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EXPONENTIAL DECAY FOR A NEUTRAL WAVE EQUATION*

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Abstract A strongly damped wave equation involving a delay of neutral type in its second order derivative is considered. It is proved that solutions decay to zero exponentially despite the fact that delays are, in general, sources of instability.

Keywords Exponential decay, modified energy, multiplier technique, neutral delay, stability.

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1. Introduction

Time-delay systems exist in abundance in engineering [11–13,16]. They appear naturally whenever there is propagation and transport of material and/or information. Delays express the deferring effect in response to an acting force. The rate of change in a delay depends upon former states. Differential equations involving delays are called Functional Differential Equations (FDEs).

In some cases, there is a need to enclose some information about the derivatives evaluated at past time. These cases form a subclass of the class of FDEs called 'Neutral Delay Differential Equations' (NDDES) [2–4, 8, 11–13, 15, 18–20, 22–28]. Some examples are

$$[u(t) - au(t-\tau)]' = f(t, u,\ u(t-\sigma)),$$
$$[u(t) - au(f(t))]' = \triangle u + g(t, u, u(\tau(t)),$$
$$u'(t) = \triangle u + f(t, u, u'(t-\tau)).$$

The study of NDDES is not only of theoretical importance. This type of FDEs appear naturally in many fields such as: Combustion, Control, Ecology and Chemical Reactors [12, 13, 15, 26].

It has been demonstrated for a long time that delay may engender instability [1, 5–7, 10, 14]. Even if the system, originally, was stable, it may become instable when inserting a delay (whatever small it is). This, of course, causes some inconvenience, discomfort and even failure of the system. Despite this irritability by 'small' delays, other NDDEs may be, on the contrary, stabilized by 'large' delays. In fact, neutral delays are sometimes intentionally added to upgrade the performance of the

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structure. Despite this fact, there is a need to find appropriate controls for NDDEs. This is not always an easy task. Different methods have been used and many results have been found. Yet, NDDEs remain not well understood.

Here, we consider the problem

$$\begin{cases} [u(t) - pu(t-\tau)]'' = \triangle u + \triangle u_t \; in \; (0,\infty) \times \Omega, \\ u(t,x) = 0, \; t \in (\tau,\infty), \; x \in \partial\Omega, \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x), \; x \in \Omega, \\ u(t,x) = \varphi(t,x), \; t \in [-\tau, 0], \; x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain of $\mathbb{R}^n$, $u_0(x)$, $u_1(x)$ and $\varphi(t,x)$ are given functions and $p, \tau > 0$. The primes, as well as the subscripts "t", denote time derivatives. We will assume the compatibility condition $\varphi(0,x) = u_0(x)$ and $\varphi_t(0,x) = u_1(x)$, $x \in \Omega$.

Second order NDDEs appear, in general, in the study of vibrating masses attached to an elastic bar and also (as the Euler equation) in some variational problems [12, 13, 15, 26]. The existence and uniqueness of a mild solution for this type of problems has been discussed in [9, 19, 27, 28]. The mild solution is strong or classical when the initial data are regular. This is proved, in fact, for more general abstract problems. The operator A (generalizing the Laplacian, with domain: the set of functions v such that $v, v_t \in H^2(\Omega)$ are absolutely continuous with Dirichlet boundary conditions) is assumed to be an infinitesimal generator of a strongly continuous cosine family of bounded linear operators. The results there are obtained using fixed point theorems. Therefore, we shall assume that the solution (and the initial data) is regular enough to justify our computation.

We brief the reader that there are many other results but for either *ordinary* differential equations (see [2–4, 12, 13, 17, 18, 20, 23–25] and the references therein), or *first* order partial differential equations [21, 22]. The few treated *second* order problems are concerned with delays in the state function or its first derivative rather than in the second derivative (which represents here the real challenge). To the best of our knowledge there are no closely related works to the present one.

It is also worth noting that several investigations appeared dealing with the oscillation phenomena for problems of exactly the same type. We do not report here these references (although one can transform second order equations into first order systems) because of the size of the paper (and to avoid being biased).

In the next section we state and prove our result on the exponential stability. We establish a range of values for the coefficient "p" for which solutions decay to zero exponentially in time.

2. Exponential decay

In this section, we shall use the standard notation for the Lebesgue and Sobolev spaces and their respective norms. We define p_* to be the positive root of

$$16C_p z^2 + 2(7 + 8C_p)z - 3 = 0, \tag{2.1}$$

where C_p is the Poincaré constant.

Our result reads as follows

Theorem 2.1. *Let us assume that* $u_0 \in H^1(\Omega)$, $u_1, \varphi_t(-\tau) \in L^2(\Omega)$ *and* $\varphi_t \in H^1([-\tau, 0]; H^1(\Omega))$. *If* $0 < p < p_*$, *then there exist two positive constants* M *(depending on the initial data) and* α *such that the 'classical' energy satisfies*

$$W(t) := \frac{1}{2}\int_\Omega \{|u_t|^2 + |\nabla u|^2\}\, dx \le Me^{-\alpha t},\ t \ge 0.$$

Proof. Multiplying both sides of the equation in (1.1) by $u_t(t) - p\, u_t(t-\tau)$ and integrating over Ω, we find for $t > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega [u_t(t) - pu_t(t-\tau)]^2\, dx = \int_\Omega [u_t(t) - pu_t(t-\tau)](\triangle u + \triangle u_t)\, dx \\
&= -\int_\Omega [\nabla u_t(t) - p\,\nabla u_t(t-\tau)].(\nabla u + \nabla u_t)(t)\, dx \\
&= -\int_\Omega \nabla u_t.\nabla u\, dx - \|\nabla u_t\|^2 + p\int_\Omega \nabla u(t).\nabla u_t(t-\tau)\, dx \\
&\quad + p\int_\Omega \nabla u_t(t).\nabla u_t(t-\tau)\, dx,
\end{aligned}$$

where $\|.\|$ denotes the L^2-norm. Next, using Young inequality, it appear that for $\delta_1, \delta_2 > 0$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega [u_t(t) - pu_t(t-\tau)]^2\, dx \le &-\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \|\nabla u_t\|^2 + p\delta_1 \|\nabla u\|^2 \\
&+ p\delta_2 \|\nabla u_t\|^2 + \frac{p}{4}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right)\|\nabla u_t(t-\tau)\|^2
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[\|u_t(t) - pu_t(t-\tau)\|^2 + \|\nabla u\|^2\right] \le &-(1 - p\,\delta_2)\|\nabla u_t\|^2 + p\delta_1\|\nabla u\|^2 \\
&+ \frac{p}{4}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right)\|\nabla u_t(t-\tau)\|^2,\ t > 0.
\end{aligned} \tag{2.2}$$

As the derivative of the *energy* functional

$$E(t) := \frac{1}{2}\left[\|u_t(t) - pu_t(t-\tau)\|^2 + \|\nabla u\|^2\right],\ t \ge 0$$

does not give a satisfactory answer regarding its dissipativity, we shall proceed to modify it by the following two functionals

$$\phi(t) := \int_\Omega u[u_t(t) - pu_t(t-\tau)]\, dx,\ t \ge 0$$

and

$$\psi(t) := \int_\Omega e^{-\beta t}\int_{t-\tau}^t e^{\beta(s+\tau)}\,|\nabla u_t(s)|^2\, ds\, dx,\ t \ge 0,\ \beta > 0.$$

Clearly, for $\delta_3, \delta_4 > 0$ and $t \ge 0$, we have

$$\begin{aligned}
\phi'(t) &= \int_\Omega u_t^2\, dx - p\int_\Omega u_t(t-\tau)u_t\, dx - \int_\Omega |\nabla u|^2\, dx - \int_\Omega \nabla u.\nabla u_t\, dx \\
&\le \|u_t\|^2 + p\delta_3\|u_t\|^2 + \frac{p}{4\delta_3}\|u_t(t-\tau)\|^2 - \|\nabla u\|^2 + \delta_4\|\nabla u\|^2 + \frac{1}{4\delta_4}\|\nabla u_t\|^2 \\
&\le (1 + p\,\delta_3)\|u_t\|^2 - (1-\delta_4)\|\nabla u\|^2 + \frac{1}{4\delta_4}\|\nabla u_t\|^2 + \frac{p}{4\delta_3}\|u_t(t-\tau)\|^2.
\end{aligned} \tag{2.3}$$

Similarly, a differentiation of $\psi(t)$, along solutions of our problem, gives

$$\psi'(t) = -\beta\,\psi(t) + e^{\beta\tau}\,\|\nabla u_t\|^2 - \|\nabla u_t(t-\tau)\|^2\,,\ t \geq 0. \tag{2.4}$$

The importance of these two functionals is clear from the appearance of $-\|\nabla u\|^2$ in the derivative of the first functional and $-\|\nabla u_t(t-\tau)\|^2$ in the derivative of the second one.

We shall work now with the *modified* energy functional

$$F(t) := E(t) + \lambda\,\phi(t) + \mu\,\psi(t),\ t \geq 0, \tag{2.5}$$

where λ and μ are two positive constants to be determined later.

From the relations (2.2)–(2.5) we see that the derivative of $F(t)$, along solutions of (1.1), is estimated as follows

$$\begin{aligned} F'(t) \leq &-(1-p\,\delta_2)\,\|\nabla u_t\|^2 + p\,\delta_1\,\|\nabla u\|^2 + \frac{p}{4}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)\|\nabla u_t(t-\tau)\|^2 \\ &+ \lambda(1+p\delta_3)\,\|u_t\|^2 - \lambda(1-\delta_4)\,\|\nabla u\|^2 + \frac{\lambda}{4\delta_4}\,\|\nabla u_t\|^2 \\ &+ \frac{\lambda p}{4\delta_3}\,\|u_t(t-\tau)\|^2 - \beta\mu\,\psi(t) + \mu\,e^{\beta\tau}\,\|\nabla u_t\|^2 - \mu\,\|\nabla u_t(t-\tau)\|^2 \end{aligned}$$

or

$$\begin{aligned} F'(t) \leq &-\left[1 - p\delta_2 - \frac{\lambda}{4\delta_4} - \mu\,e^{\beta\tau}\right]\|\nabla u_t\|^2 + \lambda(1+p\,\delta_3)\,\|u_t\|^2 \\ &-[\lambda(1-\delta_4) - p\,\delta_1]\,\|\nabla u\|^2 + \frac{\lambda p}{4\delta_3}\,\|u_t(t-\tau)\|^2 - \beta\mu\,\psi(t) \\ &-\left[\mu - \frac{p}{4}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)\right]\|\nabla u_t(t-\tau)\|^2\,,\ t>0. \end{aligned} \tag{2.6}$$

By Poincaré inequality, we may infer from (2.6), that

$$\begin{aligned} F'(t) \leq &-\left[1 - p\,\delta_2 - \frac{\lambda}{4\delta_4} - \mu\,e^{\beta\tau} - \lambda\,C_p(1+p\,\delta_3)\right]\|\nabla u_t\|^2 \\ &-[\lambda(1-\delta_4) - p\delta_1]\,\|\nabla u\|^2 - \beta\mu\,\psi(t) \\ &-\left[\mu - \frac{p}{4}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right) - \frac{p\lambda C_p}{4\delta_3}\right]\|\nabla u_t(t-\tau)\|^2\,,\ t>0. \end{aligned} \tag{2.7}$$

We need all the coefficients in (2.7) to be negative

$$\begin{cases} p\,\delta_2 + \dfrac{\lambda}{4\delta_4} + \mu\,e^{\beta\,\tau} + \lambda C_p\,(1+p\,\delta_3) < 1, \\ p\,\delta_1 < \lambda(1-\delta_4), \\ \dfrac{p\lambda C_p}{4\delta_3} + \dfrac{p}{4}\left(\dfrac{1}{\delta_1}+\dfrac{1}{\delta_2}\right) < \mu. \end{cases}$$

Let us pick $\delta_1 = \delta_2 = \frac{3}{2}$, $\delta_3 = 1$, $\delta_4 = \frac{1}{2}$, $\lambda = 4p$ and $\mu = \frac{e^{-\beta\tau}}{4}$. Then, the above conditions hold if

$$16C_p p^2 + 2(7+8C_p)p - 3 < 0.$$

The second condition is covered by this last one, which is satisfied when p is smaller than the positive root of (2.1)

$$16C_p z^2 + 2(7 + 8C_p)z - 3 = 0,$$

which is

$$p_* = \frac{-(7 + 8C_p) + \sqrt{(7 + 8C_p) + 48C_p}}{16C_p}.$$

Notice that we have ignored $e^{-\beta\tau}$ as we can make it as close to "1" as we wish by taking β small enough. Therefore, choosing $0 < p < p_*$ where p_* is the positive root of $16C_p z^2 + 2(7 + 8C_p)z - 3 = 0$ makes all the coefficients in (2.7) negative.

Clearly, for $\delta_5, \delta_6 > 0$, we have

$$\phi(t) \le \delta_5 C_p \|\nabla u\|^2 + \frac{1}{4\delta_5} \|u_t(t) - pu_t(t-\tau)\|^2, \ t > 0$$

and

$$\begin{aligned} \|u_t(t) - pu_t(t-\tau)\|^2 &\le (1 + \delta_6)\|u_t\|^2 + \left(1 + \frac{1}{\delta_6}\right)\|u_t(t-\tau)\|^2 \\ &\le (1 + \delta_6)\|u_t\|^2 + \left(1 + \frac{1}{\delta_6}\right) C_p \|\nabla u_t(t-\tau)\|^2, \ t > 0. \end{aligned}$$

These two relations show two things. First, their insertion in the right hand side of (2.7), that is adding and subtracting them with small coefficients will lead to

$$F'(t) \le -C_1 F(t), \ t > 0, \tag{2.8}$$

for some $C_1 > 0$. Second, they show that $F(t)$ is equivalent to $E(t) + \mu\,\psi(t)$. We deduce from (2.8), that

$$F(t) \le F(0)\, e^{-C_1 t}, \ t \ge 0, \tag{2.9}$$

where

$$\begin{aligned} F(0) =& \frac{1}{2}\{\|u_1 - p\varphi_t(-\tau)\|^2 + \|\nabla u_0\|^2\} \\ &+ \lambda \int_\Omega u_0[u_1 - p\varphi_t(-\tau)]\,dx + \mu \int_\Omega \int_{-\tau}^0 e^{\beta(s+\tau)} |\nabla\varphi_t(s)|^2\,ds\,dx. \end{aligned}$$

Notice that $F(0)$ is nonnegative and is well-defined as $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\varphi_t \in H^1([-\tau, 0]; H^1(\Omega))$ and $\varphi_t(-\tau) \in L^2(\Omega)$. Moreover, in view of the definition of $F(t)$ and (2.9), it is easy to see that

$$\|u_t(t) - pu_t(t-\tau)\|^2 \le 2F(0)\, e^{-C_1 t}, \ t \ge 0$$

and

$$\begin{aligned} \|u_t\| &\le \|u_t(t) - pu_t(t-\tau)\| + p\,\|u_t(t-\tau)\| \\ &\le \sqrt{2F(0)}\, e^{-\frac{C_1 t}{2}} + p\,\|u_t(t-\tau)\|, \ t \ge 0. \end{aligned} \tag{2.10}$$

Now, we need to pass to $W(t)$. To this end we must estimate $\|u_t\|$ ($\|\nabla u\|$ being already there in the definitions of $F(t)$ and $E(t)$). We claim that

$$\|u_t\| < Ae^{-\frac{C_1 t}{2}} + e^{-C_2 t} \sup_{-\tau \le \sigma \le 0} \|u_t(\sigma)\|, \ t \ge 0, \tag{2.11}$$

for some $A, C_2 > 0$ such that $A \geq \sqrt{2F(0)}$ to be determined.

For $t = 0$, as $p < 1$

$$\|u_t(0)\| \leq \sqrt{2F(0)} + p\,\|u_t(-\tau)\| < A + \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,.$$

That is, the claim is valid for $t = 0$. Let us suppose, for contradiction, that there exists a $t_* > 0$ such that

$$\|u_t(t_*)\| = Ae^{-\frac{C_1 t_*}{2}} + e^{-C_2 t_*} \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\| \tag{2.12}$$

and

$$\|u_t(t)\| < Ae^{-\frac{C_1 t}{2}} + e^{-C_2 t} \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,,\ t \in [0, t_*).$$

If $t_* < \tau$, then

$$\|u_t(t_* - \tau)\| = \|\varphi_t(t_* - \tau)\| \leq \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,.$$

Therefore, if $pe^{C_2 t_*} < 1$,

$$\begin{aligned}\|u_t(t_*)\| &= \sqrt{2F(0)}e^{-\frac{C_1 t_*}{2}} + p\,\|u_t(t_* - \tau)\| \\ &\leq Ae^{-\frac{C_1 t_*}{2}} + pe^{-C_2 t_*}e^{C_2 t_*} \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\| \\ &< Ae^{-\frac{C_1 t_*}{2}} + e^{-C_2 t_*} \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,,\end{aligned}$$

which contradicts (2.12).

In case $t_* \geq \tau$, the relation (2.10) at t_* gives

$$\begin{aligned}\|u_t(t_*)\| &\leq \sqrt{2F(0)}\,e^{-\frac{C_1 t_*}{2}} + p\,\|u_t(t_* - \tau)\| \\ &< \sqrt{2F(0)}\,e^{-\frac{C_1 t_*}{2}} + p\{Ae^{-\frac{C_1(t_*-\tau)}{2}} + e^{-C_2(t_*-\tau)} \sup\nolimits_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\} \\ &= (\sqrt{2F(0)} + pAe^{\frac{C_1\tau}{2}})\,e^{-\frac{C_1 t_*}{2}} + pe^{C_2\,\tau}e^{-C_2 t_*} \sup\nolimits_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,.\end{aligned}$$

If $pe^{C_2\tau} \leq 1$ and

$$\sqrt{2F(0)} + pA\,e^{\frac{C_1\tau}{2}} \leq A,$$

that is $A \geq \sqrt{2F(0)}\,(1 - pe^{\frac{C_1\tau}{2}})^{-1}$ with $pe^{C_1\tau/2} < 1$, then

$$\|u_t(t_*)\| < Ae^{-\frac{C_1 t_*}{2}} + e^{-C_2 t_*} \sup_{-\tau\leq\sigma\leq 0} \|u_t(\sigma)\|\,.$$

Notice that we may assume *wlog* that $pe^{\frac{C_1}{2}\,\tau} < 1$ is satisfied, for otherwise, as $p < 1$, we may consider $C_3 < \frac{C_1}{2}$ so small that $pe^{C_3\tau} < 1$. This contradicts (2.12) and proves (2.11). The proof of the theorem is complete with $\alpha = \min\{C_1, 2C_2\}$. □

Remark 2.1. The range determined here is not claimed to be optimal. Actually, we hope that this work will serve as a basis for future investigations. It would be nice to find an optimal range, that is a critical value for p or at least improve this range.

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