geq k_1$, there exists a constant $b_k > 0$ such that

$$F(x, u) \geq \frac{|u|^2}{\varrho_k^3}, \quad \forall x \in \mathbb{R} \quad \text{and} \quad |u| \geq b_k. \tag{3.17}$$

Combining (3.2), (3.16) with (3.17), for all $k \geq k_1$ and $\lambda \in [1, 2]$, we have

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x, u(x))dx \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\Omega_u^k} \frac{|u(x)|^2}{\varrho_k^3} dx \\ &\leq \frac{1}{2}\|u\|^2 - \varrho_k^2 \|u\|^2 \frac{meas(\Omega_u^k)}{\varrho_k^3} \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2 \end{aligned} \tag{3.18}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{b_k}{\varrho_k}$. Here we use the fact that $F(x, u) \geq 0$ for all $(x, u) \in \mathbb{R} \times \mathbb{R}$. For each $k \geq k_1$, if we choose $\rho_k > \max\{\sigma_k, \frac{b_k}{\varrho_k}\}$, then (3.18) implies that

$$\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} \Phi_\lambda(u) \leq -\frac{\rho_k^2}{2},$$

which deduces that the conclusion holds true. □

Now we are in the position to establish the proof of Theorem 1.1.

Proof of Theorem 1.1. Firstly, form (2.1), (2.5) and (3.2), it follows that Φ_λ maps bounded sets to bounded sets uniformly with respect to $\lambda \in [1, 2]$. In addition, (FDE)$_4$ implies that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus, $(\mathcal{A})_1$ holds. Next, using again the fact that $F(x, u) \geq 0$ for all $(x, u) \in \mathbb{R} \times \mathbb{R}$, we know that $(\mathcal{A})_2$ holds by the definition of A in (3.1). Finally, Lemmas 3.2 and 3.3 show that $(\mathcal{A})_3$ holds for $k \geq k_1$, where k_1 is given in Lemma 3.2. Therefore, for each $k \geq k_1$, applying Lemma 2.3, for almost every $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \to 0 \quad \text{and} \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as} \quad m \to \infty, \tag{3.19}$$

where

$$\zeta_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2]$$

with $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. From the proof of Lemma 3.2, we infer that

$$\zeta_k(\lambda) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall k \geq k_1 \quad \text{and} \quad \lambda \in [1, 2], \tag{3.20}$$

where $\overline{\zeta}_k = \max_{u\in B_k} \Phi_1(u)$ and $\overline{\alpha}_k = \frac{\sigma_k^2}{8} \to \infty$ as $k \to \infty$ by (3.15). In view of (3.19), for each $k \geq k_1$, we can choose a sequence $\lambda_n \to 1$ (dependent on k) and get the corresponding sequences satisfying

$$\sup_m \|u_m^k(\lambda_n)\| < \infty \quad \text{and} \quad \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0 \quad \text{as} \quad m \to \infty. \tag{3.21}$$

Claim 1. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m=1}^{\infty}$ has a strong convergent subsequence.

For notational simplicity, we will set $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$ throughout the proof of Claim 1. By (3.21), without loss of generality, we assume that

$$u_m \rightharpoonup u \quad \text{as} \quad m \to \infty \tag{3.22}$$

for some $u \in E$. According to (3.4), we have

$$\begin{aligned}\|u_m - u\|^2 =&\Phi'_{\lambda_n}(u_m)(u_m - u) - \Phi'_{\lambda_n}(u)(u_m - u)\\ &+ \lambda_n \int_{\mathbb{R}} (f(x, u_m(x)) - f(x, u(x)))(u_m(x) - u(x))dx.\end{aligned} \tag{3.23}$$

By (3.21) and (3.22), we have

$$\Phi'_{\lambda_n}(u_m)(u_m - u) \to 0 \quad \text{and} \quad \Phi'_{\lambda_n}(u)(u_m - u) \to 0 \tag{3.24}$$

as $m \to \infty$. In addition, according to (2.5), Lemma 2.2 and the Hölder inequality, we infer that

$$\begin{aligned}&|\int_{\mathbb{R}} (f(x, u_m(x)) - f(x, u(x)))(u_m(x) - u(x))dx|\\ \leq&\Big(\int_{\mathbb{R}} |f(x, u_m(x)) - f(x, u(x))|^2 dx\Big)^{1/2} \|u_m - u\|_2\\ \leq&b\Big(\int_{\mathbb{R}} (|u_m(x)| + |u(x)| + |u_m(x)|^{\nu-1} + |u(x)|^{\nu-1})^2 dx\Big)^{1/2} \|u_m - u\|_2\\ \leq&4b\Big(\int_{\mathbb{R}} (|u_m(x)|^2 + |u(x)|^2 + |u_m(x)|^{2\nu-2} + |u(x)|^{2\nu-2})dx\Big)^{1/2} \|u_m - u\|_2\\ \leq&4b\Big(\|u_m\|_2^2 + \|u\|_2^2 + \|u_m\|_{2\nu-2}^{2\nu-2} + \|u\|_{2\nu-2}^{2\nu-2}\Big)^{1/2} \|u_m - u\|_2 \to 0\end{aligned} \tag{3.25}$$

as $m \to \infty$. Here we use the fact that the boundedness of $\{u_m\}_{m\in\mathbb{N}}$ in E implies that $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}, \mathbb{R})$ and $L^{2\nu-2}(\mathbb{R}, \mathbb{R})$. On account of (3.23), (3.24) and (3.25), we obtain that $u_m \to u$ in E as $m \to \infty$. Thus, Claim 1 holds.

In view of Claim 1, without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N} \quad \text{and} \quad k \geq k_1. \tag{3.26}$$

Combining (3.26), (3.19) and (3.20), we deduce that

$$\Phi'_{\lambda_n}(u_n^k) = 0 \quad \text{and} \quad \Phi_{\lambda_n}(u_n^k) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall n \in \mathbb{N} \quad \text{and} \quad k \geq k_1. \tag{3.27}$$

Claim 2. For each $k \geq k_1$, the sequence $\{u_n^k\}_{n=1}^{\infty}$ in (3.26) is bounded in E.

As in the proof of Claim 1, for notational simplicity, we set $u_n = u_n^k$ for all $n \in \mathbb{N}$. On the contrary, if Claim 2 is not true, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{and} \quad \omega_n = \frac{u_n}{\|u_n\|} \rightharpoonup \omega \in E \quad \text{as} \quad n \to \infty. \tag{3.28}$$

According to (3.28) and Remark 2.2, passing to a subsequence if necessary, we have

$$\omega_n \to \omega \quad \text{in } L^p(\mathbb{R}, \mathbb{R}) \quad \text{for} \quad 2 \le p < \infty, \tag{3.29}$$

and

$$\omega_n(x) \to \omega(x) \quad \text{a.e.} \quad x \in \mathbb{R}. \tag{3.30}$$

When $\omega \neq 0$ occurs, $\Theta = \{x \in \mathbb{R} : \omega(x) \neq 0\}$ has a positive Lebesgue measure. Due to (3.28), it holds that

$$u_n(x) \to \infty, \quad \forall x \in \Theta. \tag{3.31}$$

Combining (3.2), (3.30), (3.31) and $(\mathrm{FDE})_1$, by Fatou's Lemma, we deduce that

$$\begin{aligned} \frac{1}{2} - \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &= \lambda \int_{\mathbb{R}} \frac{F(x, u_n(x))}{\|u_n\|^2} dx \\ &\ge \int_{\Theta} |\omega_n(x)|^2 \frac{F(x, u_n(x))}{|u_n|^2} dx \to \infty \quad \text{as} \quad n \to \infty, \end{aligned}$$

which is a contradiction to (3.27) and (3.28). When $\omega = 0$ occurs, we choose a sequence $\{s_n\}_{n\in\mathbb{N}} \subset [0, 1]$ such that

$$\Phi_{\lambda_n}(s_n u_n) = \max_{s\in[0,1]} \Phi_{\lambda_n}(s u_n). \tag{3.32}$$

For $M > 0$, let $\widetilde{\omega}_n = \sqrt{4M}\omega_n = \frac{\sqrt{4M}}{\|u_n\|} u_n$, then (3.29) yields that

$$\widetilde{\omega}_n \to \sqrt{4M}\omega = 0 \quad \text{in } L^p(\mathbb{R}, \mathbb{R}) \quad \text{for} \quad 2 \le p < \infty, \tag{3.33}$$

which, combining (2.1) with (3.29), imply that

$$\int_{\mathbb{R}} F(x, \widetilde{\omega}_n(x)) dx \le b \int_{\mathbb{R}} \Big(\frac{1}{2} |\widetilde{\omega}_n(x)|^2 + \frac{1}{\nu} |\widetilde{\omega}_n(x)|^{\nu} \Big) dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.34}$$

Note that $0 < \frac{\sqrt{4M}}{\|u_n\|} < 1$ holds by (3.28) for n large enough. Consequently, on account of (3.2), (3.32) and (3.34), we obtain that

$$\begin{aligned} \Phi_{\lambda_n}(s_n u_n) &\ge \Phi_{\lambda_n}(\widetilde{\omega}_n) \\ &= \frac{1}{2} \|\widetilde{\omega}_n\|^2 - \lambda_n \int_{\mathbb{R}} F(x, \widetilde{\omega}_n(x)) dx \\ &= 2M - \lambda_n \int_{\mathbb{R}} F(x, \widetilde{\omega}_n(x)) dx \ge M \end{aligned}$$

for n large enough. It follows that $\lim_{n\to\infty} \Phi_{\lambda_n}(s_n u_n) = \infty$. Observing that $\Phi_{\lambda_n}(0) = 0$ and $\Phi_{\lambda_n}(u_n) \in [\overline{\alpha}_k, \overline{\zeta}_k]$ in (3.27), we know that $s_n \in (0, 1)$ in (3.32) for n large enough. Moreover, one deduces that

$$0 = s_n \frac{d}{ds}\Big|_{s=s_n} \Phi_{\lambda_n}(s u_n) = \Phi'_{\lambda_n}(s_n u_n) s_n u_n, \tag{3.35}$$

which, combining (3.2), (3.4), (3.18), (3.35) with $(\mathrm{FDE})_3$, yield that

$$\begin{aligned}\Phi_{\lambda_n}(u_n) &= \Phi_{\lambda_n}(u_n) - \frac{1}{2}\Phi'_{\lambda_n}(u_n)u_n \\ &= \frac{\lambda_n}{2}\int_{\mathbb{R}} \widetilde{F}(x, u_n(x))dx \\ &\geq \frac{\lambda_n}{2\vartheta}\int_{\mathbb{R}} \widetilde{F}(x, s_n u_n(x))dx \\ &= \frac{1}{\vartheta}\Phi_{\lambda_n}(s_n u_n) - \frac{1}{2\vartheta}\Phi'_{\lambda_n}(s_n u_n)s_n u_n \\ &= \frac{1}{\vartheta}\Phi_{\lambda_n}(s_n u_n) \to \infty \quad \text{as} \quad n \to \infty,\end{aligned}$$

where ϑ is the constant in $(\mathrm{FDE})_3$, which provides a contradiction to (3.27). Thus, Claim 2 is true.

In view of Claim 2 and (3.27), for each $k \geq k_1$, using the similar arguments in the proof of Claim 1, we can also show that the sequence $\{u_n^k\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit u^k being just a critical point $\Phi = \Phi_1$. Evidently, $\Phi(u^k) \in [\overline{\alpha}_k, \overline{\zeta}_k]$ for all $k \geq k_1$. Since $\overline{\alpha}_k \to \infty$ as $k \to \infty$ in (3.20), we obtain infinitely many nontrivial critical points of Φ. Therefore, (FDE) possesses infinitely many nontrivial homoclinic solutions. □

Acknowledgements

The authors thank the editor and anonymous reviewers for their helpful and valuable comments.

References

[1] C. J. Amick and J. F. Toland, *Homoclinic orbits in the dynamic phase space analogy of an elastic strut*, European. J. Appl. Math., 1991, 3, 97–114.

[2] F. P. Bretherton, *Resonant interaction between waves: the case of discrete oscillations*, J. Fluid Mech., 1964, 20, 457–479.

[3] B. Buffoni, *Periodic and homoclinic orbits for Lorentz-Lagrangian systems via variational method*, Nonlinear Anal., 1996, 26, 443–462.

[4] B. Buffoni, M. Groves and J. F. Toland, *A plethora of capillary gravity waves with near-critical Bond and Froude numbers*, Philos. Trans. Roy. Soc. London. Ser. A, 1996, 354, 575–607.

[5] A. V. Buryak and Y. Kivshar, *Solitons due to second harmonic generation*, Phys. Lett. A, 1995, 197, 407–412.

[6] G. Chapiro, L. F. O. Faria and A. D. Maldonado, *On the existence of solutions for a class of fourth order differential equations*, J. Math. Anal. Appl., 2015, 427, 126–139.

[7] P. Coullet, C. Elphick and D. Repaux, *Nature of spatial chaos*, Phys. Rev. Lett., 1987, 58, 431–434.

[8] G. T. Dee and W. van Saarloos, *Bistable systems with propagating fronts leading to pattern formation*, Phys. Rev. Lett, 1988, 60, 2641–2644.

[9] J. Lega, J. Molonev and A. Newell, *Swift-Hohenberg for lasers*, Phys. Rev. Lett, 1994, 73, 2978–2981.

[10] F. Li, J. T. Sun, G. F. Lu and C. J. Lv, *Infinitely many homoclinic solutions for a nonperiodic fourth-order differential equation without (AR)-condition*, Appl. Math. Comput., 2014, 241, 36–41.

[11] T. X. Li, J. T. Sun and T. F. Wu, *Existence of homoclinic solutions for a fourth order differential equation with a parameter*, Appl. Math. Comput., 2015, 251, 499–506.

[12] C. Y. Li, *Homoclinic orbits of two classes of fourth order semilinear differential equations with periodic nonlinearity*, J. Appl. Math. Comput., 2008, 27, 107–116.

[13] C. Y. Li, *Remarks on homoclinic solutions for semilinear fourth-order differential equations without periodicity*, Appl. Math. J. Chinese Univ., 2009, 24, 49–55.

[14] L. A. Peletier and W. C. Troy, *Spatial Patterns: Higher Order Models in Physics and Mechnics*, Birkhäuser, Boston, 2001.

[15] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[16] A. Salvatore, *Homoclinic orbits for a special class of nonautonomous Hamiltonian systems*, Nonlinear Anal., 1997, 30, 4849–4857.

[17] J. T. Sun and T. F. Wu, *Two homoclinic solutions for a nonperiodic fourth order differential equation with a perturbation*, J. Math. Anal. Appl., 2014, 413, 622–632.

[18] J. T. Sun, T. F. Wu and F. Li, *Concentration of homoclinic solutions for some fourth-order equations with sublinear indefinite nonlinearities*, Appl. Math. Lett., 2014, 38, 1–6.

[19] S. Tersian and J. Chaparova, *Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations*, J. Math. Anal. Appl., 2001, 260, 490–506.

[20] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

[21] L. Yang, *Infinitely many homoclinic solutions for nonperiodic fourth order differential equations with general potentials*, Abst. Appl. Anal., 2014, Art. ID 435125, 7 pp.

[22] Z. H. Zhang and R. Yuan, *Homoclinic solutions for a nonperiodic fourth order differential equations without coercive conditions*, Appl. Math. Comput., 2015, 250, 280–286.

[23] W. M. Zou, *Variant fountain theorems and their applications*, Manuscripta Math., 2001, 104, 343–358.