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STRONG CONVERGENCE ANALYSIS OF A HYBRID ALGORITHM FOR NONLINEAR OPERATORS IN A BANACH SPACE

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Abstract In this paper, a hybrid algorithm is investigated for an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and a bifunction. Strong convergence of the algorithm is obtained in a strictly convex, smooth and reflexive Banach space.

Keywords Quasi-ϕ-nonexpansive mapping, equilibrium problem, generalized projection, variational inequality.

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1. Introduction-Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let S^E be the unit sphere of E. Recall that E is said to be a strictly convex space iff $\|x+y\|<2$ for all $x,y\in S^E$ and $x\neq y$. Recall that E is said to have a Gâteaux differentiable norm iff $\lim_{t\to 0}\frac{\|x\|-\|x+ty\|}{t}$ exists for each $x,y\in S^E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm if for each $y\in B_E$, the limit is attained uniformly for all $x\in S^E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for $x,y\in S^E$. In this case, we say that E is uniformly smooth.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx=\{y\in E^*:\|x\|^2=\langle x,y\rangle=\|y\|^2\}.$$

It is known

if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E;

if E is a strictly convex Banach space, then J is strictly monotone;

if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e.,continuous from the strong topology of E to the weak star topology of E;

if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^*:E^*\to E$ is the normalized duality mapping in E^*, then $J^{-1}=J^*$;

if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto;

if E is a uniformly smooth, then it is reflexive and smooth.

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It is also known that E^* is uniformly convex if and only if E is uniformly smooth. From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively.

Recall that E has the Kadec-Klee Property (hereafter KKP) if $\lim_{n\to\infty}\|x_n - x\| = 0$ as $n \to \infty$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$ as $n \to \infty$. We remark here that if E is uniformly convex, then it has the KKP; see [12] and the references therein.

Let C be a nonempty closed and convex subset of E and let $B : C \times C \to \mathbb{R}$ be a bifunction. Recall that the following equilibrium problem. Find $\bar{x} \in C$ such that $B(\bar{x}\, y) \geq 0$, $\forall y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$.

The following restrictions on bifunction B are essential in this paper.

(R-1) $B(a, a) \equiv 0, \forall a \in C$;

(R-2) $B(b, a) + B(a, b) \leq 0, \forall a, b \in C$;

(R-3) $B(a, b) \geq \limsup_{t\downarrow 0} B(tc + (1 - t)a, b)$, $\forall a, b, c \in C$;

(R-4) $b \mapsto B(a, b)$ is convex and weakly lower semi-continuous, $\forall a \in C$.

We remark here that B is said to be monotone iff $B(x, y) + B(y, x) \leq 0$ for all $x, y \in C$. $y \mapsto B(x, y)$ is convex iff

$$B(tx + (1 - t)y, z) \leq tB(x, z) + (1 - t)B(y, z)$$

for all $x, y, z \in C$ and $t \in (0, 1)$. $y \mapsto B(x, y)$ is lower semi-continuous iff $B(x, y_n) \to B(x, y)$ whenever $y_n \to y$ as $n \to \infty$. It is known that the indicator function of an open set is lower semi-continuous.

The equilibrium problem provides us a natural, novel and unified framework to study a wide class of problems arising in physics, economics, finance, transportation, network, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. It has been shown that variational inequalities, complementarity problems, fixed point problems and inclusion problems can be viewed as a special realization of the equilibrium problems; see, [3,4,9,11,19,20,23] and the references therein. Equilibrium problems have numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics. Recently, the equilibrium problem has been extensively investigated based on hybrid algorithms, in particular, the monotone hybrid algorithm; see [5,10,13,15,21,22] and the references therein.

Let T be a mapping on C. Recall that a point p is said to be a fixed point of T if and only if $p = Tp$. p is said to be an asymptotic fixed point of T if and only if C contains a sequence $\{x_n\}$, where $x_n \rightharpoonup p$ such that $x_n - Tx_n \to 0$. From now on, We use $Fix(T)$ to stand for the fixed point set and $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set.

T is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} Tx_n = y'$, then $Tx' = y'$. Let B be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if

$$\limsup_{n\to\infty} \sup_{x\in B}\{\|T^{n+1}x - T^n x\|\} = 0.$$

Next, we assume that E is a smooth Banach space which means J is single-valued. Study the functional

$$\phi(x,y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists an unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$, for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [2], Alber studied a new mapping $Proj_C$ in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, which implies from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|y\| + \|x\|)^2, \quad \forall x,y \in E \tag{1.1}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{1.2}$$

Recall that T is said to be relatively nonexpansive iff

$$\widetilde{Fix}(T) = Fix(T) \neq \emptyset, \phi(p,Tx) \le \phi(p,x), \ \forall x \in C, \ \forall p \in Fix(T).$$

T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{Fix}(T) = Fix(T) \neq \emptyset, \phi(p,T^n x) \le (\mu_n + 1)\phi(p,x), \ \forall x \in C, \ \forall p \in Fix(T), \ \forall n \ge 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

T is said to be relatively asymptotically nonexpansive in the intermediate sense iff $\widetilde{Fix}(T) = Fix(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p \in Fix(T), x \in C} \big(\phi(p,T^n x) - \phi(p,x)\big) \le 0.$$

Putting $\xi_n = \max\{0, \sup_{p\in Fix(T),x\in C} \big(\phi(p,T^n x) - \phi(p,x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$.

T is said to be quasi-ϕ-nonexpansive iff

$$Fix(T) \neq \emptyset, \phi(p,Tx) \le \phi(p,x), \ \forall x \in C, \ \forall p \in Fix(T).$$

T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$Fix(T) \neq \emptyset, \phi(p,T^n x) \le (\mu_n + 1)\phi(p,x), \ \forall x \in C, \ \forall p \in Fix(T), \ \forall n \ge 1.$$

T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense iff $Fix(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p \in Fix(T), x \in C} \big(\phi(p,T^n x) - \phi(p,x)\big) \le 0.$$

Putting $\xi_n = \max\{0, \sup_{p\in Fix(T),x\in C} \big(\phi(p,T^n x) - \phi(p,x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$.

Remark 1.1. The class of relatively asymptotically nonexpansive mappings, which was considered in [1], covers the class of relatively nonexpansive mappings [8]. The class of (asymptotically) quasi-ϕ-nonexpansive mappings [16, 17] covers the class of relatively (asymptotically) nonexpansive mappings. (Asymptotically) quasi-ϕ-nonexpansive mappings does not require the strong restriction $\widetilde{Fix}(T) = Fix(T)$. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is more desirable than the class of relatively asymptotically nonexpansive mappings in the intermediate sense. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense does not require the strong restriction $\widetilde{Fix}(T) = Fix(T)$; see [17] and the references therein.

Remark 1.2. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense [14] is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered in [7] as a non-Lipschitz continuous mappings, in the framework of Hilbert spaces.

The following lemmas also play an important role in this paper.

Lemma 1.1 ([2]). *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a closed and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

and $x_0 = \Pi_C x$ if and only if

$$\langle y - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall y \in C.$$

Lemma 1.2 ([6, 16]). *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a function with restrictions (R-1), (R-2), (R-3) and (R-4). Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that $rB(z, y) + \langle z - y, Jz - Jx \rangle \leq 0$, $\forall y \in C$ Define a mapping $W^{B,r}$ by*

$$W^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$W^{B,r}$ is single-valued quasi-ϕ-nonexpansive.*

(2) *$Sol(B) = Fix(W^{B,r})$ is closed and convex.*

Lemma 1.3. *Let E be a strictly convex, smooth and reflexive Banach space such that both E^* and E have the KKP. Let C be a convex and closed subset of E and let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C. Then $Fix(T)$ is convex.*

Proof. Let $p_1, p_2 \in Fix(T)$, and

$$p = tp_1 + (1 - t)p_2,$$

where $t \in (0, 1)$. We see that $p = Tp$. Indeed, we see from the definition of T that

$$\phi(p_1, T^n p) \leq \phi(p_1, p) + \xi_n,$$

and

$$\phi(p_2, T^n p) \leq \phi(p_2, p) + \xi_n.$$

In view of (1.2), we obtain that

$$\phi(p_1, T^n p) = \phi(p_1, p) + \phi(p, T^n p) + 2\langle p_1 - p, Jp - JT^n p\rangle, \tag{1.3}$$

and

$$\phi(p_2, T^n p) = \phi(p_2, p) + \phi(p, T^n p) + 2\langle p_2 - p, Jp - JT^n p\rangle. \tag{1.4}$$

It follows from (1.3) and (1.4) that

$$\phi(p, T^n p) \le 2\langle p - p_1, Jp - J(T^n p)\rangle + \xi_n, \tag{1.5}$$

and

$$\phi(p, T^n p) \le 2\langle p - p_2, Jp - J(T^n p)\rangle + \xi_n. \tag{1.6}$$

Multiplying t and $(1-t)$ on the both sides of (1.5) and (1.6), respectively yields that $\phi(p, T^n p) \le \xi_n$. Using (1.1), one has $\lim_{n\to\infty} \|T^n p\| = \|p\|$. Since E^* is reflexive, we may, without loss of generality, assume that $J(T^n p) \rightharpoonup v^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $v \in E$ such that $Jv = v^*$. It follows that

$$\phi(p, T^n p) = \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|J(T^n p)\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned} 0 &\ge \|p\|^2 - 2\langle p, v^*\rangle + \|v^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jv\rangle + \|Jv\|^2 \\ &= \|p\|^2 - 2\langle p, Jv\rangle + \|v\|^2 \\ &= \phi(p, v). \end{aligned}$$

This implies that $p = v$, that is, $Jp = v^*$. It follows that $J(T^n p) \rightharpoonup Jp \in E^*$. Using KKP of E^*, we obtain $\lim_{n\to\infty} \|J(T^n p) - Jp\| = 0$. Since J^{-1} is demicontinuous, we see that $T^n p \rightharpoonup p$. By virtue of KKP of E, we see $T^n p \to p$ as $n \to \infty$. Hence

$$TT^n p = T^{n+1} p \to p,$$

as $n \to \infty$. In view of the closedness of T, we obtain that $p \in Fix(T)$. This shows that $Fix(T)$ is convex. This completes the proof. □

2. Main results

Theorem 2.1. *Let E be a strictly convex, smooth and reflexive Banach space such that both E^* and E have the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ Jy_n = \alpha_n JT^n x_n + (1-\alpha_n) Jx_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \alpha_n \xi_n \ge \phi(z, u_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where

$$\xi_n = \max\{\sup_{p\in Fix(T),x\in C}\big(\phi(p,T^n x)-\phi(p,x)\big),0\},$$

$u_n \in C$ *such that*

$$r_n B(u_n,\mu) \leq \langle \mu - u_n, Ju_n - Jy_n\rangle, \quad \forall \mu \in C_n,$$

$\{\alpha_n\}$ *is a real sequence in* $[a,1]$*, where* $a \in (0,1]$ *is a real number, and* $\{r_n\} \subset [r,\infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_n\}$ *converges strongly to* $Proj_{Fix(T)\cap Sol(B)}x_1$*.*

Proof. First, we prove $Sol(B)\cap Fix(T)$ is convex and closed. Using Lemma 1.2 and Lemma 1.3, we find that $Sol(B)$ is convex and closed and $Fix(T)$ is convex. Since T is closed, one has $Fix(T)$ is also closed. So, $Proj_{Sol(B)\cap Fix(T)}x$ is well defined, for any element x in E.

Next, we prove that C_n is convex and closed. It is obvious that $C_1 = C$ is convex and closed. Assume that C_m is convex and closed for some $m \geq 1$. Let $p_1, p_2 \in C_{m+1}$. It follows that

$$p = sp_1 + (1-s)p_2 \in C_m,$$

where $s \in (0,1)$. Notice that

$$\phi(p_1,u_m) - \phi(p_1,x_m) \leq \alpha_m \xi_m,$$

and

$$\phi(p_2,u_{(m,i)}) - \phi(p_2,x_m) \leq \alpha_m \xi_m,$$

Hence, one has

$$2\langle p_1, Jx_m - Ju_m\rangle - \|x_m\|^2 + \|u_m\|^2 \leq \alpha_m\xi_m,$$

and

$$2\langle p_2, Jx_m - Ju_m\rangle - \|x_m\|^2 + \|u_m\|^2 \leq \alpha_m\xi_m.$$

Using the above two inequalities, one has

$$\phi(p,x_m) + \alpha_m\xi_m \geq \phi(z,u_m).$$

This shows that C_{m+1} is closed and convex. Hence, C_n is a convex and closed set. This proves that $Proj_{C_{n+1}}x_1$ is well defined.

Next, we prove $Sol(B)\cap Fix(T) \subset C_n$. Note that $Sol(B)\cap Fix(T) \subset C_1 = C$ is clear. Suppose that $Sol(B)\cap Fix(T)\subset C_m$ for some positive integer m. For any $w \in Sol(B)\cap Fix(T)\subset C_m$, we see that

$$\begin{aligned}
\alpha_m\xi_m + \phi(w,x_m) &\geq \alpha_m\phi(w,T^m x_m) + (1-\alpha_m)\phi(w,x_m)\\
&= \|w\|^2 - 2\alpha_m\langle w, JT^m x_m\rangle - 2(1-\alpha_m)\langle w, Jx_m\rangle\\
&\quad + \alpha_m\|T^m x_m\|^2 + (1-\alpha_m)\|x_m\|^2\\
&\geq \|(1-\alpha_m)Jx_m + \alpha_m JT^m x_m\|^2 + \|w\|^2\\
&\quad - 2\langle w, (1-\alpha_m)Jx_m + \alpha_m JT^m x_m\rangle\\
&= \phi(w,y_m)\\
&\geq \phi(w,u_m) \geq 0,
\end{aligned}$$

where

$$\xi_m = \max\{\sup_{p \in Fix(T), x \in C} \big(\phi(p, T^m x) - \phi(p, x)\big), 0\}.$$

This shows that $w \in C_{m+1}$. This implies that $Sol(B) \cap Fix(T) \subset C_n$. Using Lemma 1.1, one has

$$\langle z - x_n, Jx_1 - Jx_n \rangle \leq 0, \quad \forall z \in C_n.$$

It follows that

$$\langle w - x_n, Jx_1 - Jx_n \rangle \leq 0, \quad \forall w \in Sol(B) \cap Fix(T) \subset C_n.$$

Using Lemma 1.1 yields that

$$\phi(x_n, x_1) \leq \phi(\Pi_{Fix(T) \cap Sol(B)} x_1, x_1),$$

which implies that $\{\phi(x_n, x_1)\}$ and $\{x_n\}$ are bounded. Since E is reflexive, we may assume that $x_n \rightharpoonup \bar{x} \in C_n$. Therefore, one has

$$\phi(x_n, x_1) \leq \phi(\bar{x}, x_1).$$

This implies that

$$\begin{aligned} \phi(\bar{x}, x_1) &\leq \liminf_{n \to \infty} (\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1 \rangle) \\ &= \liminf_{n \to \infty} \phi(x_n, x_1) \leq \phi(\bar{x}, x_1). \end{aligned}$$

It follows that

$$\lim_{n \to \infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1).$$

Hence, we have

$$\lim_{n \to \infty} \|x_n\| = \|\bar{x}\|.$$

Using the KKP of the spaces, one obtains that x_n converges strongly to $\bar{x}$ as $n \to \infty$. On the other hand, we find that

$$\phi(x_{n+1}, x_1) \geq \phi(x_n, x_1),$$

which shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. Therefore, one has $\lim_{n \to \infty} \phi(x_n, x_1)$ exists. It follows that

$$\phi(x_{n+1}, x_1) - \phi(x_n, x_1) \geq \phi(x_{n+1}, x_n) \geq 0.$$

Therefore, we have

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0.$$

Since $x_{n+1} \in C_{n+1}$, one sees that

$$\phi(x_{n+1}, x_n) + \alpha_n \xi_n \geq \phi(x_{n+1}, u_n) \geq 0.$$

It follows that

$$\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0.$$

Hence, one has

$$\lim_{n \to \infty} (\|u_n\| - \|x_{n+1}\|) = 0.$$

This implies that

$$\lim_{n\to\infty}\|Ju_n\| = \lim_{n\to\infty}\|u_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Ju_n\}$ is bounded. Assume that Ju_n converges weakly to $u^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Ju = u^*$. It follows that

$$\phi(x_{n+1}, u_n) + 2\langle x_{n+1}, Ju_n\rangle = \|x_{n+1}\|^2 + \|Ju_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has

$$\begin{aligned}0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, u^*\rangle + \|u^*\|^2\\ &= \|\bar{x}\|^2 + \|Ju\|^2 - 2\langle \bar{x}, Ju\rangle\\ &= \phi(\bar{x}, u) \ge 0.\end{aligned}$$

That is, $\bar{x} = u$, which in turn implies that $J\bar{x} = u^*$. Hence, $Ju_n \rightharpoonup J\bar{x} \in E^*$. Using the KKP, we obtain $\lim_{n\to\infty} Ju_n = J\bar{x}$. Since J^{-1} is demi-continuous and E has the KKP, one gets $u_n \to \bar{x}$, as $n \to \infty$. On the other hand, one has

$$0 \le \phi(x_{n+1}, y_n) \le \alpha_n\xi_n + \phi(x_{n+1}, x_n).$$

Hence, one has

$$\lim_{n\to\infty}\phi(x_{n+1}, y_n) = 0.$$

Hence, one has

$$\lim_{n\to\infty}(\|y_n\| - \|x_{n+1}\|) = 0.$$

This implies that

$$\lim_{n\to\infty}\|Jy_n\| = \lim_{n\to\infty}\|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Jy_n\}$ is bounded. Assume that Jy_n converges weakly to $y^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n\rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has

$$\begin{aligned}0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2\\ &= \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy\rangle\\ &= \phi(\bar{x}, y) \ge 0.\end{aligned}$$

That is, $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. Hence,

$$Jy_n \rightharpoonup J\bar{x} \in E^*.$$

Using the KKP, we obtain

$$\lim_{n\to\infty} Jy_n = J\bar{x}.$$

Since J^{-1} is demi-continuous and E has the KKP, one gets

$$y_n \to \bar{x} \in C_n, \quad \text{as } n \to \infty.$$

Next, we show that $\bar{x} \in Fix(T) \cap Sol(B)$. Note the fact

$$Jy_n - Jx_n = \alpha_n(JT^n x_n - Jx_n),$$

and the restriction on $\{\alpha_n\}$, one has

$$\lim_{n\to\infty} \|Jx_n - JT^n x_n\| = 0.$$

This implies that

$$\lim_{n\to\infty} \|JT^n x_n - J\bar{x}\| = 0.$$

Since J^{-1} is demicontinuous, one has $T^n x_n \rightharpoonup \bar{x}$. Since

$$|\|T^n x_n\| - \|\bar{x}\|| \le \|J(T^n x_n) - J\bar{x}\|,$$

one has $\|T^n x_n\| \to \|\bar{x}\|$, as $n \to \infty$. Since E has the KKP, we obtain

$$\lim_{n\to\infty} \|\bar{x} - T^n x_n\| = 0.$$

Since T is also uniformly asymptotically regular, one has

$$\lim_{n\to\infty} \|\bar{x} - T^{n+1} x_n\| = 0.$$

That is, $T(T^n x_n) \to \bar{x}$. Using the closedness of T, we find $T\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T)$.

Next, we show that $\bar{x} \in Sol(B)$. Since

$$\langle y - u_n, Ju_n - Jy_n\rangle \ge r_n B(y, u_n), \quad \forall y \in C_n,$$

we see that

$$\|y - u_n\|\|Ju_n - Jy_n\| \ge r_n B(y, u_n).$$

In view of (R-4), one has

$$B(y, \bar{x}) \le 0.$$

For $0 < s < 1$, define

$$y^s = sy + (1 - s)\bar{x}.$$

It follows that $y^s \in C$, which yields that $B(y^s, \bar{x}) \le 0$. It follows from the (R-1) and (R-4) that

$$0 = B(y^s, y^s) \le sB(y^s, y) + (1 - s)B(y^s, \bar{x}) \le sB(y^s, y).$$

That is, $B(y^s, y) \ge 0$. Letting $s \downarrow 0$, we obtain from (R-3) that $B(\bar{x}, y) \ge 0$, $\forall y \in C$. This implies that $\bar{x} \in Sol(B)$. This completes the proof that $\bar{x} \in Sol(B) \cap Fix(T)$.

Finally, we prove $\bar{x} = Proj_{Sol(B)\cap Fix(T)} x_1$. Note the fact

$$\langle w - x_n, Jx_1 - Jx_n\rangle \le 0, \quad \forall w \in Sol(B) \cap Fix(T).$$

It follows that

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \ge 0, \quad \forall w \in Fix(T) \cap Sol(B).$$

Using Lemma 1.1, we find that $\bar{x} = Proj_{Fix(T)\cap Sol(B)} x_1$. This completes the proof. □

Remark 2.1. Theorem 2.1, which mainly improves the corresponding results in [14, 17, 19, 21, 24] unify the recent results on hybrid algorithms. The algorithm is more efficient since u_n is searched monotonicially in C_n instead of always in C. The framework of the space is only smooth. To be more clear, we remove the uniform smoothness. The typical example of the space in Theorem 2.1 is a reflexive, strictly convex and smooth Musielak-Orlicz space.

Corollary 2.1. *Let E be a strictly convex, smooth and reflexive Banach space such that both E^* and E have the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \alpha_n \xi_n \geq \phi(z, u_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where

$$\xi_n = \max\{\sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\},$$

$u_n \in C_n$ such that $r_n B(u_n, \mu) \leq \langle \mu - u_n, Ju_n - JT^n x_n\rangle$, $\forall \mu \in C_n$, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $Proj_{Fix(T)\cap Sol(B)} x_1$.

If T is the identity operator, we have the following result.

Corollary 2.2. *Let E be a strictly convex, smooth and reflexive Banach space such that both E^* and E have the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that $Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, u_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $u_n \in C_n$ such that $r_n B(u_n, \mu) \leq \langle \mu - u_n, Ju_n - Jx_n\rangle$, $\forall \mu \in C_n$, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $Proj_{Sol(B)} x_1$.

3. Applications

In this section, we consider solutions of a variational inequality and give some deduced results in the framework Hilbert spaces.

Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the the following variational inequality. Finding a point $x \in C$ such that

$\langle x - y, Ax\rangle \le 0$, $\forall y \in C$. The symbol $Nc(x)$ stand for the normal cone for C at a point $x \in C$; that is,

$$Nc(x) = \{x^* \in E^* : \langle x - y, x^*\rangle \ge 0,\ \forall y \in C\}.$$

From now on, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Theorem 3.1. *Let E be a strictly convex, smooth and reflexive Banach space such that both E^* and E have the KKP. Let C be a convex and closed subset of E. Let $A : C \to E^*$ be a single valued, monotone and hemicontinuous operator and let B be a function with (R-1), (R-2), (R-3) and (R-4). Assume that $Sol(B) \cap VI(C, A)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following process*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \forall i \in \Lambda, \\ x_1 = Proj_{C_1} x_0, \\ z_n = VI(C, A + \frac{1}{r}(J - Jx_n)), \\ Jy_n = \alpha_n Jz_n + (1 - \alpha_n)Jx_n, \quad n \ge 1, \\ C_{n+1} = \{w \in C_n : \phi(w, x_n) \ge \phi(w, u_n\}, \\ x_{n+1} = Proj_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where $\xi_n = \max\{\sup_{p\in Fix(T),x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\}$, $u_n \in C_n$ such that $r_n B(u_n, \mu) \le \langle \mu - u_n, Ju_n - Jy_n\rangle$, $\forall \mu \in C_n$, $\{\alpha_n\}$ is a real sequence in $[a, 1]$, where $a \in (0, 1]$ is a real number, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $Proj_{VI(C,A)\cap Sol(B)} x_1$.

Proof. Define a new operator M by

$$Mx = \begin{cases} Ax + Nc(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Hence, M is maximal monotone and $M^{-1}(0) = VI(C, A)$ [18], where $M^{-1}(0)$ stands for the zero point set of M. For each $r > 0$, and $x \in E$, we see that there exists an unique x_r in the domain of M such that $Jx \in Jx_r + rM(x_r)$, where $x_r = (J + rM)^{-1}Jx$. Notice that

$$z_n = VI(C, \frac{1}{r}(J - Jx_n) + A),$$

which is equivalent to

$$\langle z_n - y, Az_n + \frac{1}{r}(Jz_n - Jx_n)\rangle \le 0, \quad \forall y \in C,$$

that is,

$$\frac{1}{r}\big(Jx_n - Jz_n\big) \in Nc(z_n) + Az_n.$$

This implies that $z_n = (J + rM)^{-1}Jx_n$. From [16], we find that $(J + rM)^{-1}J$ is closed quasi-ϕ-nonexpansive with $Fix((J + rM)^{-1}J) = M^{-1}(0)$. Using Theorem 2.1, we find the desired conclusion immediately. □

In the framework of Hilbert spaces, one has

$$\sqrt{\phi(x, y)} = \|x - y\|, \quad \forall x, y \in E.$$

The generalized projection is reduced to the metric projection and the class of asymptotically-ϕ-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense. Using Theorem 3.1, we find the following results.

Theorem 3.2. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = P_{C_1}x_0, \\ y_n = \alpha_n T^n x_n + (1-\alpha_n)x_n, \\ C_{n+1} = \{z \in C_n : \|z - x_n\|^2 + \alpha_n \xi_n \geq \|z - u_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \end{cases}$$

where $\xi_n = \max\{\sup_{p \in Fix(T), x \in C}\big(\|p - T^n x\|^2 - \|p - x\|^2\big), 0\}$, $u_n \in C_n$ such that $r_n B(u_n, \mu) \leq \langle \mu - u_n, u_n - y_n \rangle$, $\forall \mu \in C_n$, $\{\alpha_n\}$ is a real sequence in $[a, 1]$, where $a \in (0, 1]$ is a real number, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $P_{Fix(T)\cap Sol(B)}x_1$.

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