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PHRAGMÉN-LINDELÖF ALTERNATIVE FOR THE LAPLACE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS*

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Abstract This paper investigates the spatial behavior of the solutions of the Laplace equation on a semi-infinite cylinder when dynamical nonlinear boundary conditions are imposed on its lateral side. We prove a Phragmén-Lindelöf alternative for the solutions. To be precise, we see that the solutions increase in an exponential way or they decay as a polynomial. To give a complete description of the decay in this last case we also obtain an upper bound for the amplitude term by means of the boundary conditions. In the last section we sketch how to generalize the results to a system of two elliptic equations related with the heat conduction in mixtures.

Keywords Phragmén-Lindelöf alternative, dynamic boundary conditions, Laplace equation.

MSC(2010) 35B30, 35B40, 35B53, 35J65.

1. Introduction

The analysis of the spatial stability of the solutions for the equilibrium equations in elasticity is a relevant issue when the deformations of a cylinder are studied. It is closely related with Saint-Venant's principle and it has deserved a lot of interest recently.

Results about spatial stability have been extended to dynamical elastic problems and to dynamical thermal problems. In fact, the interest for the spatial stability has gone beyond thermomechanics and, nowadays, it is the aim of study in different types of partial differential equations and/or systems. It is worth noting that the mathematical framework where such results are considered is the Phragmén-Lindelöf principle which proposes an increase/decay alternative for the solutions.

The spatial behavior of elliptic [3], parabolic [6, 8], hyperbolic [1, 4, 9] equations and/or combinations of them [12] have been already obtained. However, there are many aspects yet which need to be studied and clarified. In this note we want to pay attention to the Laplace equation with nonlinear dynamical boundary conditions. That is, when a certain nonlinear ordinary dynamical differential equation is satisfied at the lateral boundary of the cylinder where the Laplace equation is satisfied. As far as the authors know, there are no results in the literature on spatial behavior of solutions when such kind of boundary conditions are assumed. We

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obtain a Phragmén-Lindelöf type alternative for the solutions of the problem. In fact, we see that the solutions either blow-up in an exponential way when the large variable becomes unbounded, or they decay as a polynomial when the large variable is increasing. It is appropriate to recall several results when nonlinear boundary conditions are imposed (see the papers by Horgan and Payne [7] and Leseduarte and Quintanilla [11]). We follow a similar approach. However, our results only apply when the nonlinear term is super-linear, but not in the sub-linear case.

The plan of the paper is the following. In the next section we propose the problem we are dealing with. A Phragmén-Lindelöf alternative is obtained in Section 3. When the solutions decay, our estimate is impractical if we do not have some information on the amplitude term. In Section 4 we obtain an upper bound for the amplitude term when the solutions decay. Section 5 is devoted to give an illustrative example. In the last section we propose an extension of the results to a system of two linear elliptic equations that are related with the heat conduction in mixtures.

2. Preliminaries

We want to investigate the spatial asymptotic behavior of the solutions of the Laplace equation with nonlinear dynamic boundary conditions. Therefore, we consider a semi-infinite cylinder $R=[0,\infty)\times D$, where D is a two-dimensional bounded domain smooth enough to apply the divergence theorem.

We consider a problem related with the Laplace equation

$$\Delta u = 0 \ \text{ on } \ R\times(0,t). \tag{2.1}$$

To define the boundary conditions, we suppose that $\partial D=\Omega_1\cup\Omega_2$, where $\Omega_1\cap\Omega_2=\emptyset$ and such that the measure of Ω_2 is positive. On Ω_1, we impose that

$$\frac{\partial u}{\partial n}+f_1(u)=0 \ \text{ on } \ [0,\infty)\times\Omega_1\times(0,t); \tag{2.2}$$

on Ω_2 we suppose that

$$\frac{\partial u}{\partial n}+s(u)u_t+f_2(u)=0 \ \text{ on } \ [0,\infty)\times\Omega_2\times(0,t); \tag{2.3}$$

and on the finite end of the cylinder we impose that

$$u(0,x_2,x_3,\tau)=g(x_2,x_3,\tau) \ \text{ on } \ \{0\}\times D\times(0,t). \tag{2.4}$$

From now on, we assume that

$$f_1(u)u\geq 0 \tag{2.5}$$

for every u, and that there exists a positive constant C such that

$$f_2(u)u+\omega S_1(u)\geq C|u|^{2p}, \tag{2.6}$$

where $p\geq 1$, ω is a positive constant large enough and

$$S_1(u)=\int_0^u \eta s(\eta)\,d\eta\geq 0. \tag{2.7}$$

As we impose dynamic boundary conditions on Ω_2, we need to assume also initial conditions on Ω_2. We suppose that

$$u = 0 \quad \text{on} \quad [0,\infty) \times \Omega_2 \times \{0\}. \tag{2.8}$$

In this paper we will use the following notation:

$$\begin{aligned} &D(z) = \{z\} \times D; \;\; \Omega_1(z) = \{z\} \times \Omega_1; \;\; \Omega_2(z) = \{z\} \times \Omega_2; \\ &R(z) = \{\boldsymbol{x} \in R, x_1 \geq z\}; \;\; \Omega_i^*(z) = \{\boldsymbol{x} \in [0,\infty) \times \Omega_i, x_1 \geq z\}. \end{aligned}$$

In the analysis we will need to use the generalized Poincaré inequality. We recall that there exists a positive constant C_1 such that (see [2], p281)

$$\int_D |u|^2 \, da \leq C_1 \left[\int_D |\nabla u|^2 \, da + \left| \int_{\Omega_2} u \, dl \right|^2 \right], \tag{2.9}$$

for every smooth function u and for a two dimensional domain D. It is worth noting that the precise value of the constant C_1 depends on the domain D and on the subset of the boundary Ω_2.

3. Spatial estimates

In this section we obtain an alternative of the Phragmén-Lindelöf type for the solutions of the problem determined by (2.1)–(2.4) and (2.8). From now on, we consider a positive constant, ω, such that condition (2.6) is satisfied. We define the function

$$\Phi(z,t) = -\int_0^t \int_{D(z)} \exp(-\omega\tau) u u_{,1} \, da \, d\tau. \tag{3.1}$$

We note that for $z \geq z_0$, the function $\Phi(z,t)$ may be expressed as

$$\begin{aligned} \Phi(z,t) - \Phi(z_0,t) = &-\int_0^t \int_{z_0}^z \int_D \exp(-\omega\tau) |\nabla u|^2 \, dx \, d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_1} \exp(-\omega\tau) f_1(u) u \, da \, d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_2} \exp(-\omega\tau) \left[f_2(u) u + \omega S_1(u) \right] da \, d\tau \\ &- \int_{z_0}^z \int_{\Omega_2} \exp(-\omega t) S_1(u) \, da. \end{aligned} \tag{3.2}$$

Notice that if

$$\lim_{z\to\infty} \Phi(z,t) = 0, \tag{3.3}$$

then the relation (3.2) implies that

$$\begin{aligned} \Phi(z,t) = &\int_0^t \int_{R(z)} \exp(-\omega\tau) |\nabla u|^2 \, dx \, d\tau + \int_0^t \int_{\Omega_1^*(z)} \exp(-\omega\tau) f_1(u) u \, da \, d\tau \\ &+ \int_0^t \int_{\Omega_2^*(z)} \exp(-\omega\tau) \left[f_2(u) u + \omega S_1(u) \right] da \, d\tau \\ &+ \int_{\Omega_2^*(z)} \exp(-\omega t) S_1(u) \, da. \end{aligned} \tag{3.4}$$

From (3.2) we also see that

$$\begin{aligned}\frac{\partial \Phi}{\partial z} = &- \int_0^t \int_{D(z)} \exp(-\omega\tau)|\nabla u|^2 \, da \, d\tau - \int_0^t \int_{\Omega_1(z)} \exp(-\omega\tau) f_1(u) u \, dl \, d\tau \\ &- \int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u) \right] dl \, d\tau \\ &- \exp(-\omega t) \int_{\Omega_2(z)} S_1(u) \, dl.\end{aligned} \tag{3.5}$$

In view of the Schwarz inequality, from (3.1) we find

$$\begin{aligned}|\Phi(z,t)| \leq &\left(\int_0^t \int_{D(z)} \exp(-\omega\tau) u^2 \, da \, d\tau \right)^{1/2} \\ &\times \left(\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2 \, da \, d\tau \right)^{1/2}.\end{aligned} \tag{3.6}$$

We have that

$$\begin{aligned}\int_0^t \int_{D(z)} \exp(-\omega\tau) u^2 \, da \, d\tau \leq &\int_0^t \int_{D(z)} u^2 \, da \, d\tau \\ \leq &C_1 \int_0^t \left[\int_{D(z)} u_{,\alpha} u_{,\alpha} \, da + \left| \int_{\Omega_2(z)} u \, dl \right|^2 \right] d\tau \\ \leq &C_1 \int_0^t \left[\int_{D(z)} u_{,\alpha} u_{,\alpha} \, da + M_1 \int_{\Omega_2(z)} u^2 \, dl \right] d\tau,\end{aligned} \tag{3.7}$$

where

$$M_1 = \left[\text{measure } (\Omega_2) \right]^{1/2},$$

and Greek sub-indices are restricted to two and three.

From (3.6) and (3.7), it follows that

$$\begin{aligned}|\Phi(z,t)| \leq &C_1^{1/2} \left[\int_0^t \int_{D(z)} u_{,\alpha} u_{,\alpha} \, da \, d\tau + M_1 \int_0^t \int_{\Omega_2(z)} u^2 \, dl \, d\tau \right]^{1/2} \\ &\times \left[\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2 \, da \, d\tau \right]^{1/2}.\end{aligned} \tag{3.8}$$

But

$$\int_0^t \int_{\Omega_2(z)} |u|^2 \, dl \, d\tau \leq M_2 \left(\int_0^t \int_{\Omega_2(z)} |u|^{2p} \, dl \, d\tau \right)^{1/p}, \tag{3.9}$$

where $p \geq 1$ and

$$M_2 = \left[t \text{ measure } (\Omega_2) \right]^{p/(p-1)}.$$

We obtain that

$$
\begin{aligned}
&\left(\int_0^t \int_{\Omega_2(z)} |u|^2 \, dl \, d\tau\right)^{1/2} \\
&\le \exp(\omega t)^{1/(2p)} M_2^{1/2} \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau)|u|^{2p} \, dl \, d\tau\right)^{1/(2p)} \\
&\le M_3 \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u)\right] dl \, d\tau\right)^{1/(2p)},
\end{aligned}
\tag{3.10}
$$

where

$$M_3 = C^{-1/(2p)} \exp(\omega t)^{1/(2p)} M_2^{1/2}.$$

Then, it follows that

$$
\begin{aligned}
\Phi(z,t) \le &\left[2M_4 \left(\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,\alpha} u_{,\alpha} \, da \, d\tau\right)^{1/2}\right. \\
&\left. + M_5 \left(\int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau) \left[f_2(u)u + \omega S_1(u)\right] \, dl \, d\tau\right)^{1/(2p)}\right] \\
&\times \left[\int_0^t \int_{D(z)} \exp(-\omega\tau) u_{,1}^2 \, da \, d\tau\right]^{1/2},
\end{aligned}
\tag{3.11}
$$

where

$$M_4 = \frac{1}{2} C_1^{1/2} \exp(\omega t), \quad M_5 = C_1^{1/2} M_1 M_3.$$

After some standard manipulations we arrive at (see [7, p128])

$$|\Phi(z,t)| \le M_4 \left[-\frac{\partial \Phi}{\partial z}\right] + M_6 \left[-\frac{\partial \Phi}{\partial z}\right]^{(p+1)/(2p)}, \tag{3.12}$$

where

$$M_6 = p^{1/2}(p+1)^{-(p+1)/(2p)} M_5.$$

Consequences of the estimate (3.12) have been studied by Horgan and Payne (see [7, p134]). It can be proved that either there exists a positive constant Q_1 (see [7, p135]) such that

$$-\Phi(z,t) \ge \hat{C}_1 Q_1 \exp\left(\frac{z - z_0}{\hat{C}_1}\right), \quad z \ge z_0, \tag{3.13}$$

where

$$\hat{C}_1 = M_4 + M_6(2-\beta)\beta^{-1}\hat{\sigma}_2, \quad \beta = \frac{2p}{p+1}$$

and $\hat{\sigma}_2$ is an arbitrary positive constant, or the decay estimate (see [7, p136])

$$
\begin{aligned}
\Phi(z,t) \le &\hat{C}_2 \left\{\left[2\hat{C}_3(p+1)\right]^{-1} (p-1) \left[z + \hat{Q}(0)\right]\right\}^{-(p+1)/(p-1)} \\
&+ \hat{C}_3 \left\{\left[2\hat{C}_3(p+1)\right]^{-1} (p-1) \left[z + \hat{Q}(0)\right]\right\}^{-2(p+1)/(p-1)}
\end{aligned}
\tag{3.14}
$$

holds, where

$$\hat{C}_2 = M_4(2-\beta)\hat{\sigma}_1^{-(\beta-1)/(2-\beta)} + M_6, \quad \hat{C}_3 = M_4(\beta-1)\hat{\sigma}_1$$

and $\hat{\sigma}_1$ is an arbitrary positive constant and

$$\begin{aligned}\hat{Q}(0) =& 2\hat{C}_3(p+1)\left\{\left[\Phi(0,t)\hat{C}_3^{-1} + \frac{\hat{C}_2^2}{4\hat{C}_3^2}\right]^{1/2} - \frac{\hat{C}_2}{2\hat{C}_3}\right\}^{-(p-1)/(p+1)} \\ &- \hat{C}_3(p+1)\left\{\left[\Phi(0,t)\hat{C}_3^{-1} + \frac{\hat{C}_2^2}{4\hat{C}_3^2}\right]^{1/2} - \frac{\hat{C}_2}{2\hat{C}_3}\right\}^{2/(p+1)} .\end{aligned}$$

We note that estimate (3.14) implies that (3.3) holds and then the function $\Phi(z,t)$ is determined by (3.4).

Our results can be summarize by means of the following theorem.

Theorem 3.1. *Let $u(\boldsymbol{x},t)$ be a solution of the problem determined by (2.1)–(2.4) and (2.8). Then either the function*

$$\begin{aligned}&\int_0^t\int_0^z\int_D |\nabla u|^2 dx\, d\tau + \int_0^t\int_0^z\int_{\Omega_1} f_1(u)u\, da\, d\tau \\ &+ \int_0^t\int_0^z\int_{\Omega_2}[f_2(u)u + \omega S_1(u)]\, da\, d\tau + \int_0^z\int_{\Omega_2} S_1(u)\, da\end{aligned}$$

becomes unbounded in an exponential way when z tends to infinite, or the function

$$\begin{aligned}&\int_0^t\int_{R(z)} |\nabla u|^2 dx\, d\tau + \int_0^t\int_{\Omega_1^*(z)} f_1(u)u\, da\, d\tau \\ &+ \int_0^t\int_{\Omega_2^*(z)}[f_2(u)u + \omega S_1(u)]\, da\, d\tau + \int_{\Omega_2^*(z)} S_1(u)\, da\end{aligned}$$

decays at least as fast as $z^{-(p+1)/(p-1)}$ when z tends to infinite.

For the particular case $p = 1$, we can improve the estimates. From the estimate (3.12), we see that

$$|\Phi(z,t)| \leq (M_4 + M_6)\left(-\frac{\partial\Phi}{\partial z}\right). \tag{3.15}$$

It is well known that this inequality implies an alternative of the following type (see [3]):

The function $\Phi(z,t)$ satisfies the estimate

$$-\Phi(z,t) \geq Q_1^* \exp\left(\frac{z-z_0}{M_4+M_6}\right),\ z \geq z_0, \tag{3.16}$$

where Q_1^* is a positive constant, or the decay estimate

$$\Phi(z,t) \leq \Phi(0,t)\exp\left(-\frac{z}{M_4+M_6}\right),\ z \geq 0 \tag{3.17}$$

is satisfied. We note that estimates (3.16) and (3.17) give an alternative of exponential type.

4. The amplitude term

To make clear the estimates obtained in the previous section, we require a bound for $\Phi(0,t)$ in terms of the boundary conditions at the end of the cylinder $x_3 = 0$. Otherwise, the decay estimate obtained at (3.14) would be impractical because the dependence of the amplitude on the data would not be explicit. To make calculations easier, we assume in this section that $\Omega_1 = \emptyset$ and that $f_2(u) = 0$. Furthermore, we impose that

$$mS_1(u) \geq |S_2(u)|^{p_1}, \; m > 0, \; p_1 > 1, \tag{4.1}$$

where $S_2(u) = \int_0^u s(\eta)\,d\eta$.

We note that

$$\begin{aligned}\Phi(0,t) = &\int_0^t \int_R \exp(-\omega\tau)|\nabla u|^2\,dx\,d\tau + \omega \int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)S_1(u)\,da\,d\tau \\ &+ \exp(-\omega t)\int_{\Omega_2^*(0)} S_1(u)\,da = -\int_0^t \int_{D(0)} \exp(-\omega\tau) g u_{,1}\,da\,d\tau,\end{aligned} \tag{4.2}$$

where $g(x_2, x_3, t)$ was considered at (2.4). We now define

$$h(x_1, x_2, x_3, s) = g(x_2, x_3, s)\exp(-bx_1), \tag{4.3}$$

where b is an arbitrary positive constant. We have that

$$\begin{aligned}\Phi(0,t) &= \int_0^t \int_R \exp(-\omega\tau)h_{,i}u_{,i}\,dx\,d\tau + \int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)hS(u)u_\tau\,da\,d\tau \\ &= \int_0^t \int_R \exp(-\omega\tau)h_{,i}u_{,i}\,dx\,d\tau + \omega\int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)hS_2(u)\,da\,d\tau \\ &\quad - \int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)h_s S_2(u)\,da\,d\tau + \exp(-\omega t)\int_{\Omega_2^*(0)} hS_2(u)\,da.\end{aligned} \tag{4.4}$$

We see that

$$\begin{aligned}\Phi(0,t) \leq &\left[\int_0^t \int_R \exp(-\omega\tau)h_{,i}h_{,i}dx\,d\tau\right]^{1/2} \left[\int_0^t \int_R \exp(-\omega\tau)u_{,i}u_{,i}dx\,d\tau\right]^{1/2} \\ &+ \omega\left[\int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)h^{q_1}\,da\,d\tau\right]^{1/q_1} \\ &\times \left[\int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)|S_2(u)|^{p_1}\,da\,d\tau\right]^{1/p_1} \\ &+ \left[\int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau\right]^{1/q_1} \\ &\times \left[\int_0^t \int_{\Omega_2^*(0)} \exp(-\omega\tau)|S_2(u)|^{p_1}\,da\,d\tau\right]^{1/p_1} \\ &+ \exp(-\omega t)\left[\int_{\Omega_2^*(0)} h^{q_1}da\right]^{1/q_1} \left[\int_{\Omega_2^*(0)} |S_2(u)|^{p_1}da\right]^{1/p_1},\end{aligned} \tag{4.5}$$

where $q_1^{-1}+p_1^{-1}=1$.

Using the arithmetic-geometric mean inequality and Young's inequality, we find that

$$\begin{aligned}\Phi(0,t) \leq &\frac{1}{2}\int_0^t\int_R \exp(-\omega\tau)u_{,i}u_{,i}\,dx\,d\tau \\ &+\frac{1}{2}\int_0^t\int_R \exp(-\omega\tau)h_{,i}h_{,i}\,dx\,d\tau \\ &+\frac{1}{2}\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)S_1(u)\,da\,d\tau \\ &+\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{\omega^{q_1}}{p_1}\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h^{q_1}\,da\,d\tau \\ &+\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau \\ &+\frac{1}{2}\exp(-\omega_1 t)\int_{\Omega_1^*(0)} S_1(u)\,da \\ &+\left(\frac{2m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\int_{\Omega_1^*(0)} \exp(-\omega t)h^{q_1}\,da.\end{aligned} \tag{4.6}$$

We obtain that

$$\begin{aligned}\Phi(0,t) \leq &\int_0^t\int_R \exp(-\omega\tau)h_{,i}h_{,i}dx\,d\tau \\ &+2\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{\omega^{q_1}}{p_1}\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h^{q_1}\,da\,d\tau \\ &+2\left(\frac{4m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau \\ &+2\left(\frac{2m}{p_1}\right)^{q_1/p_1}\frac{1}{p_1}\exp(-\omega_1 t)\int_{\Omega_1^*(0)} h^{q_1}\,da.\end{aligned} \tag{4.7}$$

We have

$$h_{,1}(x_1,x_2,x_3,\tau) = -bg(x_2,x_3,\tau)\exp(-bx_1), \tag{4.8}$$

$$h_{,\alpha}(x_1,x_2,x_3,\tau) = g_{,\alpha}(x_2,x_3,\tau)\exp(-bx_1), \tag{4.9}$$

$$h_\tau(x_1,x_2,x_3,\tau) = g_\tau(x_2,x_3,\tau)\exp(-bx_1). \tag{4.10}$$

Therefore,

$$\int_0^t\int_R \exp(-\omega\tau)h_{,i}h_{,i}\,dx\,d\tau = \int_0^t\int_{D(0)} \exp(-\omega\tau)\left(\frac{g_{,\alpha}g_{,\alpha}}{2b}+\frac{b}{2}\,g^2\right)da\,d\tau, \tag{4.11}$$

$$\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h^{q_1}\,da\,d\tau = \int_0^t\int_{\partial D(0)} \exp(-\omega\tau)\frac{g^{q_1}}{q_1 b}\,dl\,d\tau, \tag{4.12}$$

$$\int_0^t\int_{\Omega_1^*(0)} \exp(-\omega\tau)h_\tau^{q_1}\,da\,d\tau = \int_0^t\int_{\partial D(0)} \exp(-\omega\tau)\frac{g_\tau^{q_1}}{q_1 b}\,dl\,d\tau \tag{4.13}$$

and

$$\int_{\Omega_1^*(0)} h^{q_1} da = \int_{\partial D(0)} \frac{g^{q_1}}{q_1 b} dl. \tag{4.14}$$

We then obtain

$$\begin{aligned}\Phi(0,t) \le & \int_0^t \int_{D(0)} \exp(-\omega\tau) \left(\frac{g_{,\alpha} g_{,\alpha}}{2b} + \frac{b}{2} g^2 \right) da\, d\tau \\ & + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{\omega^{q_1}}{p_1 q_1 b} \int_0^t \int_{\partial D(0)} \exp(-\omega\tau)|g|^{q_1} dl\, d\tau \\ & + 2\left(\frac{4m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1 q_1 b} \int_0^t \int_{\partial D(0)} \exp(-\omega\tau)|g_{,\tau}|^{q_1} dl\, d\tau \\ & + 2\left(\frac{2m}{p_1}\right)^{q_1/p_1} \frac{1}{p_1 q_1 b} \exp(-\omega t) \int_{\partial D(0)} |g|^{q_1} dl.\end{aligned} \tag{4.15}$$

We can optimize the right hand side of (4.15) with respect to b, but it does seem an easy task.

5. An example

In this section we give an exemple for the boundary conditions satisfying (2.5)–(2.7) and (4.1). We try to illustrate the obtained results.

First we consider the conditions (2.5)–(2.7). We can take, for instance, $f_1(u) = a_1(1-\cos u)u^{-1}$, where $a_1 \ge 0$; $f_2(u) = a_2(1-\cos u)u^{-1} - b_2|u|^k u$, where $a_2 \ge 0$, $b_2 \ge 0$ and $k \ge 0$ and $s(u) = s_1|u|^k$, with $s_1 > 0$. In this case we have that $f_1(u)u = a_1(1-\cos u) \ge 0$ and (2.5) holds. Moreover,

$$S_1(u) = \frac{s_1}{k+2}|u|^{k+2}$$

and condition (2.7) holds. Regard to the condition (2.6), if we take $\bar{C} > 0$ and

$$\omega = \frac{(k+2)b_2}{s_1} + \bar{C},$$

we have

$$\begin{aligned}f_2(u)u + \omega S_1(u) &= a_2(1-\cos u) - b_2|u|^{k+2} + \left[\frac{(k+2)b_2}{s_1} + \bar{C}\right] \frac{s_1}{k+2}|u|^{k+2} \\ &= a_2(1-\cos u) + \frac{s_1\bar{C}}{k+2}|u|^{k+2} \ge C|u|^{k+2} \ge C|u|^{2p},\end{aligned}$$

where $C = \dfrac{s_1\bar{C}}{k+2}$ and $p = \dfrac{k+2}{2}$. So, (2.6) holds.

We now give some explicit values for the parameters obtained in the estimates of Section 3. We assume that Ω_2 is such that mesure$(\Omega_2) = 1$, $b_2 = 0$, $k = 2$, $\bar{C} = 2$ and $s_1 = 1$. Therefore, $\omega = 2$, $p = 2$ and $C = 1/2$. We also consider $\hat{\sigma}_1 = \hat{\sigma}_2 = 1$. With these values, we obtain

$$M_1 = 1, \quad M_2 = t^2, \quad M_3 = 2^{1/4} t \exp(2t)^{1/4}, \quad M_4 = \frac{1}{2} C_1^{1/2} \exp(2t),$$

$$M_5 = 2^{1/4} C_1^{1/2} t \exp(2t)^{1/4}, \quad M_6 = \left(\frac{2}{3}\right)^{3/4} C_1^{1/2} t \exp(2t)^{1/4}.$$

We note that Q_1 is an arbitrary constant, but we do not have any *a priori* bound. Thus, estimates (3.13) and (3.14) hold for

$$\hat{C}_1 = \frac{1}{2} C_1^{1/2} \exp(2t) + \frac{1}{2} C_1^{1/2} \left(\frac{2}{3}\right)^{3/4} t \exp(2t)^{1/4},$$
$$\hat{C}_2 = \frac{1}{3} C_1^{1/2} \exp(2t) + C_1^{1/2} \left(\frac{2}{3}\right)^{3/4} t \exp(2t)^{1/4}, \quad \text{and} \quad \hat{C}_3 = \frac{1}{6} C_1^{1/2} \exp(2t).$$

The proposed example also satisfies condition (4.1). As we pointed out before, we assume that $\Omega_1 = \emptyset$ and $f_2(u) = 0$. When $s(u) = s_1 |u|^k$, with $s_1 > 0$ and $k \geq 0$ condition (4.1) is always satisfied. In fact, in this case we have that

$$|S_2(u)| = \frac{s_1}{k+1} |u|^{k+1}.$$

If we take

$$p_1 = \frac{k+2}{k+1} > 1 \quad \text{and} \quad m = \frac{s_1^{p_1 - 1}(k+2)}{(k+1)^{p_1}} > 0,$$

we get

$$m \frac{s_1}{k+2} |u|^{k+2} \geq \left(\frac{s_1}{k+1}\right)^{p_1} |u|^{(k+1)p_1}$$

and condition (4.1) holds.

6. Extension to the system

In this section we sketch how to extend the Phragmén-Lindelöf alternative to a system of equations related with the heat conduction in a mixture of rigid solids. In place of the equation (2.1) we will consider the system

$$\left.\begin{aligned} k_{11}\Delta u_1 + k_{12}\Delta u_2 - \alpha(u_1 - u_2) = 0, \\ k_{21}\Delta u_1 + k_{22}\Delta u_2 - \alpha(u_1 - u_2) = 0, \end{aligned}\right\} \tag{6.1}$$

where we asume that the matrix

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \tag{6.2}$$

is positive definite and that α is a positive constant.

It is worth recalling that this system determines the temperatures in a mixture of isotropic and homogeneous heat conducting materials (see [5, 10, 13]). To define the boundary conditions, we assume that

$$u_1 - u_2 = 0 \ \text{ on } \ (0, \infty) \times \partial D \times (0, t) \tag{6.3}$$

together with

$$q_i n_i + f_1^*(u_1, u_2) = 0 \ \text{ on } \ [0, \infty) \times \Omega_1 \times (0, t) \tag{6.4}$$

and on Ω_2 we assume that

$$q_i n_i + m_1(u_1, u_2) u_{1,t} + m_2(u_1, u_2) u_{2,t} + f_2^*(u_1, u_2) = 0 \ \text{ on } \ [0, \infty) \times \Omega_2 \times (0, t), \tag{6.5}$$

where

$$q_i = q_i^{(1)} + q_i^{(2)}, \quad q_i^{(1)} = k_{11}u_{1,i} + k_{12}u_{2,i}, \quad q_i^{(2)} = k_{21}u_{1,i} + k_{22}u_{2,i} \tag{6.6}$$

and

$$f_1^*(u,u)u \geq 0, \quad f_2^*(u,u)u + \omega M_1(u) \geq C|u|^{2p}, \tag{6.7}$$

with

$$M_1 = \int_0^u \eta\big[m_1(\eta,\eta) + m_2(\eta,\eta)\big]\,d\eta \geq 0$$

and C a positive constant. In this situation, the analysis starts by considering the function

$$\begin{aligned}\Phi(z,t) = -\int_0^t \int_{D(z)} &\exp(-\omega\tau)\big[\,(k_{11}u_{1,1} + k_{12}u_{2,1})\,u_1 \\ &+ (k_{21}u_{1,2} + k_{22}u_{2,2})\,u_2\big]\,da\,d\tau.\end{aligned} \tag{6.8}$$

We note that, for $z \geq z_0$,

$$\begin{aligned}\Phi(z,t) - \Phi(z_0,t) = &-\int_0^t \int_{z_0}^z \int_D \exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2 + (k_{12}+k_{21})\nabla u_1 \nabla u_2 \\ &+ k_{22}|\nabla u_2|^2 + \alpha(u_1-u_2)^2\big]\,dx\,d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_1} \exp(-\omega\tau) f_1^*(u,u)u\,da\,d\tau \\ &- \int_0^t \int_{z_0}^z \int_{\Omega_2} \exp(-\omega\tau)\,[f_2^*(u,u)u + \omega M_1(u)]\,da\,d\tau \\ &- \int_{z_0}^z \int_{\Omega_2} \exp(-\omega t) M_1(u)\,da.\end{aligned} \tag{6.9}$$

It is worth noting that

$$\begin{aligned}\frac{\partial \Phi}{\partial z} = &-\int_0^t \int_{D(z)} \exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2 + (k_{12}+k_{21})\nabla u_1 \nabla u_2 \\ &+ k_{22}|\nabla u_2|^2 + \alpha(u_1-u_2)^2\big]\,da\,d\tau \\ &- \int_0^t \int_{\Omega_1(z)} \exp(-\omega\tau) f_1^*(u,u)u\,dl\,d\tau \\ &- \int_0^t \int_{\Omega_2(z)} \exp(-\omega\tau)\big[f_2^*(u,u)u + \omega M_1(u)\big]\,dl\,d\tau \\ &- \int_{\Omega_2(z)} \exp(-\omega t) M_1(u)\,dl.\end{aligned} \tag{6.10}$$

An analysis similar to the one proposed at Section 3 allows us to obtain the inequality

$$|\Phi(z,t)| \leq M_4^* \left[-\frac{\partial \Phi}{\partial z}\right] + M_6^* \left[-\frac{\partial \Phi}{\partial z}\right]^{(p+1)/(2p)}, \tag{6.11}$$

where M_4^* and M_6^* are calculable positive constants depending on the parameters of the problem and the time. This estimate allows us to get an alternative of the type (3.13) and (3.14). Therefore, we can obtain a similar result to the one given at Theorem 3.1. To be precise, we can prove that either the function

$$\int_0^t \int_{R(z)} \exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2 + (k_{12}+k_{21})\nabla u_1 \nabla u_2 + k_{22}|\nabla u_2|^2 + \alpha(u_1-u_2)^2\big]\,dx\,d\tau + \int_0^t \int_{\Omega_1^*(z)} \exp(-\omega\tau) f_1^*(u,u)u\,da\,d\tau + \int_0^t \int_{\Omega_2^*(z)} \exp(-\omega\tau)\,[f_2^*(u,u)u + \omega M_1(u)]\,da\,d\tau + \int_{\Omega_2^*(z)} \exp(-\omega t) M_1(u)\,da$$

decays as a polynomial, or the function

$$\int_0^t \int_{z_0}^z \int_D \exp(-\omega\tau)\big[k_{11}|\nabla u_1|^2 + (k_{12}+k_{21})\nabla u_1 \nabla u_2 + k_{22}|\nabla u_2|^2 + \alpha(u_1-u_2)^2\big]\,dx\,d\tau + \int_0^t \int_{z_0}^z \int_{\Omega_1} \exp(-\omega\tau) f_1^*(u,u)u\,da\,d\tau + \int_0^t \int_{z_0}^z \int_{\Omega_2} \exp(-\omega\tau)\,[f_2^*(u,u)u + \omega M_1(u)]\,da\,d\tau + \int_{z_0}^z \int_{\Omega_2} \exp(-\omega t) M_1(u)\,da$$

increases in an exponential way.

Estimates for the amplitude term can be obtained in a similar way as in Section 5.

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