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# BIFURCATION FROM TWO EQUILIBRIA OF STEADY STATE SOLUTIONS FOR NON-REVERSIBLE AMPLITUDE EQUATIONS*

Yancong Xu[1,†], Rui Xu[1], and Yu Yang[2]

**Abstract** In this paper, bifurcation and stability of two kinds of constant stationary solutions for non-reversible amplitude equations on a bounded domain with Neumann boundary conditions are investigated by using the perturbation theory and weak nonlinear analysis. The asymptotic behaviors and local properties of two explicit steady state solutions, and pitch-fork bifurcations are also obtained if the bounded domain is regarded as a parameter. In addition, the stability of a new increasing or decaying local steady state solution with oscillations are analyzed.

**Keywords** Bifurcation, stability, perturbed amplitude equations, eigenvalue problems.

**MSC(2010)** 37G25, 37G20, 37C29.

## 1. Introduction

In recent years, bifurcation and stability of steady state solutions for partial differential equations on a bounded domain have attracted a lot of attention, in particular, the occurrence of patterns of symmetric dynamical systems was investigated [1–3, 5–24, 26]. However, it is very important to study the dynamics of steady-state solutions of symmetry-breaking systems [4, 25]. To study pattern formation [17, 18] with a conserved quantity, the appropriate symmetric properties of the system play a very important role. Sometimes the pursuit of pattern formation in various fields of science can be a result of the study of the amplitude equations. For example, Blömker & Mohammed [3] have discussed rigorously the amplitude equations describing the essential dynamics using the natural separation of timescales near a change of stability, the impact of degenerate noise on the dominant behavior, and see that additive noise has the potential to stabilize the dynamics of the dominant modes and higher order corrections to the amplitude equation. In order to manifest the dynamics of steady state solutions, many boundary conditions are imposed on the corresponding systems. Ghergu [13] studied a Gierer-Meinhardt

†the corresponding author. Email address: Yancongx@163.com(Y. Xu)
1Department of Mathematics, Hangzhou Normal University, No. 16, Xuelin Street, Hangzhou, China
2School of Science and Technology, Zhejiang International Studies University, No. 140, Wensan Road, Hangzhou, China
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type system with Dirichlet boundary conditions on a smooth bounded domain, and obtained some conditions to guarantee the existence, regularity and boundary behavior of steady state solutions.

In this paper, we consider steady-state solutions of the following perturbed parabolic partial differential equations:

$$\begin{cases} u_t = u_{xx} + u - u^3 - uv + \delta u_x, \\ v_t = \sigma v_{xx} + \mu(u^2)_{xx}, \end{cases} \tag{1.1}$$

on the cylindrical domain $Q = (0, L) \times R^+$ with the boundary conditions $u_x = v_x = 0$, at $x = 0, L$. Set $\sigma > 0$, and $\mu, \delta$ may have either sign. It is easy to see that the amplitude equation (1) is invariant if

$$u \to -u, v \to v,$$

but breaks the reversible symmetry

$$x \to -x.$$

System (1.1) with $\delta = 0$ has been studied under various backgrounds. For example, hydrodynamics in Fauve [11], thermosolutal convection, magnetoconvection, rotating convection in [6, 17], cell pattern in Coullet& Iooss [5]. Also, it has been studied in details in Norbury, et al. [20] with different multiscales, some explicit localized spike solutions and their stability are also given. If a domain is large enough, the instability of rolls undergoes a supercritical pitch-fork bifurcation in [18, 24]. Li & Chen [15] has investigated the asymptotic behaviors of steady state solution branches bifurcating from the equilibria of the two-dimensional K-S equation.

The dynamics of equation (1.1) with $\delta = 0$ has been discussed by Shi & Gao [24], the asymptotic periodic stationary solutions are obtained, and if the length L of the domain is regarded as bifurcation parameter, branches of nontrivial solutions are shown by using the perturbation method. So, it is necessary to study the static bifurcations of non-symmetric system (1.1) with $\delta \neq 0$. We will find the increasing or decaying asymptotic steady solutions with oscillations and the corresponding bifurcation diagrams with the bounded domain $L$.

In this paper, we investigate the dynamics of the perturbed version of system (1.1) with the perturbed term by applying the method developed by [8–10,15,21,22]. Since the perturbation term $\delta u_x$ destroys the reversibility of system (1.1), compared to the case of one equation [15, 24], then the discussion here is more intricate. In system (1.1), $u$ can be complex, that is, $u = re^{ix}$. The additional phase space $x$ makes analytic analysis very complicated. So we restrict our attention to the invariant subspace in which $u$ is real.

We consider the steady-state solutions of system (1.1) with Neumann boundary condition as follows:

$$\begin{cases} u_{xx} + u - u^3 - uv + \delta u_x = 0, & 0 < x < L, \\ \sigma v_{xx} + \mu(u^2)_{xx} = 0, & 0 < x < L, \\ u_x(0) = u_x(L) = 0, \\ v_x(0) = v_x(L) = 0. \end{cases} \tag{1.2}$$

It is not easy to obtain all the explicit solutions of system (1.2). Here we only discuss the stability and bifurcation of the constant solutions. Clearly, system (1.2)

has two types of constant stationary solutions $u_0$ and $v_0$, i.e.,
$(i)$ $u_0 = 0,\ v_0 = c,\quad (c \in R)$,
$(ii)$ $u_0 = c,\ v_0 = 1 - c^2,\quad (c \in R, c \neq 0)$.

By letting $u = u_1, \dot{u_1} = u_2, v = u_3, \dot{u_3} = u_4$, system (1.2) becomes

$$\begin{cases} \dot{u_1} = u_2, \\ \dot{u_2} = -u_1 + u_1^3 + u_1 u_3 - \delta u_2, \\ \dot{u_3} = u_4, \\ \dot{u_4} = -\frac{2\mu}{\sigma}(u_2^2 - u_1^2 + u_1^4 + u_1^2 u_3 - \delta u_1 u_2). \end{cases} \tag{1.3}$$

The constant stationary solutions of system (1.3) are given by:
$(E_1)$ $u_1 = 0,\ u_2 = 0,\ u_3 = c, u_4 = 0,\quad (c \in R)$,
$(E_2)$ $u_1 = c,\ u_2 = 0,\ u_3 = 1 - c^2, u_4 = 0,\quad (c \in R, c \neq 0)$.

In order to make a comparison, we first carry out linear stability analysis of two equilibria from the point of view of dynamical system. The Jacobian matrix of system (1.3) is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 + 3u_1^2 + u_3 & -\delta & u_1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2\mu}{\sigma}(-2u_1 + 4u_1^3 + 2u_1 u_3 - \delta u_2) & \frac{-2\mu}{\sigma}(2u_2 - \delta u_1) & \frac{-2\mu}{\sigma}u_1^2 & 0 \end{pmatrix}.$$

For $J(E_1)$, we get

$$\lambda_{1,2} = 0(double), \quad \lambda_{3,4} = \frac{-\delta \pm \sqrt{\delta^2 - 4(1-c)}}{2}.$$

If $\delta = 0$ and $c < 1$, then we obtain a pair of pure imaginary $\lambda_{3,4} = \pm i$; if $c > 1$, $\delta > 0$ or $\delta < 0$, then there exist an unstable eigenvalue($\lambda_3 > 0$) and a stable real eigenvalue($\lambda_4 < 0$); if $c < 1$ and $\delta < 0$, then there exists a pair of conjugate imaginary with $\Re(\lambda_3) > 0$ and $\Re(\lambda_4) > 0$; if $c < 1$ and $\delta > 0$, then there exists a pair of conjugate imaginary with $\Re(\lambda_3) < 0$ and $\Re(\lambda_4) < 0$.

For $J(E_2)$, we obtain

$$\lambda^2(\lambda^2 + \delta\lambda + \frac{2\mu c^2}{\sigma} - 2c^2) = 0,$$

and

$$\lambda_{1,2}^* = 0(double), \quad \lambda_{3,4}^* = \frac{-\delta \pm \sqrt{\delta^2 - 8c^2(\frac{\mu}{\sigma} - 1)}}{2}.$$

When $\delta = 0$ and $\frac{\mu}{\sigma} > 1$, there exist a pair of pure imaginary $\lambda_{3,4}^* = \pm c\sqrt{2(\frac{\mu}{\sigma} - 1)}i$; if $\frac{\mu}{\sigma} < 1$, $\delta > 0$ or $\delta < 0$, then there exist an unstable eigenvalue($\lambda_3^* > 0$) and a stable real eigenvalue($\lambda_4^* < 0$); if $\frac{\mu}{\sigma} > 1$ and $\delta < 0$, then there exists a pair of conjugate imaginary with $\Re(\lambda_3^*) > 0$ and $\Re(\lambda_4^*) > 0$; if $\frac{\mu}{\sigma} > 1$ and $\delta > 0$, then there exists a pair of conjugate imaginary with $\Re(\lambda_3^*) < 0$ and $\Re(\lambda_4^*) < 0$.

As discussed above, we know that the equilibria $E_1$ and $E_2$ are highly degenerate, therefore, the dynamics near the equilibria are very complicated. Hence, in this paper, we will focus on the qualitative behavior of steady state solutions of system (1.2) with Neumann boundary conditions, especially, the stability and bifurcation of solutions about bounded domain.

This paper is outlined as follows. In Section 2, we reduce the steady state of system (1.2) and rescale it to a single ordinary differential equation. In Section 3, we use the weakly nonlinear analysis to discuss the bifurcation. Finally, some brief conclusions are given in Section 4.

## 2. Stability and bifurcation of $(u_0, v_0) = (0, c)(c \in R)$

In this section, we first simplify the steady state system (1.2) and rescale it to a single ordinary differential equation. By using the perturbation theory, we discuss the branches bifurcated from the solution and the stability of the bifurcated solution.

By integrating both sides of the second equation of system (1.2) twice, and using the Neumann boundary conditions, we have

$$\sigma v + \mu u^2 = d,$$

where $d$ is an arbitrary constant. Obviously, $v = \frac{d}{\sigma} = c$ for $u = 0$. Then system (1.2) becomes

$$\begin{cases} u_{xx} + \delta u_x + (1-c)u + (\frac{\mu}{\sigma} - 1)u^3 = 0, & 0 < x < L, \\ u_x(0) = u_x(L) = 0. \end{cases} \tag{2.1}$$

Define $X$=$\{u \in H^2[0, L] | u'(0) = u'(L) = 0\}$. Linearizing around the trivial solution $u = 0$ and taking a new differential operator

$$\Gamma_1 = \frac{\partial^2}{\partial x^2} + \delta \frac{\partial}{\partial x} + (1-c)I,$$

we obtain the following corresponding eigenvalue problem

$$\begin{cases} \varphi_{xx} + \delta\varphi_x + (1-c)\varphi = \lambda\varphi, & 0 < x < L, \\ \varphi_x(0) = \varphi_x(L) = 0. \end{cases} \tag{2.2}$$

The eigenvalues $\lambda$ of (2.2) are given by

$$\lambda = \lambda_m = -(\frac{m\pi}{L})^2 - \frac{\delta^2}{4} + 1 - c, m \in N.$$

Then the corresponding eigenvectors are $\sqrt{2}\cos(\frac{m\pi x}{L} - \frac{\pi}{4})e^{-\frac{\delta}{2}x}, m \in N$, which is a decaying or increasing amplitude function with oscillations, it reflects that the perturbation term $\delta u_x$ change the center into a focus.

When $c \geq 1$, the eigenvalues are all negative except for only one zero eigenvalue. So we get that if $c \geq 1$, the stationary solution $u = 0$ is neutrally stable. When $c < 1 - \frac{\delta^2}{4}$, there is at least one positive eigenvalue $1 - c - \frac{\delta^2}{4}$, i.e., if $c < 1 - \frac{\delta^2}{4}, u = 0$ is linearly unstable.

In the following, we discuss the bifurcation from the stationary solution $u = 0$ when $c < 1 - \frac{\delta^2}{4}$. For simplicity, we introduce the variables $\widetilde{x}$ and $\widetilde{u}$, $L\widetilde{x} = x$, $\widetilde{u}(\widetilde{x}) = u(x)$, and then drop the tildes, system (2.1) becomes

$$\begin{cases} u_{xx} + L\delta u_x + L^2(1-c)u + L^2(\frac{\mu}{\sigma} - 1)u^3 = 0, & 0 < x < 1, \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{2.3}$$

The corresponding eigenvalue problem at $u = 0$ becomes

$$\begin{cases} \varphi_{xx} + L\delta\varphi_x + L^2(1-c)\varphi = \lambda'\varphi, & 0 < x < 1, \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{2.4}$$

It is clear to see that $\lambda' = L^2\lambda$.

We now take $L$ as a bifurcation parameter. Since $L = \frac{2\pi}{\sqrt{4-4c-\delta^2}} (c < 1 - \frac{\delta^2}{4})$ is a bifurcation point, we want to know how many solutions bifurcate from this trivial solution and their asymptotic expression of system (2.3).

Define

$$L = \frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}} + \gamma(\epsilon)\epsilon, \tag{2.5}$$

$$u = \epsilon\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^2 a(x,\epsilon), \tag{2.6}$$

where $\epsilon$ is a small parameter, $a(x,\epsilon) \in \{\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x}\}^{\perp}$.

Substituting (2.5) and (2.6) into (2.3), we get

$$\begin{aligned} &(\epsilon\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^2 a(x,\epsilon))'' \\ &+ \delta(\frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}} + \gamma(\epsilon)\epsilon)(\epsilon\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^2 a(x,\epsilon))' \\ &+ (1-c)(\frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}} + \gamma(\epsilon)\epsilon)^2(\epsilon\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^2 a(x,\epsilon)) \\ &+ (\frac{\mu}{\sigma} - 1)(\frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}} + \gamma(\epsilon)\epsilon)^2(\epsilon\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^2 a(x,\epsilon))^3 = 0, \end{aligned} \tag{2.7}$$

where $'$ and $''$ denote $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$, respectively.

Taking the coefficient of $\epsilon^2$ in (2.7) to be equal to zero, we have

$$\begin{aligned} &a_0''(x) + \frac{\pi\delta}{\sqrt{1-c-(\frac{\delta}{2})^2}}a_0'(x) + \frac{\pi^2(1-c)}{1-c-(\frac{\delta}{2})^2}a_0(x) \\ &\qquad + \frac{2\pi(1-c)}{\sqrt{1-c-(\frac{\delta}{2})^2}}\gamma(0)\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} = 0, \end{aligned} \tag{2.8}$$

where $a_0(x) = a(x,0)$.

Taking the inner product on both sides of (2.8) with $\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x}$, we get $\gamma(0) = 0$. Since $a_0(x) \in \{\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x}\}^{\perp}$ and satisfies the boundary conditions, we obtain $a_0(x) = 0$.

Assuming the coefficient of $\epsilon^3$ in (2.7) vanishes, we get

$$\begin{aligned} &a_1''(x) + \frac{\pi\delta}{\sqrt{1-c-(\frac{\delta}{2})^2}}a_1'(x) + \frac{\pi^2(1-c)}{1-c-(\frac{\delta}{2})^2}a_1(x) \\ &\qquad + \frac{2\pi(1-c-\frac{\delta^2}{2})}{\sqrt{1-c-(\frac{\delta}{2})^2}}\gamma_1(0)\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} \end{aligned}$$

$$+ (\frac{\mu}{\sigma} - 1)\frac{\pi^2}{1-c-(\frac{\delta}{2})^2}\cos^3(\pi x - \frac{\pi}{4})e^{-\frac{3\delta}{2}x}$$
$$- \frac{2\delta\pi^2}{\sqrt{1-c-(\frac{\delta}{2})^2}}\gamma_1(0)\sin(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} = 0, \quad (2.9)$$

where $a_1(x) = \frac{\partial a(x,\epsilon)}{\partial \epsilon}|_{\epsilon=0}$ and $\gamma_1(0) = \gamma'(0)$.

Taking the inner product on both sides of (2.9) with $\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x}$, we have

$$\gamma_1(0) = \frac{\pi}{4\sqrt{1-c-(\frac{\delta}{2})^2}}(\frac{\mu}{\sigma} - 1)\frac{F}{G}, \quad (2.10)$$

where

$$F = 2(384\pi^6 + 256\pi^5\delta + 208\pi^4\delta^2 + 80\pi^3\delta^3 + 32\pi^2\delta^4 + 4\pi\delta^5 + \delta^6), \quad (2.11)$$
$$G = (4(c-1)\pi\delta + 4\pi^2(2c-2+\delta^2) + \delta^2(2c-2+\delta^2))(\delta^4 + 64\pi^2 + 20\delta^2). \quad (2.12)$$

For simplicity, we let $D = \frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}}$. Then (2.9) becomes

$$a_1''(x) + \delta D a_1'(x) + D^2(1-c)a_1(x) + 2D(1-c-(\frac{\delta^2}{2}))\gamma_1(0)\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x}$$
$$+ (\frac{\mu}{\sigma} - 1)D^2\cos^3(\pi x - \frac{\pi}{4})e^{-\frac{3\delta}{2}x} - 2D\delta\pi\gamma_1(0)\sin(\pi x - \frac{\pi}{4})e^{-\frac{\delta}{2}x} = 0,$$

and

$$\gamma_1(0) = \frac{D}{4}(\frac{\mu}{\sigma} - 1)\frac{F}{G}.$$

Substituting (2.10) to (2.9), we have

$$a_1(x) = \sqrt{\rho^2 + \omega^2}\cos(\pi x - \alpha) + \sqrt{p^2 + q^2}\cos(3\pi x - \beta),$$

where $\cos(\alpha) = \frac{\rho}{\sqrt{\rho^2+\omega^2}}$, $\sin(\alpha) = \frac{\omega}{\sqrt{\rho^2+\omega^2}}$ and $\cos(\beta) = \frac{p}{\sqrt{p^2+q^2}}$, $\sin(\beta) = \frac{q}{\sqrt{p^2+q^2}}$,

$$\begin{aligned}\rho =& \frac{De^{-\frac{1}{2}x(\delta - D\delta + D\sqrt{-4+4c+\delta^2})}F(-2+2c+\delta^2)(\mu-\sigma)}{2A}(2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}\\ &\times \sqrt{-4+4c+\delta^2}(4(c-1)D^2 + 4\pi^2 - 4\pi\delta - \delta^2 + 2D\delta(2\pi+\delta)))\\ &+ \frac{De^{-\frac{1}{2}x(\delta - D\delta + D\sqrt{-4+4c+\delta^2})}F\pi\delta(\mu-\sigma)}{A}(2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}\sqrt{-4+4c+\delta^2}\\ &\times (4(c-1)D^2 + 4\pi^2 + 4\pi\delta - \delta^2 + 2D\delta(-2\pi+\delta)))\\ &+ \frac{6\sqrt{2}D^2(\mu-\sigma)}{B}((c-1)^3D^6 + 81\pi^6 + (c-1)^2D^5\pi\delta + 81D\pi^5\delta\\ &+ 9D^3\pi^3\delta(-2+2c+\delta^2) + 9D^2\pi^4(-11+11c+\delta^2)\\ &+ D^4\pi^2(19 + 19c^2 - 9\delta^2 + c(9\delta^2 - 38))),\end{aligned} \quad (2.13)$$

$$\begin{aligned}\omega =&\frac{De^{-\frac{1}{2}x(\delta-D\delta+D\sqrt{-4+4c+\delta^2})}F(-2+2c+\delta^2)(\mu-\sigma)}{2A}\\&\times(2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}\sqrt{-4+4c+\delta^2}(4(c-1)D^2+4\pi^2+4\pi\delta-\delta^2+2D\delta(-2\pi+\delta)))\\&+\frac{De^{-\frac{1}{2}x(\delta-D\delta+D\sqrt{-4+4c+\delta^2})}F\pi\delta(\mu-\sigma)}{A}(-2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}\\&\times\sqrt{-4+4c+\delta^2}(4(c-1)D^2+4\pi^2-4\pi\delta-\delta^2+2D\delta(2\pi+\delta)))\\&+\frac{6\sqrt{2}D^2(\mu-\sigma)}{B}((c-1)^3D^6+81\pi^6-(c-1)^2D^5\pi\delta-81D\pi^5\delta\\&-9D^3\pi^3\delta(-2+2c+\delta^2)+9D^2\pi^4(-11+11c+\delta^2)\\&+D^4\pi^2(19+19c^2-9\delta^2+c(9\delta^2-38))),\end{aligned}\tag{2.14}$$

$$\begin{aligned}p =&\frac{2\sqrt{2}D^2(\mu-\sigma)}{B}(-(c-1)^3D^6-9\pi^6+3(c-1)^2D^5\pi\delta+3D\pi^5\delta+D^2\pi^4(19-19c-9\delta^2)\\&+3D^3\pi^3\delta(-2+2c+\delta^2)+D^4\pi^2(-11-11c^2+\delta^2-c(\delta^2-22))),\end{aligned}\tag{2.15}$$

$$\begin{aligned}q =&\frac{2\sqrt{2}D^2(\mu-\sigma)}{B}((c-1)^3D^6+9\pi^6+3(c-1)^2D^5\pi\delta+3D\pi^5\delta-D^2\pi^4(19-19c-9\delta^2)\\&+3D^3\pi^3\delta(-2+2c+\delta^2)-D^4\pi^2(-11-11c^2+\delta^2-c(\delta^2-22))),\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}A =&\sqrt{2}G\sqrt{-4+4c+\delta^2}(16(c-1)^2D^4+16(c-1)D^3\delta^2+(4\pi^2+\delta^2)^2-4D(4\pi^2\delta^2+\delta^4)\\&+4D^2(\delta^2(2-2c+\delta^2)+4\pi^2(-2+2c+\delta^2)))\sigma,\\B =&(-2D^2+2cD^2+2\pi^2+D^2\delta^2-D^2\delta\sqrt{-4+4c+\delta^2})(-2D^2+2cD^2+18\pi^2\\&+D^2\delta^2-D^2\delta\sqrt{-4+4c+\delta^2})(-2D^2+2cD^2+2\pi^2+D^2\delta^2+D^2\delta\sqrt{-4+4c+\delta^2})\\&\times(-2D^2+2cD^2+18\pi^2+D^2\delta^2+D^2\delta\sqrt{-4+4c+\delta^2})\sigma.\end{aligned}$$

According to the above discussions, we have shown that for $L$ near $\frac{2\pi}{\sqrt{4-4c-\delta^2}}$, there are nontrivial steady state solution branches of system (2.3) bifurcated from the trivial solution $u=0$:

$$\begin{cases}u(\epsilon)=\epsilon\cos(\pi x-\frac{\pi}{4})e^{-\frac{\delta}{2}x}+\epsilon^3(\sqrt{\rho^2+\omega^2}\cos(\pi x-\alpha)\\\qquad\qquad+\sqrt{p^2+q^2}\cos(3\pi x-\beta))+o(\epsilon^3),\\L=\frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}}+\frac{D}{4}(\frac{\mu}{\sigma}-1)\frac{F}{G}\epsilon^2+o(\epsilon^2).\end{cases}\tag{2.17}$$

The bifurcations of system (2.3) bifurcated from the trivial solution $u=0$ are illustrated (see Fig. 1 and Fig. 2).

We can also discuss the bifurcation and stability of equilibria of system (2.3) by letting

$$\begin{cases}\dot{u_1}=u_2,\\\dot{u_2}=-L^2(1-c)u_1-L^2(\frac{\mu}{\sigma}-1)u_1^3-L\delta u_2.\end{cases}$$

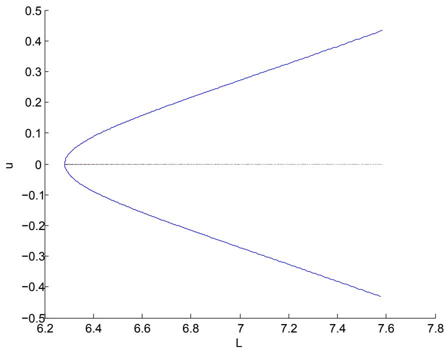


**Figure 1.** The supercritical case $\mu < \sigma$ for $x = 1, c = 0.5, \mu = 0.05, \sigma = 0.1, \delta = 1.$

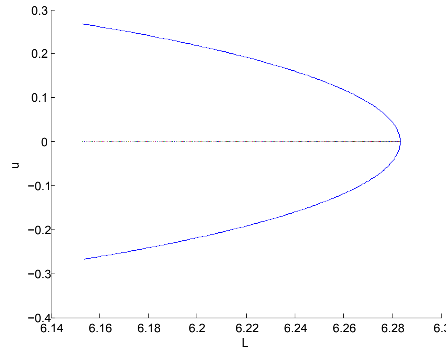


**Figure 2.** The subcritical case $\mu > \sigma$ for $x = 1, c = 0.5, \mu = 0.1, \sigma = 0.05, \delta = 1.$

By setting $u_2 = 0, (1-c)u_1 + (\frac{\mu}{\sigma} - 1)u_1^3 = 0$, we obtain that there is only one equilibrium $(u_1, u_2) = (0,0)$ for $\frac{c-1}{\mu-\sigma} < 0$; three equilibria $(u_1, u_2) = (0,0), (u_1, u_2) = (\pm\sqrt{\frac{(c-1)\sigma}{\mu-\sigma}}, 0)$ for $\frac{c-1}{\mu-\sigma} > 0$. If $\delta^2 - 4(1-c) > 0$, then $(u_1, u_2) = (0,0)$ is a saddle; if $\delta^2 - 4(1-c) < 0$, then $(u_1, u_2) = (0,0)$ is a focus. The parameter $\delta$ controls the stability of the equilibrium $(u_1, u_2) = (0,0)$.

When $c < 1$, there are three equilibria for $\mu < \sigma$; when $c > 1$, there is a unique equilibrium for $\mu > \sigma$. Furthermore, when $c < 1$ and $\mu < \sigma$, the system undergoes a supercritical pitch-fork bifurcation; when $c < 1$ and $\mu > \sigma$, it undergoes a subcritical pitch-fork bifurcation. From the boundary conditions, we conclude that the dynamics of an explicit solution connecting those equilibria. As mentioned above, if $c < 1$, then $(u_1, u_2) = (0,0)$ is a center for $\delta = 0$, and $(u_1, u_2) = (0,0)$ is a focus for $\delta \neq 0$. Clearly, the eigenvector $\sqrt{2}\cos(\frac{m\pi x}{L} - \frac{\pi}{4})e^{-\frac{\delta}{2}x}, (m \in N)$ is an explicit solution connecting the equilibrium $(u_1, u_2) = (0,0)$ and the other saddle.

**Theorem 2.1.** *For $L$ near $\frac{2\pi}{\sqrt{4-4c-\delta^2}}$, there are nontrivial steady state solution branches of system (1.2) bifurcated from the stationary solution $(0, c)$:*

$$\begin{cases} u(\epsilon) = \epsilon\cos(\dfrac{\pi x}{L} - \dfrac{\pi}{4})e^{-\frac{\delta}{2}x} + \epsilon^3(\sqrt{\rho^2+\omega^2}\cos(\dfrac{\pi x}{L} - \alpha) \\ \qquad\qquad + \sqrt{p^2+q^2}\cos(\dfrac{3\pi x}{L} - \beta)) + o(\epsilon^3), \\ v(\epsilon) = c - \dfrac{\mu}{\sigma}\cos^2(\dfrac{\pi x}{L} - \dfrac{\pi}{4})e^{-\delta x}\epsilon^2 + o(\epsilon^2), \\ L = \dfrac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}} + \dfrac{\pi}{4\sqrt{1-c-(\frac{\delta}{2})^2}}(\dfrac{\mu}{\sigma} - 1)\dfrac{F}{G}\epsilon^2 + o(\epsilon^2), \end{cases} \tag{2.18}$$

*where $F, G, \rho, \omega, p$ and $q$ are defined in (2.11), (2.12), (2.13), (2.14) and (2.15), respectively.*

Next, we study the stability of the nontrivial solution given above. Consider the eigenvalue problem of system (2.3) on $u(\epsilon)$

$$F_u(u(\epsilon), L(\epsilon))\phi(\epsilon) = \lambda(\epsilon)\phi(\epsilon). \tag{2.19}$$

Since $\lambda(0) = 0, \phi(0) = \sqrt{2}\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta x}{2}}$, we assume

$$\phi(\epsilon) = \sqrt{2}\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta x}{2}} + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \tag{2.20}$$

$$\lambda(\epsilon) = \lambda_1\epsilon + \lambda_2\epsilon^2 + \cdots. \tag{2.21}$$

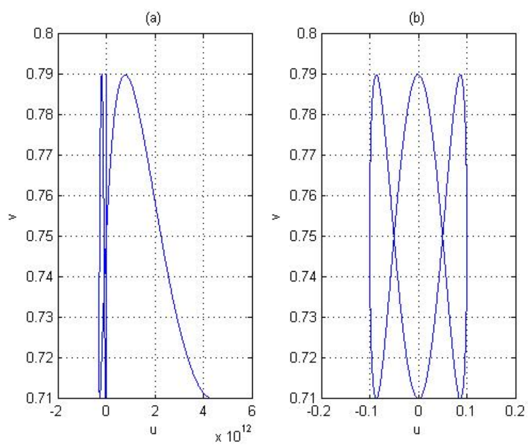


**Figure 3.** Solution profiles of (2.17) are illustrated in phase space for $\epsilon = 0.1, L = 2\pi, \mu = 2, \sigma = 1, c = 0.75$. Non-symmetric solution for $\delta \neq 0$ (see Figure (a)), and symmetric solution for $\delta = 0$ (see Figure (b)).

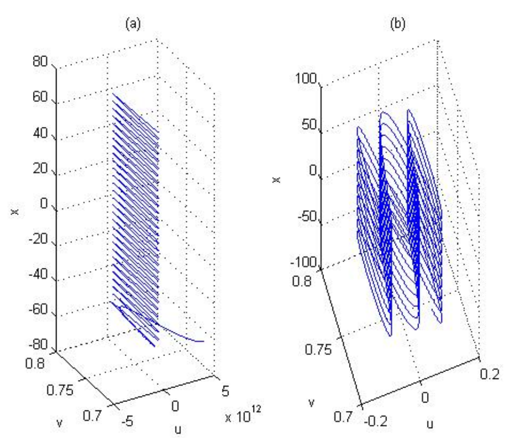


**Figure 4.** Three-dimensional solution profiles of (2.17) are illustrated in phase space for $\epsilon = 0.1, L = 2\pi, \mu = 2, \sigma = 1, c = 0.75$. Three-dimensional non-symmetric solution for $\delta \neq 0$ (see Figure (a)), and three-dimensional symmetric solution for $\delta = 0$ (see Figure (b)).

By substituting (2.19) and (2.20) to (2.18), we obtain

$$\frac{\partial^2}{\partial x^2}\phi(\epsilon) + L\delta\frac{\partial}{\partial x}\phi(\epsilon) + L^2(1-c)\phi(\epsilon) + 3L^2(\frac{\mu}{\sigma} - 1)u^2(\epsilon)\phi(\epsilon) - \lambda(\epsilon)\phi(\epsilon) = 0. \tag{2.22}$$

On the branches given in (2.16), equating the coefficient of $\epsilon$ in (2.21) to 0, we have

$$\phi_1'' + \frac{2\pi\delta}{\sqrt{4-4c-\delta^2}}\phi_1' + \frac{4\pi^2(1-c)}{4-4c-\delta^2}\phi_1 - \sqrt{2}\lambda_1\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta x}{2}} = 0. \tag{2.23}$$

According to (2.22) and the boundary conditions, we get $\lambda_1 = 0, \phi_1 = 0$.
In view of the second-order terms of $\epsilon$ in (2.21), we get

$$\begin{aligned}
&\phi_2'' + \frac{2\pi\delta}{\sqrt{4-4c-\delta^2}}\phi_2' + \frac{4\pi^2(1-c)}{4-4c-\delta^2}\phi_2 + [\frac{3\sqrt{2}}{8}\pi\delta(1-\frac{\mu}{\sigma})\frac{1}{\sqrt{1-c-(\frac{\delta}{2})^2}(1-c-\delta\pi)} \\
&\quad + \frac{3\sqrt{2}}{4}\pi^2(1-\frac{\mu}{\sigma})(1-c)\frac{1}{(1-c-(\frac{\delta}{2})^2)(1-c-\delta\pi)} + \frac{9\sqrt{2}}{4}\pi^2(\frac{\mu}{\sigma}-1) - \sqrt{2}\lambda_2] \\
&\quad e^{-\frac{\delta x}{2}}\cos(\pi x - \frac{\pi}{4}) + \frac{3\sqrt{2}}{4}\pi^2(\frac{\mu}{\sigma}-1)\frac{1}{1-c-(\frac{\delta}{2})^2}e^{-\frac{\delta x}{2}}\cos(3\pi x - \frac{3\pi}{4}) = 0.
\end{aligned} \tag{2.24}$$

Taking some calculations, we have

$$\begin{aligned}
\lambda_2 = &\frac{3}{8}\pi\delta(1-\frac{\mu}{\sigma})\frac{1}{\sqrt{1-c-(\frac{\delta}{2})^2}(1-c-\delta\pi)} \\
&+ \frac{3}{4}\pi^2(1-\frac{\mu}{\sigma})(1-c)\frac{1}{(1-c-(\frac{\delta}{2})^2)(1-c-\delta\pi)} + \frac{9}{4}\pi^2(\frac{\mu}{\sigma}-1), \\
\phi_2 = &c_1 e^{\frac{1}{2}x(-D\delta - D\sqrt{-4+4c+\delta^2})} + c_2 e^{\frac{1}{2}x(-D\delta + D\sqrt{-4+4c+\delta^2})} + \sqrt{\zeta^2+\xi^2}\cos(3\pi x - \gamma),
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
&\cos(\gamma) = \frac{\zeta}{\sqrt{\zeta^2+\xi^2}}, \quad \sin(\gamma) = \frac{\xi}{\sqrt{\zeta^2+\xi^2}}, \\
&E = 2\sqrt{-4+4c+\delta^2}(16(c-1)^2D^4 + 16(c-1)D^3\delta^2 + (36\pi^2+\delta^2)^2 \\
&\qquad - 4D(36\pi^2\delta^2+\delta^4) + 4D^2(\delta^2(2-2c+\delta^2) + 36\pi^2(-2+2c+\delta^2)))\sigma, \\
&D = \frac{\pi}{\sqrt{1-c-(\frac{\delta}{2})^2}}, \\
&\zeta = \frac{2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}}{E}\sqrt{-4+4c+\delta^2} \\
&\qquad \times (4cD^2 - 4D^2 + 36\pi^2 + 12\pi\delta - \delta^2 + 2D\delta^2 - 12D\pi\delta), \\
&\xi = \frac{-2De^{\frac{1}{2}Dx(-\delta+\sqrt{-4+4c+\delta^2})}}{E}\sqrt{-4+4c+\delta^2} \\
&\qquad \times (4cD^2 - 4D^2 + 36\pi^2 - 12\pi\delta - \delta^2 + 2D\delta^2 + 12D\pi\delta).
\end{aligned} \tag{2.26}$$

Therefore, on the branches, the eigenvalue of $F_u(u(\epsilon), L(\epsilon))$ is

$$\begin{aligned}
\lambda(\epsilon) = &(\frac{3}{8}\pi\delta(1-\frac{\mu}{\sigma})\frac{1}{\sqrt{1-c-(\frac{\delta}{2})^2}(1-c-\delta\pi)} \\
&+ \frac{3}{4}\pi^2(1-\frac{\mu}{\sigma})(1-c)\frac{1}{(1-c-(\frac{\delta}{2})^2)(1-c-\delta\pi)} + \frac{9}{4}\pi^2(\frac{\mu}{\sigma}-1))\epsilon^2 + o(\epsilon^2).
\end{aligned}$$

The corresponding eigenfunction is

$$\begin{aligned}
\phi = \sqrt{2}\cos(\pi x - \frac{\pi}{4})e^{-\frac{\delta x}{2}} &+ (c_1 e^{\frac{1}{2}x(-D\delta - D\sqrt{-4+4c+\delta^2})} + c_2 e^{\frac{1}{2}x(-D\delta + D\sqrt{-4+4c+\delta^2})} \\
&+ \sqrt{\zeta^2+\xi^2}\cos(3\pi x - \gamma))\epsilon^2 + o(\epsilon^2),
\end{aligned}$$

where $\zeta$ and $\xi$ are given by (2.25), respectively.

When $1-\delta\pi < c < 1-\frac{\delta^2}{4}$, we obtain that the eigenvalue of the linearized operator $F_u(u(\epsilon), v(\epsilon))$ at the nontrivial solutions given in (2.16) is negative if $\mu < \sigma$, and positive if $\mu > \sigma$. Finally, for small $\epsilon$, if $\mu < \sigma$, then the corresponding solution branches are stable; if $\mu > \sigma$, then the corresponding solution branches are unstable. Thus, we get the theorem as follows:

**Theorem 2.2.** *Assume $1-\delta\pi < c < 1-\frac{\delta^2}{4}$, for small $\epsilon$, then the solution branches bifurcated from $(0, c)$ are stable if $\mu < \sigma$; the solution branches bifurcated from $(0, c)$ are unstable if $\mu > \sigma$.*

## 3. Stability and bifurcation of $(u_0, v_0) = (c, 1-c^2)(c \in R, c \neq 0)$

In this section, by using the method of weakly nonlinear analysis, rather than the method used in Section 2, we study the bifurcation. If we use the same method we will find that the eigenvalue here is too complicated relating to the parameters $\mu, \sigma, L$ and so we can not simply set $L$ for bifurcation parameter anymore.

We first discuss the stability of solution $(c, 1-c^2)$. Linearizing system (1.2) at the equilibrium $(c, 1-c^2)$, we obtain the corresponding eigenvalue problem

$$\begin{cases} \varphi_{xx} + \delta\varphi_x - 2c^2\varphi - c\psi = \lambda\varphi, 0 < x < L, \\ 2\mu c\varphi_{xx} + \sigma\psi_{xx} = \lambda\psi, 0 < x < L, \\ \varphi_x(0) = \varphi_x(L) = 0, \\ \psi_x(0) = \psi_x(L) = 0. \end{cases} \tag{3.1}$$

Set

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} a_m \\ b_m \end{pmatrix} \cos(\frac{m\pi x}{L} - \frac{\pi}{4}). \tag{3.2}$$

Substituting (3.2) to (3.1), we get the following characteristic equation

$$\begin{aligned} \lambda^2 + [(\frac{m\pi}{L})^2 + c^2 + (\frac{m\pi}{L})^2\sigma + \delta\frac{m\pi}{L}\tan(\frac{m\pi x}{L} - \frac{\pi}{4})]\lambda + [(\frac{m\pi}{L})^4 + c^2\sigma(\frac{m\pi}{L})^2 \\ - 2\mu c^2(\frac{m\pi}{L})^2 + \delta\sigma(\frac{m\pi}{L})^3\tan(\frac{m\pi x}{L} - \frac{\pi}{4})] = 0. \end{aligned} \tag{3.3}$$

Since

$$\tan(\frac{m\pi x}{L} - \frac{\pi}{4}) = \frac{\tan(\frac{m\pi x}{L}) - 1}{1 + \tan(\frac{m\pi x}{L})} = 1 - \frac{2}{1 + \tan(\frac{m\pi x}{L})}, \tag{3.4}$$

and $\frac{m}{L}$ is the period of $\tan(\frac{m\pi x}{L} - \frac{\pi}{4})$, we have $\lim_{x \to \frac{L}{2m} + \frac{kL}{m}} \tan(\frac{m\pi x}{L}) = \infty, k = 1, 2, 3\cdots$. This yields that $\tan(\frac{m\pi x}{L} - \frac{\pi}{4}) \to 1$ as $x \to \frac{L}{2m} + \frac{kL}{m}$. From the discussion above and equation (3.3), it is asymptotic instability if

$$(\frac{m\pi}{L})^4 + c^2\sigma(\frac{m\pi}{L})^2 - 2\mu c^2(\frac{m\pi}{L})^2 + \delta\sigma(\frac{m\pi}{L})^3 < 0.$$

It is easy to see that the stationary solution is side unstable if $\mu > (\frac{m\pi}{L})^2\frac{\sigma}{2c^2} + \frac{\sigma}{2} + \frac{m\pi}{L}\frac{\delta\sigma}{2c^2}$, $m \in N$. Thus, under the condition $x \to \frac{L}{2m} + \frac{kL}{m}$, we have the following theorem.

**Theorem 3.1.** *If* $\mu > (\frac{m\pi}{L})^2\frac{\sigma}{2c^2}+\frac{\sigma}{2}+\frac{m\pi}{L}\frac{\delta\sigma}{2c^2}$*, then the stationary solution* $(c, 1-c^2)$ *is asymptotically unstable; if* $\mu < (\frac{m\pi}{L})^2\frac{\sigma}{2c^2}+\frac{\sigma}{2}+\frac{m\pi}{L}\frac{\delta\sigma}{2c^2}$*, then the stationary solution* $(c, 1-c^2)$ *is asymptotically stable.*

To determine what the bifurcation is, we use the weakly nonlinear analysis to discuss the nonlinear development of the instability.

Denote $u = c + a(x,t), v = 1 - c^2 + b(x,t)$, the nonlinear equations for $a$ and $b$ become

$$\begin{cases} a_t = a_{xx} + \delta a_x - 2c^2 a - 3ca^2 - a^3 - ab - cb, \\ b_t = \sigma b_{xx} + \mu(2ac + a^2)_{xx}. \end{cases} \tag{3.5}$$

By introducing a small parameter $\epsilon$ to rescale the parameters

$$\begin{aligned} &\mu = \mu_0 + \epsilon^2\mu_2, \\ &T = \epsilon^2 t, \\ &a = \epsilon a_1 + \epsilon^2 a_2 + O(\epsilon^3), \\ &b = \epsilon b_1 + \epsilon^2 b_2 + O(\epsilon^3), \end{aligned}$$

where $a_i$ and $b_i$ are functions of $x$ and $T$. By considering a single mode,

$$\begin{aligned} &a_1 = a_{11}(T)\cos(\frac{m\pi x}{L} - \frac{\pi}{4}), \\ &b_1 = b_{11}(T)\cos(\frac{m\pi x}{L} - \frac{\pi}{4}), \qquad m \in N, \end{aligned}$$

from the terms of order $\epsilon$ in (3.5), we have

$$cb_{11}(T) + (2c^2 + (\frac{m\pi}{L})^2 + \frac{\delta m\pi}{L}\tan(\frac{m\pi x}{L} - \frac{\pi}{4}))a_{11}(T) = 0,$$

$$\mu_0 = (1 + \frac{1}{2c^2}(\frac{m\pi}{L})^2)\sigma.$$

It follows from $o(\epsilon^2)$ that

$$\begin{aligned} &a_2 = a_{20}(T) + a_{22}(T)\cos(\frac{2m\pi x}{L} - \frac{\pi}{2}), \\ &b_2 = b_{22}(T)\cos(\frac{2m\pi x}{L} - \frac{\pi}{2}), \end{aligned}$$

where

$$\begin{aligned} &a_{20}(T) = [\frac{1}{4c^3}(\frac{m\pi}{L})^2 - \frac{1}{4c}]a_{11}^2(T), \\ &a_{22}(T) = \frac{3m\pi + 2\delta L\tan(\frac{m\pi x}{L} - \frac{\pi}{4})}{4c(3m\pi + 2\delta L\tan(\frac{2m\pi x}{L} - \frac{\pi}{2}))}a_{11}^2(T). \\ &b_{22}(T) = (-\frac{1}{2}a_{11}^2 - 2ca_{22})(1 + \frac{1}{2c^2}(\frac{m\pi}{L})^2). \end{aligned}$$

By using the condition $x \to \frac{L}{2m} + \frac{kL}{m}$, we get $cb_{11}(T) + (2c^2 + (\frac{m\pi}{L})^2 + \frac{\delta m\pi}{L})a_{11}(T) = 0$, $a_{22}(T) = 0$, $b_{22}(T) = -\frac{1}{2}(1 + \frac{1}{2c^2}(\frac{m\pi}{L})^2)a_{11}^2(T)$.

From the above solvability condition at $o(\epsilon^3)$, we obtain the equation on the slow timescale $T$:

$$[\frac{\delta m\pi}{L} + 2c^2 + (\frac{m\pi}{L})^2]\frac{da_{11}(T)}{dT} = 2c^2\mu_2(\frac{m\pi}{L})^2 a_{11}(T) + \frac{\sigma(\frac{m\pi}{L})^4((\frac{m\pi}{L})^2 - 1)}{2c^2}a_{11}^3(T). \tag{3.6}$$

Define

$$\kappa = \frac{2c^2\mu_2(\frac{m\pi}{L})^2}{\frac{\delta m\pi}{L} + 2c^2 + (\frac{m\pi}{L})^2}, \tag{3.7}$$

$$\nu = \frac{\sigma(\frac{m\pi}{L})^2((\frac{m\pi}{L})^4 + c^2(\frac{m\pi}{L})^2 - 2c^4)}{4c^4[\frac{\delta m\pi}{L} + 2c^2 + (\frac{m\pi}{L})^2]}. \tag{3.8}$$

Then (3.6) becomes the following equation

$$\frac{da_{11}(T)}{dT} = \kappa a_{11}(T) + \nu a_{11}^3(T), \tag{3.9}$$

which can be regarded as the normal form for the pitchfork bifurcation. If $\nu < 0$, then the bifurcation is a supercritical case; if $\nu > 0$, then the bifurcation is a subcritical case. Hence, by using the condition $x \to \frac{L}{2m} + \frac{kL}{m}$, we get the following result.

**Theorem 3.2.** *If $m < \frac{cL}{\pi}$, $\delta > -\frac{2Lc^2+m^2\pi^2L}{m\pi}$, or $m > \frac{cL}{\pi}$, $\delta < -\frac{2Lc^2+m^2\pi^2L}{m\pi}$, then the bifurcation from the stationary solution $(c, 1-c^2)$ is a supercritical pitchfork bifurcation; if $m > \frac{cL}{\pi}$, $\delta > -\frac{2Lc^2+m^2\pi^2L}{m\pi}$, or $m < \frac{cL}{\pi}$, $\delta < -\frac{2Lc^2+m^2\pi^2L}{m\pi}$, then the bifurcation from the stationary solution $(c, 1-c^2)$ is a subcritical pitchfork bifurcation.*

**Remark 3.1.** The perturbation term destroys the reversibility of amplitude system, and changes the type of equilibrium (i.e., the center is changed into a focus). Then steady state solutions near the equilibria are decaying or increasing with oscillations.

# 4. Conclusions

In this paper, the explicit steady state solutions and their bifurcations of a perturbed amplitude equation are investigated under Neumann boundary conditions on a bounded domain (0, L). The perturbed term plays an important role in changing the stability and bifurcation of amplitude equation by destroying the reversibility of amplitude equation, for example, it makes the corresponding zero equilibrium (0,0) changing from a center into a focus and the other two equilibria keep unchanged. The dynamics are very interesting and complicate by adding the term. It is interesting and challenging to generalize one space dimensions to several space dimensions.

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