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SUB-MANIFOLD AND TRAVELING WAVE SOLUTIONS OF ITO'S 5TH-ORDER MKDV EQUATION*

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Abstract In this paper, we study Ito's 5th-order mKdV equation with the aid of symbolic computation system and by qualitative analysis of planar dynamical systems. We show that the corresponding higher-order ordinary differential equation of Ito's 5th-order mKdV equation, for some particular values of the parameter, possesses some sub-manifolds defined by planar dynamical systems. Some solitary wave solutions, kink and periodic wave solutions of the Ito's 5th-order mKdV equation for these particular values of the parameter are obtained by studying the bifurcation and solutions of the corresponding planar dynamical systems.

Keywords Ito's 5th-order mKdV equation, traveling wave solutions, subequations, planar dynamical systems.

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1. Introduction

In recent decades, with the availability of computer algebra packages (for example, Maple, Matlab), various methods have been proposed to seek exact solutions of nonlinear partial differential equations (NLPDEs), especially for those higher-order NLPDEs arising from fluid mechanics, elasticity, mathematical biology, or other real applications. Also, some valuable methods have been developed to construct exact traveling wave solutions for nonlinear wave equations, for example, the inverse scattering method, Bäcklund transformation method, Darboux transformation method, Hirota bilinear method, tanh-function method, invariant subspace method etc. Some special functions such as Jacobi elliptic functions, hyperbolic functions and so on or integrable ordinary differential equations (ODEs) like linear ODEs, Riccati equation, etc., have been used to study the solutions of nonlinear wave equations [7,9,10,12–14,16,19,20]. The tanh-function [7,13] and exp-function methods [9] and some generalized forms of these methods have been developed and applied to search for solitary wave solutions. The Jacobi elliptic function expansion

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method [14, 16] was proposed to find periodic wave solutions of NLPDEs. Some qualitative analysis methods or numerical simulation methods have been applied to study the solutions of some NLPDEs [1,2,4–6,11,15,17,18,21,22]. The solitary wave solutions, periodic wave solutions, wave front solutions and even certain singular traveling wave solutions, such as compacton, peakon or cuspon, are always of physical significance. Therefore, it is vital to find exact expressions or even prove the existence of such solutions for a better understanding of some physical phenomena in wave transmission.

In the present paper, with the aid of symbolic computation and qualitative analysis of planar dynamical system we study the subequations and exact traveling wave solutions of Ito's 5th-order mKdV equation which was proposed in [10] and is given by

$$u_t + (6u^5 + 10\alpha(u^2u_{xx} + uu_x^2) + u_{4x})_x = 0, \tag{1.1}$$

where α is a real constant.

Some solitary wave solutions of (1.1) were obtained in [13] and some periodic wave solutions were presented in [14] by using the Jacobi elliptic-function method. The modified Jacobi elliptic function expand method was applied to re-investigate Ito's 5th-order mKdV equation (1.1) with $\alpha = -1$ in [16]. To investigate the traveling wave solutions of (1.1), we introduce a new variable $\xi = x - ct$ and afterward integrate the derived ODE once with respect to ξ, and then we have

$$u^{(4)} + 10\alpha(u^2u'' + uu'^2) - cu + 6u^5 = g, \tag{1.2}$$

where g is the constant of integration and $'$ represents the derivative with respective to ξ. Clearly, $u(x,t) = u(x-ct)$ is a traveling wave solution with wave speed c if and only if $u(\xi)$ satisfies the ODE (1.2) for an arbitrary constant g. Therefore, one has to study the exact solutions of the ODE (1.2) to obtain the traveling wave solutions of (1.1). Note that (1.2) is a 4th-order ODE which corresponds to a dynamical system in four-dimensional space. However, we know that it is very difficult to study the phase portraits of four-dimensional dynamical system. Therefore, it might be an effective way to seek exact solutions by studying the invariant sets of this system in a lower-dimensional space, which has been successfully applied to study the higher-order ODEs [21, 22].

The outline of the paper is as follows. In Section 2, we show that equation (1.2) admits an invariant set determined by a first-order ODE of the form $u'^2 = P_4(u)$ which is named as subequation of (1.2), where $P_4(u)$ is a quartic polynomial in u. In Section 3, we derive the traveling wave solutions of the Ito's 5th-order mKdV equation (1.1) by studying the bifurcation and exact solutions of its subequation obtained in Section 2. Some discussions and conclusions are presented in Section 4.

2. Subequations of euqation (1.2)

Suppose that $P_m(u)$ is a polynomial of degree m in u. If $u(\xi)$ is a solution of the solvable first-order ODE

$$u'^2 = P_m(u), \tag{2.1}$$

then it satisfies

$$u'' = \frac{1}{2}P'_m(u) \tag{2.2}$$

and

$$u^{(4)} = \frac{1}{2}P_m'''(u)P_m(u) + \frac{1}{4}P_m'(u)P_m''(u). \tag{2.3}$$

Note that ODE (1.2) is of the form

$$F(u, u'^2, u'', u^{(4)}) = 0, \tag{2.4}$$

where F is polynomial function. Substituting (2.1)-(2.3) into (2.4) gives the equation

$$F(u, P_m(u), \frac{1}{2}P_m'(u), \frac{1}{2}P_m'''(u)P_m(u) + \frac{1}{4}P_m'(u)P_m''(u)) = 0, \tag{2.5}$$

from which one concludes that (2.5) is an identical equation in u if $u(\xi)$ satisfies (2.4) provided that it is a solution of (2.1). Note that the left-hand side of (2.5) is a polynomial in u. Collecting the coefficients of the same powers of u and equating them to 0, gives a system of algebraic equations in the coefficients of the undetermined polynomial $P_m(u)$. By solving these algebraic equations, one can determine the polynomial $P_m(u)$ and then the solvable first-order ODE (2.1), from which certain solutions of the higher-order equation (2.4) can be obtained. As in [21,22], we call (2.1) a subequation of (2.4) which determines a sub-manifold of equation (2.4).

Clearly, $m = 4$ for (1.2), which can be seen by balancing the highest degree of u in (2.5), that is to say, we choose

$$\left(\frac{du}{d\xi}\right)^2 = a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0 \tag{2.6}$$

as the potential subequation with $a_4 \neq 0$. Then we know that $u(\xi)$ satisfies (1.2) provided that it is a solution of (2.6) if $a_0, \cdots, a_4$ satisfy the following algebraic equations:

$$\begin{aligned}
&4a_4^2 + 5\alpha a_4 + 1 = 0,\\
&(5\alpha + 6a_4)a_3 = 0,\\
&3a_3^2 + 8a_2a_4 + 8\alpha a_2 = 0,\\
&a_2a_3 + 2a_1a_4 + 2\alpha a_1 = 0,\\
&24a_0a_4 + 9a_1a_3 - 2c + 20\alpha a_0 + 2a_2^2 = 0,\\
&a_1a_2 + 6a_0a_3 - 2g = 0.
\end{aligned} \tag{2.7}$$

Thus, solving system (2.7) with Maple, one can find the possible subequations of (1.2) in the form (2.6). We now present the result in the following theorem.

Theorem 2.1. *Equation (1.2) has a sub-manifold determined by a first-order ODE (2.6) if $a_i, i = 0, ..., 4$, c and g satisfy one of the following assumptions:*

(A_1) for $\alpha = -1$, $a_4 = 1$, $a_3 = 0$, $c = 2a_0 + a_2^2$ $g = \frac{1}{2}a_1a_2$, a_0, a_1 and a_2 arbitrary;

(A_2) for $\alpha = 1$, $a_4 = -1$, $a_3 = 0$, $c = a_2^2 - 2a_0$ $g = -\frac{1}{2}a_1a_2$, a_0, a_1 and a_2 arbitrary;

(A_3) for $\alpha = \frac{3\sqrt{2}}{5}$, $a_4 = -\frac{\sqrt{2}}{2}$, $a_2 = -\frac{15\sqrt{2}}{8}a_3^2$, $a_1 = \frac{75}{8}a_3^3$, $c = \frac{1575}{32}a_3^4$, $g = 3a_0a_3 - \frac{1125\sqrt{2}}{128}a_3^5$, a_0 and a_3 arbitrary;

(A_4) for $\alpha = -\frac{3\sqrt{2}}{5}$, $a_4 = \frac{\sqrt{2}}{2}$, $a_2 = \frac{15\sqrt{2}}{8}a_3^2$, $a_1 = \frac{75}{8}a_3^3$, $c = \frac{1575}{32}a_3^4$, $g = 3a_0a_3 + \frac{1125\sqrt{2}}{128}a_3^5$, a_0 and a_3 arbitrary;

(A_5) for arbitrary $\alpha \geq \frac{4}{5}$ or $\alpha \leq -\frac{4}{5}$, $a_4 = \frac{-5}{8}\alpha \pm \frac{\sqrt{25\alpha^2-16}}{8}$, $a_3 = a_2 = a_1 = 0$, $c = \frac{1}{2}(5\alpha + 3\sqrt{25\alpha^2 - 16})a_0$, $g = 0$, a_0 arbitrary.

The above theorem tells one that a function $u(\xi)$ satisfies (1.2) provided that it is a solution of (2.6) if one of the previously mentioned assumptions (A_1)–(A_5) is satisfied. Therefore the traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha \geq \frac{4}{5}$ or $\alpha \leq -\frac{4}{5}$ might be derived by investigating the sub-manifold of the fourth-order ODE (1.2).

3. Traveling wave solutions of Ito's 5th-order mKdV equation

Clearly, if a function $u(\xi)$ satisfies equation (2.6), then it satisfies the following planar dynamical system:

$$\begin{cases} u' = v, \\ v' = 2a_4u^3 + \dfrac{3}{2}a_3u^2 + a_2u + \dfrac{1}{2}a_1, \end{cases} \tag{3.1}$$

which is a Hamiltonian system with Hamiltonian

$$H(u,v) = \frac{1}{2}\left[v^2 - \left(a_4u^4 + a_3u^3 + a_2u^2 + a_1u\right)\right]. \tag{3.2}$$

The solution of (2.6) is fully determined by the energy curve $h = \frac{1}{2}a_0$, i.e., $H(u,v) = \frac{1}{2}a_0$.

According to dynamical system theorems [3], one knows that only bounded orbits of system (3.1) correspond to its bounded solutions. Due to the fact that the bounded orbits of an analytic Hamiltonian system could only be periodic orbits surrounding center, heteroclinic orbits or homoclinic orbits, we only need to study the case when the dynamical system has at least one center if we only focus on the bounded nontrivial solutions of system (3.1).

In order to investigate the bounded exact traveling wave solutions of Ito's 5th-order mKdV equation (1.1), we study the bounded orbits determined by $H(u,v) = \frac{1}{2}a_0$ and $\alpha, a_0, \cdots, a_4$ and g satisfy one of the conditions of Theorem 2.1.

3.1. Traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha = -1$

In this subsection we study the traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha = -1$.

Theorem 3.1. *Assume that $\alpha = -1$. Then the following statements hold for Ito's 5th-order mKdV equation.*

(1) For arbitrary u_0,

$$u(\xi) = u_0\left(1 + 3a - \frac{12q^2e^{u_0q\xi}}{9e^{2u_0q\xi} + 24(1+3a)e^{u_0q\xi} + 8(2+3a)}\right) \tag{3.3}$$

with $a > 0$ and

$$u(\xi) = u_0\left(1 + 3a + \frac{12q^2e^{u_0q\xi}}{9e^{2u_0q\xi} - 24(1+3a)e^{u_0q\xi} + 8(2+3a)}\right) \tag{3.4}$$

with $-\frac{2}{3} < a < -\frac{1}{2}$ *are two families of solitary wave solutions of (1.1). Here* $\xi = x - ct$, $q = 3\sqrt{2a(1+2a)}$ *and* $c = 6u_0^4(81a^4 + 126a^3 + 84a^2 + 30a + 5)$.

(2) For arbitrary u_0 *and* $0 < a < 2$,

$$u(\xi) = u_0\left(1 - \frac{12a^2 e^{au_0\xi}}{9e^{2au_0\xi} + 24e^{au_0\xi} + 16 - 4a^2}\right) \tag{3.5}$$

are a family of solitary wave solutions of (1.1), where $\xi = x - ct$ *and* $c = u_0^4(a^4 - 10a^2 + 30)$.

(3) For arbitrary u_0,

$$u(x,t) = \pm u_0 \tanh\left(u_0(x - 6u_0^4 t)\right) \tag{3.6}$$

are a family of kink and a family of anti-kink wave solutions of (1.1).

(4) Suppose $e_\pm = \frac{1}{2}\left(-1 \pm \sqrt{1-4b}\right)$ *for* $b < \frac{1}{4}$. *For arbitrary* u_0 *and* Q_0

$$u(\xi) = u_0\left(1 + 3Q_1 + \frac{3(Q_2 - Q_1)(Q_0 - Q_1)}{Q_0 - Q_1 - (Q_0 - Q_2)sn^2(u_0\Omega\xi, q)}\right), \tag{3.7}$$

are a family of periodic traveling wave solutions of (1.1), where $Q_0 \in (e_-, 0)$ *when* $b < 0$, $Q_0 \in (e_+, 0)$ *when* $0 < b \le \frac{2}{9}$ *and* $Q_0 \in (e_-, e_+)$ *when* $\frac{2}{9} < b < \frac{1}{4}$. *Here* $Q_1 < Q_2 < Q_3$ *are three roots of equation* $Q^3 + (\frac{4}{3} + Q_0)Q^2 + (2b + {Q_0}^2 + \frac{4}{3}Q_0)Q + 2bQ_0 + \frac{4}{3}{Q_0}^2 + {Q_0}^3 = 0$, $\Omega = \frac{3}{2}\sqrt{(Q_3 - Q_2)(Q_0 - Q_1)}$, $q = \sqrt{\frac{(Q_3-Q_1)(Q_0-Q_2)}{(Q_3-Q_2)(Q_0-Q_1)}}$, $\xi = x - ct$ *and* $c = 6u_0^4(-54Q_0^4 - 72Q_0^3 - 108bQ_0^2 + 54b^2 - 30b + 5)$.

Proof. According to Theorem 2.1, we see that (1.2) with $\alpha = -1$ admits the subequation in the form (2.6) if $a_0, ..., a_4$, c and g satisfy (A_1) $a_4 = 1, a_3 = 0$, $c = 2a_0 + a_2^2$, $g = \frac{1}{2}a_1a_2$, a_0, a_1 and a_2 arbitrary. Now we firstly study the bounded orbits determined by $H(u,v) = \frac{1}{2}a_0$ of system (3.1) with $a_4 = 1$ and $a_3 = 0$, i.e.,

$$\begin{cases} u' = v, \\ v' = 2u^3 + a_2u + \dfrac{1}{2}a_1 \end{cases} \tag{3.8}$$

for arbitrary a_1 and a_2.

Suppose $f(u_0) = 2u_0^3 + a_2u_0 + \frac{1}{2}a_1 = 0$, that is to say that $(u_0, 0)$ is an equilibrium point of system (3.8). For (3.8) with $a_1 = 0$ and $a_2 \ge 0$, it is easy to check that $(u_0, 0) = (0, 0)$ is the unique equilibrium point which is a saddle and thus there is no bounded solution can be derived. However, for (3.8) with $a_1 = 0$ and $a_2 < 0$ or with $a_1 \ne 0$, we can choose $u_0 \ne 0$, then the rescaling $\bar{u} = \frac{1}{3u_0}(u - u_0)$, $\bar{v} = \frac{\sqrt{2}}{18u_0^2}v$ and $\eta = 3\sqrt{2}u_0\xi$ transforms (3.8) into the system

$$\begin{cases} \dot{\bar{u}} = \bar{v}, \\ \dot{\bar{v}} = \bar{u}^3 + \bar{u}^2 + b\bar{u}, \end{cases} \tag{3.9}$$

where $b = \frac{6u_0^2 + a_2}{18u_0^2}$ and $\dot{}$ represents the derivative with respect to the new variable η. We now study the bounded orbits of system (3.9) determined by

$$H_1(\bar{u}, \bar{v}) = \frac{1}{2}\left[\bar{v}^2 - \left(\frac{1}{2}\bar{u}^4 + \frac{2}{3}\bar{u}^3 + b\bar{u}^2\right)\right] = \frac{a_0 - a_2{u_0}^2 - 3{u_0}^4}{324{u_0}^4}. \tag{3.10}$$

The bifurcation points of system (3.9) are $b = \frac{1}{4}$, $b = \frac{2}{9}$ and $b = 0$. When $b > \frac{1}{4}$, system (3.9) has only one equilibrium point which is a saddle and thus it has no bounded orbits. It has one saddle and one cusp if $b = \frac{1}{4}$ or $b = 0$. So system (3.9) has no bounded solutions if $b \geq \frac{1}{4}$ or $b = 0$. Let $e_\pm = \frac{1}{2}(-1 \pm \sqrt{1-4b})$. Then system (3.9) has three equilibrium points, viz., $(0,0)$, $(e_+, 0)$ and $(e_-, 0)$ when $b < \frac{1}{4}$ and $b \neq 0$. The point $(0,0)$ is a center but $(e_+, 0)$ and $(e_-, 0)$ are saddle and $H_1(0,0) < H_1(e_+, 0) < H_1(e_-, 0)$ when $b < 0$. Hence, there is a homoclinic orbit connecting $(e_+, 0)$, which is the boundary of a family of closed orbits surrounding the center $(0,0)$ if $b < 0$. The point $(e_+, 0)$ is a center whereas $(0,0)$ and $(e_-, 0)$ are saddle when $0 < b < \frac{1}{4}$. For $b = \frac{2}{9}$, $H_1(e_+, 0) < H_1(0,0) = H_1(e_-, 0)$, so there are two heteroclinic orbits connecting $(0,0)$ and $(e_-, 0)$, which are the boundary of a family of closed orbits surrounding the center $(e_+, 0)$. For $0 < b < \frac{2}{9}$, $H_1(e_+, 0) < H_1(0,0) < H_1(e_-, 0)$, so there is a homoclinic orbit connecting $(0,0)$, which is the boundary of a family of closed orbits surrounding the center $(e_+, 0)$. For $\frac{2}{9} < b < \frac{1}{4}$, $H_1(e_+, 0) < H_1(e_-, 0) < H_1(0,0)$, so there is a homoclinic orbit connecting $(e_-, 0)$, which is the boundary of a family of closed orbits surrounding the center $(e_+, 0)$.

Case (1) $b < 0$.

From the above analysis, we see that the corresponding Hamiltonian of the homoclinic orbit is determined by $h = H_1(e_+, 0)$. Note that in this case $e_+ = \frac{1}{2}(-1 + \sqrt{1-4b}) > 0$. By substituting $h = H_1(e_+, 0)$ in (3.10), one derives

$$\frac{d\bar{u}}{d\eta} = \pm(\bar{u} - e_+)\sqrt{\frac{1}{2}(\bar{u} - r_+)(\bar{u} - r_-)}\,, \tag{3.11}$$

where $r_\pm = -\frac{1}{6} - \frac{1}{2}\sqrt{1-4b} \pm \frac{1}{3}\sqrt{1 + 3\sqrt{1-4b}}$.

Solving equation (3.11)(refer to the formula in [8]) yields the bounded solution

$$\bar{u}(\eta) = e_+ - \frac{72q_+^2 e^{q\eta}}{9e^{2q_+\eta} + 24(1+3e_+)e^{q_+\eta} + 8(2+3e_+)}, \tag{3.12}$$

where $q_+ = \sqrt{e_+(1+2e_+)}$. From (3.10) solving $h = H_1(e_+, 0)$ for a_0 and recalling that $c = 2a_0 + a_2^2$, one has $c = 6u_0^4(81e_+^4 + 126e_+^3 + 84e_+^2 + 30e_+ + 5)$. For simplicity we denote e_+ by a. Therefore, we obtain (3.3).

Case (2) $0 < b < \frac{2}{9}$.

The bounded solution of (3.9) corresponding to $h = H_1(0,0)$ is

$$\bar{u}(\eta) = \frac{-72be^{\sqrt{b}\eta}}{9e^{2\sqrt{b}\eta} + 24e^{\sqrt{b}\eta} + 16 - 72b}. \tag{3.13}$$

Denote $3\sqrt{2b}$ by a, then $0 < a < 2$ and thus we obtain (3.5) and prove statement (2).

Case (3) $b = \frac{2}{9}$.

The heteroclinic orbits are determined by $h = H_1(0,0)$. The bounded solutions of (3.9) corresponding to $h = H_1(0,0)$ are

$$\bar{u}(\eta) = \pm\frac{1}{3}\tanh\left(\frac{\sqrt{2}}{6}\eta\right) - \frac{1}{3} \tag{3.14}$$

from which we obtain (3.6) and prove statement (3).

Case (4) $\frac{2}{9} < b < \frac{1}{4}$.

The homoclinic orbit is determined by $h = H_1(e_-, 0)$. Note that $-\frac{2}{3} < e_- < -\frac{1}{2}$ when $\frac{2}{9} < b < \frac{1}{4}$. The bounded solution of (3.9) corresponding to $h = H_1(e_-, 0)$ is

$$\bar{u}(\eta) = e_- + \frac{72q_-^2 e^{q_- \eta}}{9e^{2q_- \eta} - 24(1+3e_-)e^{q_- \eta} + 8(2+3e_-)}, \tag{3.15}$$

where $q_- = \sqrt{e_-(1+2e_-)}$. From (3.10) solving $h = H_1(e_-, 0)$ for a_0 and substituting in $c = 2a_0 + a_2^2$ gives $c = 6u_0^4(81e_-^4 + 126e_-^3 + 84e_-^2 + 30e_- + 5)$. For simplicity denote e_- by a and then we obtain (3.4) and prove statement (1).

For arbitrary $Q_0 \in (e_-, 0)$ when $b < 0$, $Q_0 \in (e_+, 0)$ when $0 < b \leq \frac{2}{9}$ and $Q_0 \in (e_-, e_+)$ when $\frac{2}{9} < b < \frac{1}{4}$, $H_1(\bar{u}, \bar{v}) = h$ with $h = H_1(Q_0, 0)$ determines the periodic orbit of (3.9). Suppose that $Q_1 < Q_2 < Q_3$ are three roots of equation $Q^3 + (\frac{4}{3} + Q_0)Q^2 + (\frac{4}{3}Q_0 + {Q_0}^2 + 2a_1)Q + 2a_1 Q_0 + \frac{4}{3}{Q_0}^2 + {Q_0}^3 = 0$, then from $H_1(\bar{u}, \bar{v}) = H_1(Q_0, 0)$, we have

$$\frac{d\bar{u}}{d\eta} = \pm\sqrt{\frac{1}{2}(\bar{u} - Q_0)(\bar{u} - Q_1)(\bar{u} - Q_2)(\bar{u} - Q_3)}\ . \tag{3.16}$$

From (3.16), we get the following periodic solutions of (3.9):

$$\bar{u}(\eta) = Q_1 + \frac{(Q_2 - Q_1)(Q_0 - Q_1)}{Q_0 - Q1 - (Q_0 - Q_2)(sn(\Omega\eta, q))^2}, \tag{3.17}$$

where $\Omega = \frac{\sqrt{2}}{4}\sqrt{(Q_3 - Q_2)(Q_0 - Q_1)}$ and $q = \sqrt{\frac{(Q_3 - Q_1)(Q_0 - Q_2)}{(Q_3 - Q_2)(Q_0 - Q_1)}}$. Solving $H_1(Q_0, 0) = \frac{a_0 - 3{u_0}^4 - a_2\,{u_0}^2}{324{u_0}^4}$ for a_0 and substituting in $c = 2a_0 + a_2^2$ yields $c = 6u_0^4(-54Q_0^4 - 72Q_0^3 - 108bQ_0^2 + 54b^2 - 30b + 5)$. Then we obtain (3.7) from (3.17) and the conclusion is proven. This completes the proof of the theorem.

□

3.2. Traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha = 1$

In this subsection we study the traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha = 1$.

Theorem 3.2. *Assume that $\alpha = 1$. Then the following conclusion holds for Ito's 5th-order mKdV equation.*

(1) For arbitrary u_0,

$$u(x,t) = u_0\left(\frac{2}{1 + u_0^2(x - \frac{15}{8}u_0^4 t)^2} - \frac{1}{2}\right) \tag{3.18}$$

and

$$u(x,t) = u_0\left(1 - \frac{4}{1 + 4u_0^2(x - 30u_0^4 t)^2}\right) \tag{3.19}$$

are two families of solitary wave solutions of (1.1).

(2) For arbitrary u_0 and $-\frac{1}{2} < a < 0$,

$$u(\xi) = u_0\left(1 + 3a + \frac{12q^2 e^{qu_0\xi}}{8(2+3a) + 24(1+3a)e^{qu_0\xi} + 9e^{2qu_0\xi}}\right) \tag{3.20}$$

and

$$u(\xi)=u_0\left(1+3a-\frac{12q^2e^{qu_0\xi}}{8(2+3a)-24(1+3a)e^{qu_0\xi}+9e^{2qu_0\xi}}\right) \tag{3.21}$$

are two families of solitary wave solutions of (1.1), where $\xi=x-ct, q=-3\sqrt{-2a(1+2a)}$ *and* $c=6u_0^4(81a^4+126a^3+84a^2+30a+5)$.

(3) For arbitrary u_0 *and* $a<0$,

$$u(\xi)=u_0\left(1+\frac{12a^2e^{au_0\xi}}{4(4+a^2)+24e^{au_0\xi}+9e^{2au_0\xi}}\right) \tag{3.22}$$

and

$$u(\xi)=u_0\left(1-\frac{12a^2e^{au_0\xi}}{4(4+a^2)-24e^{au_0\xi}+9e^{2au_0\xi}}\right) \tag{3.23}$$

are two families of solitary wave solutions of (1.1), where $\xi=x-ct$ *and* $c=u_0^4(a^4+10a^2+30)$.

(4) For arbitrary u_0, Q_0 *and* $b<-\frac{1}{4}$ *or arbitrary* $Q_0\notin\{-\frac{1}{2},\frac{1}{6}\}$ *and* $b=-\frac{1}{4}$,

$$u(\xi)=u_0\left(1+3Q_1+\frac{3q(Q_0-Q_1)}{q+p\left(ns(\Omega\xi,k)+cs(\Omega\xi,k)\right)^2}\right) \tag{3.24}$$

are a family of periodic traveling wave solutions of (1.1). Here Q_1 *is the real root of the equation*

$$Q_1^3+(Q_0+\frac{4}{3})Q_1^2+(Q_0^2+\frac{4}{3}Q_0-2b)Q_1+(Q_0^3+\frac{4}{3}Q_0^2-2bQ_0)=0, \tag{3.25}$$

$p=\frac{1}{3}\sqrt{9Q_1^2+Q_1(18Q_0+12)+27Q_0^2+24Q_0-18b}$, $k=\frac{1}{2}\sqrt{\frac{(Q_0-Q_1)^2-(p-q)^2}{pq}}$, $\Omega=-3u_0\sqrt{pq}$ $q=\frac{1}{3}\sqrt{27Q_1^2+Q_1(18Q_0+24)+9Q_0^2+12Q_0-18b}$, $\xi=x-ct$ *and* $c=6u_0^4(-54Q_0^4-72Q_0^3+108bQ_0^2+54b^2+30b+5)$.

(5) For arbitrary u_0, $b>-\frac{1}{4}$ *and* $\frac{1}{2}(-1-\sqrt{1+4b})<Q_3<\frac{1}{2}(-1+\sqrt{1+4b})$,

$$u(\xi)=u_0\left(1+3Q_3-\frac{3(Q_3-Q_1)(Q_3-Q_2)}{(Q_1-Q_2)sn^2(\Omega\xi,k)+(Q_3-Q_1)}\right) \tag{3.26}$$

and

$$u(\xi)=u_0\left(1+3Q_4-\frac{3(Q_4-Q_2)(Q_4-Q_1)}{(Q_2-Q_1)sn^2(\Omega\xi,k)+(Q_4-Q_2)}\right) \tag{3.27}$$

are two families of periodic wave solutions of (1.1). Here $\xi=x-ct$ *and* $c=6u_0^4(-54Q_3^4-72Q_3^3+108bQ_3^2+54b^2+30b+5)$, $\Omega=-\frac{3}{2}u_0\sqrt{(Q_1-Q_3)(Q_2-Q_4)}$, $k=\sqrt{\frac{(Q_1-Q_2)(Q_3-Q_4)}{(Q_1-Q_3)(Q_2-Q_4)}}$, Q_1, Q_2 *and* Q_4 *are three roots of the equation*

$$Q^3+(\frac{4}{3}+Q_3)Q^2+(Q_3^2+\frac{4}{3}Q_3-2b)Q+Q_3^3+\frac{4}{3}Q_3^2-2bQ_3=0, \tag{3.28}$$

where $Q_1<Q_2<Q_4$.

(6) For arbitrary ω *and* $1>k>0$,

$$u(\xi)=\omega k\left(-1+\frac{2}{1+(ns(\omega\xi,k)+cs(\omega\xi,k))^2}\right) \tag{3.29}$$

are a family of periodic wave solutions of (1.1), where $\xi=x-(6k^4-6k^2+1)\omega^4t$.

Proof. The proof is similar to the proof of Theorem 3.1. We first note that (1.2) with $\alpha = 1$ admits the subequation in the form (2.6) if $a_0, \ldots, a_4$ and g satisfy (A_2) $a_4 = -1,\ a_3 = 0,\ c = a_2^2 - 2a_0\ g = -\frac{1}{2}a_1 a_2,\ a_0,\ a_1$ and a_2 arbitrary. Now we study the bounded orbits determined by $H(u,v) = \frac{1}{2}a_0$ of system (3.1) with $a_4 = -1, a_3 = 0$, i.e.,

$$\begin{cases} u' = v, \\ v' = -2u^3 + a_2 u + \dfrac{a_1}{2}, \end{cases} \tag{3.30}$$

for arbitrary a_1 and a_2.

Similarly, we assume that $(u_0, 0)$ is an equilibrium point of system (3.15), i.e., $-2u_0^3 + a_2 u_0 + \frac{a_1}{2} = 0$. Obviously, $u_0 \neq 0$ if $a_1 \neq 0$. We also can choose $u_0 \neq 0$ if $a_1 = 0$ and $a_2 > 0$, then the rescaling $\bar{u} = \frac{1}{3u_0}(u - u_0),\ \bar{v} = -\frac{\sqrt{2}}{18u_0^2}v$ and $\eta = -3\sqrt{2}u_0\xi$ transforms (3.30) into the system

$$\begin{cases} \dot{\bar{u}} = \bar{v}, \\ \dot{\bar{v}} = -\bar{u}^3 - \bar{u}^2 + b\bar{u}, \end{cases} \tag{3.31}$$

where $b = \frac{a_2 - 6u_0^2}{18u_0^2}$ and $\dot{}$ denotes the derivative with respect to the new variable η. We study the bounded orbits of system (3.31) determined by

$$H_2(\bar{u}, \bar{v}) = \frac{1}{2}\left[\bar{v}^2 - \left(-\frac{1}{2}\bar{u}^4 - \frac{2}{3}\bar{u}^3 + b\bar{u}^2\right)\right] = \frac{6u_0^4 - 2a_2 {u_0}^2 + a_2^2 - c}{648{u_0}^4}. \tag{3.32}$$

Clearly, it has only one equilibrium point $(0,0)$, which is a center if $b < -\frac{1}{4}$. Thus if $b < -\frac{1}{4}$ all the orbits of system (3.31) are closed curves which correspond to the periodic solutions of this system. It has a cusp and a center if $b = 0$ or $b = -\frac{1}{4}$ and thus all the closed orbits not passing through the cusp correspond to the periodic solutions and the orbits passing through the cusp correspond to solitary wave solutions. For $b > -\frac{1}{4}$ and $b \neq 0$, system (3.31) has three equilibrium points $(0,0)$ and $(\bar{e}_{1\pm}, 0)$, where $\bar{e}_{1\pm} = \frac{1}{2}(-1 \pm \sqrt{1+4b})$. These are two center points and a saddle. There are two homoclinic orbits connecting the saddle which are the boundary curves of the two families of closed orbits surrounding the two centers.

Case (1) $b = -\frac{1}{4}$.

The phase orbits of system are all periodic orbits except the one passing through the singular point $(-\frac{1}{2}, 0)$. The orbit connecting the singular point $(-\frac{1}{2}, 0)$ is given by

$$\bar{u}(\eta) = \frac{12}{18 + \eta^2} - \frac{1}{2}. \tag{3.33}$$

From (3.32), we see that $H_2(-\frac{1}{2}, 0) = \frac{1}{192} = \frac{6u_0^4 - 2a_2{u_0}^2 + a_2^2 - c}{648{u_0}^4}$. Thus, $c = \frac{15}{8}u_0^4$ and from (3.33), we obtain (3.18).

Case (2) $b = 0$.

The phase orbits of system are all periodic orbits except the one passing through the singular point $(0,0)$. The orbit connecting the singular point $(0,0)$ is given by

$$\bar{u}(\eta) = -\frac{12}{9 + 2\eta^2}. \tag{3.34}$$

From (3.32), we obtain $H_2(0,0) = 0 = \frac{6u_0^4 - 2a_2{u_0}^2 + a_2^2 - c}{648{u_0}^4}$. Thus we have $c = 30u_0^4$. From (3.34), we obtain (3.19) and statement (1) is proved.

Case (3) $-\frac{1}{4} < b < 0$.

The homoclinic orbits are determined by $h_2 = H_2(\bar{e}_{1+}, 0)$. The bounded solutions of (3.31) corresponding to these two homoclinic orbits are given by

$$\bar{u}(\eta) = \bar{e}_{1+} + \frac{72\Omega_1^2 e^{\Omega_1 \eta}}{8(2+3\bar{e}_+) + 24(1+3\bar{e}_{1+})e^{\Omega_1\eta} + 9e^{2\Omega_1\eta}} \tag{3.35}$$

and

$$\bar{u}(\eta) = \bar{e}_{1+} - \frac{36\Omega_1^2 e^{\Omega_1 \eta}}{8(2+3\bar{e}_+) - 24(1+3\bar{e}_{1+})e^{\Omega_1\eta} + 9e^{2\Omega_1\eta}}, \tag{3.36}$$

where $\Omega_1 = \sqrt{-\bar{e}_{1+}(1+2\bar{e}_{1+})}$. Note that $-\frac{1}{2} < \bar{e}_{1+} < 0$ when $-\frac{1}{4} < b < 0$. Denote $\bar{e}_{1+}$ by a and from above results, we obtain (3.20) and (3.21), then conclusion (2) is proved.

Case (4) $b > 0$.

The homoclinic orbits are determined by $h = H_2(0,0)$. The bounded solutions of (3.31) corresponding to these two orbits are

$$\bar{u}(\eta) = \frac{72be^{\sqrt{b}\eta}}{16 + 72b + 24e^{\sqrt{b}\eta} + 9e^{2\sqrt{b}\eta}} \tag{3.37}$$

and

$$\bar{u}(\eta) = \frac{-72be^{\sqrt{b}\eta}}{16 + 72b - 24e^{\sqrt{b}\eta} + 9e^{2\sqrt{b}\eta}}. \tag{3.38}$$

Let $-3\sqrt{2b} = a$, then $a < 0$, so we obtain (3.22) and (3.23) and conclusion (3) is proved.

However, we also know that equation (1.2) with $\alpha = 1$ admits the subequation in the form (2.6) with $a_4 = -1$, $a_3 = a_1 = 0$, $c = a_2^2 - 2a_0$ and $g = 0$ for arbitrary a_0 and a_2, i.e.,

$$\left(\frac{du}{d\xi}\right)^2 = -u^4 + a_2 u^2 + a_0. \tag{3.39}$$

Let $a_2 = (2k^2-1)\omega^2$ and $a_0 = k^2\omega^4(1-k^2)$ for arbitrary ω and $1 > k > 0$, then (3.39) can be rewritten as

$$\frac{du}{d\xi} = \pm\sqrt{(k\omega - u)(u - k\omega)(u^2 + (1-k^2)\omega^2)}. \tag{3.40}$$

Solving (3.40) yields (3.29) which is a family of periodic wave solutions of (1.1). This completes the proof of the theorem.

□

3.3. Traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha = \frac{3\sqrt{2}}{5}$

In this subsection we show that Ito's 5th-order mKdV equation with $\alpha = \frac{3\sqrt{2}}{5}$ has a family of periodic wave solutions determined by a sub-manifold of its associated higher-order ODE.

According to Theorem 2.1, we know that (1.2) with $\alpha = \frac{3\sqrt{2}}{5}$ admits the subequation in the form (2.6) if and only if $a_0, \ldots, a_4, g$ and c satisfy condition (A_3), i.e., $a_4 = -\frac{\sqrt{2}}{2}$, $a_2 = -\frac{15\sqrt{2}}{8}a_3^2$, $a_1 = \frac{75}{8}a_3^3$, $g = -3a_0a_3 + \frac{1125\sqrt{2}}{128}a_3^5$, $c = \frac{1575}{32}a_3^4$, a_0 and a_3 arbitrary. System (3.1) with coefficients satisfying (A_3) is written as

$$\begin{cases} u' = v, \\ v' = -\sqrt{2}u^3 + \dfrac{3}{2}a_3u^2 - \dfrac{15\sqrt{2}}{8}a_3^2u + \dfrac{75}{16}a_3^3, \end{cases} \tag{3.41}$$

for arbitrary a_3. Let $e_0 = \lambda a_3$, where λ satisfies the cubic algebraic equation

$$-\sqrt{2}\lambda^3 + \frac{3}{2}\lambda^2 - \frac{15\sqrt{2}}{8}\lambda + \frac{75}{16} = 0, \tag{3.42}$$

then $(e_0, 0)$ is the unique equilibrium point of system (3.38) which is a center. In fact, it is easy to check that (3.42) has a unique root and the characteristic values of (3.41) at $(e_0, 0)$ are two conjugate imaginary numbers, which implies that system (3.41) has a unique center for arbitrary a_3. Consequently, for arbitrary a_0 and a_3, $H(u, v) = \frac{1}{2}a_0$ defines a family of periodic orbits around the center $(e_0, 0)$, where $H(u, v)$ is determined by (3.2). Under this condition (A_5) for arbitrary $u_0 < e_0$, the right hand of the first-order ODE (2.6) can be rewritten as

$$a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0 = \frac{\sqrt{2}}{2}(u_1 - u)(u - u_0)((u - m)^2 + n^2), \tag{3.43}$$

where u_1 is the real root of the equation

$$\begin{aligned} &u^3 + (u_0 - \sqrt{2}a_3)u^2 + ({u_0}^2 - \sqrt{2}a_3u_0 + \frac{15}{4}{a_3}^2)u \\ &+ \frac{75}{8}\sqrt{2}{a_3}^3{u_0}^3 - \sqrt{2}a_3{u_0}^2 + \frac{15}{4}{a_3}^2u_0 = 0 \end{aligned} \tag{3.44}$$

and $u_0 < e_0 < u_1$, $m = \frac{1}{2}(u_0 + u_1 - \sqrt{2}a_3)$ and $n^2 = \frac{1}{4}(13a_3^2 - 5u_0^2 - 14u_0u_1 - 5u_1^2 + 6\sqrt{2}a_3(u_0 + u_1))$.

Thus, from (2.6) we have

$$\frac{du}{\sqrt{(u_1 - u)(u - u_0)((u - m)^2 + n^2)}} = \sqrt{\frac{\sqrt{2}}{2}}d\xi. \tag{3.45}$$

Solving (3.45) for $u(\xi)$ yields

$$u(\xi) = u_0 + \frac{q(u_1 - u_0)}{q + p(ns(\Omega\xi, k) + cs(\Omega\xi, k))^2}, \tag{3.46}$$

where $\xi = x - \frac{1575}{32}a_3^4t$, $p = \frac{1}{2}\sqrt{15a_3^2 + 4\sqrt{2}a_3(2u_0 + u_1) - 4u_0^2 - 16u_0u_1 - 4u_1^2}$, $q = \frac{1}{2}\sqrt{15a_3^2 + 4\sqrt{2}a_3(u_0 + 2u_1) - 4u_0^2 - 16u_0u_1 - 4u_1^2}$, $\Omega = \sqrt{\frac{\sqrt{2}}{2}pq}$ and $k = \frac{1}{2}\sqrt{\frac{(u_1 - u_0)^2 - (p - q)^2}{pq}}$.

Theorem 3.3. *For arbitrary u_0, Ito's 5th-order mKdV equation with $\alpha = \frac{3\sqrt{2}}{5}$ has a family of periodic wave solutions defined as (3.46).*

Remark 3.1. For the case when $\alpha = -\frac{3\sqrt{2}}{5}$, even though we know that (1.2) admits the subequation in the form (2.6) if and only if $a_0, \ldots, a_4, g$ and c satisfy condition (A_4). However, system (3.1) with coefficients satisfying condition (A_4) has only one equilibrium point which is a saddle, therefore it has no bounded nontrivial solutions. Therefore, no bounded nontrivial traveling wave solutions could be found here.

3.4. Traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha \geq \frac{4}{5}$

For equation (1.1) with $\alpha \geq \frac{4}{5}$ or $\alpha \leq -\frac{4}{5}$, according to Theorem 2.1, we know that (1.2) admits the subequation in the form (2.6) if $a_0, \ldots, a_4$, c and g satisfy condition (A_5) $a_4 = -\frac{5\alpha}{8} \pm \frac{\sqrt{25\alpha^2-16}}{8}$, $a_3 = a_2 = a_1 = 0$, $c = \frac{1}{2}(5\alpha \pm 3\sqrt{25\alpha^2 - 16})a_0$, and $g = 0$ for arbitrary a_0. Obviously, system

$$\begin{cases} u' = v, \\ v' = 2a_4u^3 \end{cases} \tag{3.47}$$

has only one equilibrium point $(0, 0)$ which is a center when $a_4 < 0$ and is a saddle when $a_4 > 0$. It is easy to check that $-\frac{5\alpha}{8} \pm \frac{\sqrt{25\alpha^2-16}}{8} > 0$ when $\alpha \leq -\frac{4}{5}$ and $-\frac{5\alpha}{8} \pm \frac{\sqrt{25\alpha^2-16}}{8} < 0$ when $\alpha \geq \frac{4}{5}$. So we know that (3.47) with $a_4 = -\frac{5\alpha}{8} \pm \frac{\sqrt{25\alpha^2-16}}{8}$ has no bounded nontrivial solutions if $\alpha \leq -\frac{4}{5}$ and has a family of periodic solutions if $\alpha \geq \frac{4}{5}$. By careful computations, the following explicit periodic traveling wave solutions of Ito's 5th-order mKdV equation with $\alpha \geq \frac{4}{5}$ can be obtained.

Theorem 3.4. *For $\alpha \geq \frac{4}{5}$ and arbitrary $u_0 > 0$,*

$$u(\xi) = u_0\left(-1 + \frac{2}{1 + (ns(\Omega\xi, \frac{\sqrt{2}}{2}) + cs(\Omega\xi, \frac{\sqrt{2}}{2}))^2}\right) \tag{3.48}$$

is a family of periodic wave solutions of (1.1). Here $\Omega = \frac{1}{2}u_0\sqrt{5\alpha \pm \sqrt{25\alpha^2 - 16}}$ and $\xi = x - \frac{1}{4}(5\alpha \mp 3\sqrt{25\alpha^2 - 16})\Omega^2u_0^2\ t$.

4. Conclusion and Discussion

In this paper we obtained traveling wave solutions of the Ito's 5th-order mKdV equation with $\alpha = -1$ or $\alpha \geq \frac{4}{5}$. By using the traveling wave variable we transformed this equation into a 4th-order nonlinear ordinary differential equation which is associated with a dynamical system in 4-dimensional space. Generally speaking, it is very difficult to study dynamical systems in higher dimensional space. However, with the aid of symbolic computation system, we obtained sub-manifolds which are determined by some planar dynamical systems of the corresponding dynamical system of Ito's 5th-order mKdV equation with $\alpha \geq \frac{4}{5}$ or $\alpha = -1$. By using bifurcation and dynamical system theorem [3], all possible bounded real solutions of the involving planar dynamical systems were studied and then some exact solitary wave solutions, kink and anti-kink wave solutions and some periodic wave solutions were obtained. The known results on the real bounded traveling wave solutions of Ito's 5th-order mKdV equation in the literature [13, 14, 16] were also recovered. It is worth pointing out that the method proposed in this paper, which combines the symbolic computation system and qualitative analysis, might be applied to study the traveling wave solutions of other higher-order nonlinear partial differential equations.

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