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EXISTENCE OF THE GLOBAL SMOOTH SOLUTION TO A FRACTIONAL NONLINEAR SCHRÖDINGER SYSTEM IN ATOMIC BOSE-EINSTEIN CONDENSATES

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Abstract In this paper, the fractional nonlinear Schrödinger equations for atomic Bose-Einstein condensates are studied. By using the Galërkin method and *a priori* estimates, the existence and uniqueness of global smooth solution are obtained.

Keywords Fractional Schrödinger equations, the Galërkin method, *a priori* estimates, global smooth solution

MSC(2010) 35K15, 35K55.

1. Introduction

In this paper we consider the following fractional nonlinear coupled Schrödinger system [17]

$$\begin{cases} i\hbar u_t = \left(\frac{\hbar^2}{2M}(-\Delta)^\alpha + \lambda_u|u|^2 + \lambda|v|^2\right) u + \sqrt{2}\beta\overline{u}v, \\ i\hbar v_t = \left(\frac{\hbar^2}{4M}(-\Delta)^\alpha + \varepsilon + \lambda_v|v|^2 + \lambda|u|^2\right) v + \frac{\beta}{\sqrt{2}}u^2, \end{cases} \tag{1.1}$$

with the initial condition and periodic boundary condition

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), x \in \Omega, \tag{1.2}$$

$$u(x+2L,t) = u(x,t), \quad v(x+2L,t) = v(x,t), \quad x \in \Omega, \quad t \geq 0, \tag{1.3}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$, $\frac{1}{2} < \alpha < 1$, $i = \sqrt{-1}$, $L > 0$, $\Omega = (-L, L)$, $\hbar$ is Planck constant, $M > 0$ is the mass of a single atom, $\lambda_u, \lambda_v, \lambda$ represent the strengths of the atom-atom, molecule-molecule and atom-molecule interactions, respectively and ε, β are any real constants.

Nonlinear Schrödinger equations have been used to analyze several physical situations, and have attracted the attention of researchers, especially in optics and hydrodynamics. In optics, systems of coupled nonlinear equations can be used to describe the propagation of light along birefringent optical fibers [11].

Fractional differential equations are extensively used in modeling phenomena in various fields of science and engineering [3]. Rida etal. [15] have studied nonlinear Schrödinger equation of fractional order. The investigation of the exact solutions of nonlinear evolution equations play an important role in the study of nonlinear

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physical phenomena. Then many authors have considered the fractional nonlinear Schrödinger equation. In 2008, Guo boling, Han yongqian and Xin jie [9] proved the existence and uniqueness of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation by using energy method. In 2011, Jiaqian Hu, Jie Xin , Hong Lu [11] considered a class of systems of fractional nonlinear Schrödinger equations. They proved the existence and uniqueness of the global solution to the periodic boundary value problem by using the Galërkin method. Further discussion can be found in Refs [2, 10, 12, 13, 16].

As far as we know, the fractional nonlinear Schrödinger system (1.1) has not yet been fully studied. In this paper, we prove the existence and uniqueness of the global solution to the periodic boundary value problem for a class of system of fractional nonlinear Schrödinger equations by using the Galërkin method.

Before starting the main results, we review the notations and the calculus inequalities used in this paper.

To simplify the notation in this paper, we shall denote by $\int U(x)dx$ the integration $\int_\Omega U(x)dx$, C is a generic constant and may assume different values in different formulates. And denote $L^p = L^p(\Omega)$ be the Banach space endowed with the norm $\|\cdot\|_{L^p}$, when $p = 2$, $L^2(\Omega)$ denote the Hilbert space with the usual scalar product $(\cdot,\cdot)$. Here, (u, v) denotes the integral $\int uvdx$ as usual.

The Fourier transform $\hat{f} = F(f)$ of a tempered distribution $f(x)$ on $\mathbb{R}^d$ is defined as

$$F(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx,$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$. For $\forall\alpha \in \mathbb{R}$, $(-\Delta)^{\frac{\alpha}{2}} f$ can be defined as

$$F((-\Delta)^{\frac{\alpha}{2}} f)(\xi) = |\xi|^\alpha \hat{f}(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^\alpha f(x)e^{-ix\cdot\xi}dx.$$

Using the Fourier inverse transform, $(-\Delta)^{\frac{\alpha}{2}} f$ can be denoted as

$$(-\Delta)^{\frac{\alpha}{2}} f = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^\alpha \hat{f}(\xi)e^{ix\cdot\xi}d\xi.$$

Then the Sobolev space H^α is

$$H^\alpha = H^\alpha(\mathbb{R}^d) = \left\{f \in S'(\mathbb{R}^d : \hat{f}) \text{ is a function and } \|f\|^2_{H^\alpha} < \infty\right\},$$

where

$$\|f\|^2_{H^\alpha} := \int_{\mathbb{R}^d}(1 + |\xi|^2)^\alpha|\hat{f}(\xi)|^2 d\xi < \infty.$$

Under this definition, it is clear to see that H^α is a Banach space. And if φ and ϕ belong to H^α, combining the Parseval's identity we conclude the following equation

$$\int_{\mathbb{R}^d}(-\Delta)^\alpha\varphi \cdot \phi dx = \int_{\mathbb{R}^d}(-\Delta)^{\alpha_1}\varphi \cdot (-\Delta)^{\alpha_2}\phi dx,$$

where α_1 and α_2 are nonnegative and $\alpha_1 + \alpha_2 = \alpha$.

These concepts can be easily generalized to the periodic case, and we make no explicit distinctions about the notations for the two situations.

The following auxiliary lemmas will be needed.

Lemma 1.1. *(The Gagliardo-Nirenberg inequality) Assuming $u \in L^q(\mathbb{R})$, $\partial_x^m u \in L^r(\mathbb{R})$, $1 \le q, r \le \infty$. Let p and α satisfy*

$$\frac{1}{p} = j + \theta(\frac{1}{r} - m) + (1-\theta)\frac{1}{q}; \quad \frac{j}{m} \le \theta \le 1.$$

Then

$$\|\partial_x^j u\|_p \le C(p, m, j, q, r)\|\partial_x^m u\|_r^\theta \|u\|_q^{1-\theta}. \tag{1.4}$$

In particular, as $m = \alpha, j = 0, p = 4, r = 2, q = 2$, we have

$$\|u\|_4^4 \le C\|(-\Delta)^{\frac{\alpha}{2}} u\|_2^{\frac{1}{\theta}} \|u\|_2^{4-\frac{1}{\theta}}. \tag{1.5}$$

Lemma 1.2. *(The Gronwall inequality) Let c be a constant, and $b(t), u(t)$ be non-negative continuous functions in the interval [0,T] satisfying*

$$u(t) \le c + \int_0^t b(\tau)u(\tau)d\tau, \qquad t \in [0, T].$$

Then $u(t)$ satisfies the estimate

$$u(t) \le c\exp\left(\int_0^t b(\tau)d\tau\right), \qquad \text{for} \quad t \in [0, T]. \tag{1.6}$$

Theorem 1.1. *Let $u_0(x) \in H^\alpha_{per}(\Omega), v_0(x) \in H^\alpha_{per}(\Omega)$, and $0 < \alpha < 1$. Then for $\forall T > 0$, the system (1.1)–(1.3) has a global weak solution*

$$(u, v) \in L^\infty\left([0, T); H^\alpha_{per}(\Omega)\right)^2, \qquad (u_t, v_t) \in L^\infty\left([0, T); H^{-\alpha}_{per}(\Omega)\right)^2. \tag{1.7}$$

Theorem 1.2. *Let $u_0(x) \in H^{4\alpha}_{per}(\Omega), v_0(x) \in H^{4\alpha}_{per}(\Omega)$, and $\frac{1}{2} < \alpha < 1$. Then for $\forall T > 0$, the system (1.1)–(1.3) has a uniquely global smooth solution*

$$(u, v) \in L^\infty\left([0, T); H^{4\alpha}_{per}(\Omega)\right)^2, \qquad (u_t, v_t) \in L^\infty\left([0, T); H^{2\alpha}_{per}(\Omega)\right)^2. \tag{1.8}$$

Theorem 1.3. *Let $u_0(x) \in H^{4\alpha}_{per}(\mathbb{R}), v_0(x) \in H^{4\alpha}_{per}(\mathbb{R})$, and $\frac{1}{2} < \alpha < 1$. Then the system (1.1)–(1.3) has a uniquely global smooth solution*

$$(u, v) \in L^\infty\left([0, \infty); H^{4\alpha}_{per}(\mathbb{R})\right)^2, \qquad (u_t, v_t) \in L^\infty\left([0, \infty); H^{2\alpha}_{per}(\mathbb{R})\right)^2. \tag{1.9}$$

2. *A priori* estimates

In this section, we give the demonstration of *a priori* estimates that guarantee the existence of the global smooth solution of the system (1.1)-(1.3).

Lemma 2.1. *Let $u_0(x) \in L^2(\Omega), v_0(x) \in L^2(\Omega)$ and (u, v) be a solution of the system (1.1) with initial data (u_0, v_0), then we have the identity*

$$\|u(x, t)\|_2^2 + 2\|v(x, t)\|_2^2 \equiv \|u_0(x)\|_2^2 + 2\|v_0(x)\|_2^2. \tag{2.1}$$

Proof. Taking the inner product for the first equation of the system (1.1) with $\overline{u}$ and the second equation with $\overline{v}$, respectively, and integrating the resulting equations with respect to x on Ω, and then taking the imaginary part of the resulting equations, we obtain

$$\begin{cases} \frac{\hbar}{2}\frac{d}{dt}\|u\|_2^2 = \sqrt{2}\beta \mathrm{Im}\int (\overline{u})^2 v dx, \\ \frac{\hbar}{2}\frac{d}{dt}\|v\|_2^2 = \frac{\beta}{\sqrt{2}}\mathrm{Im}\int u^2\overline{v}dx. \end{cases} \tag{2.2}$$

□

Multiplying the second equation of the system (2.2) by 2 and then sum up the first equation, it follows that

$$\frac{\hbar}{2}\frac{d}{dt}\|u\|_2^2 + \hbar\frac{d}{dt}\|v\|_2^2 = 0,$$

which implies the identity (2.1).

This completes the proof of Lemma 2.1.

Lemma 2.2. *Let* $u_0 \in H^\alpha_{per}(\Omega), v_0 \in H^\alpha_{per}(\Omega)$, $0 < \alpha < 1$, *then for the solution* (u, v) *of the system (1.1), we can get*

$$\sup_{0\le t\le\infty}(\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}v\|_2^2) \le C, \tag{2.3}$$

where C *is a constant depending only on* $\|u_0\|_{H^\alpha_{per}}, \|v_0\|_{H^\alpha_{per}}$.

Proof. The inner product is taken to the first equation of the system (1.1) with $\overline{u}_t$ and the second equation with $\overline{v}_t$, and then integrating and taking the real part of the resulting equations, we get

$$\begin{cases} 0 = \frac{\hbar^2}{4M}\frac{d}{dt}\int |(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{\lambda_u}{4}\frac{d}{dt}\int |u|^4dx + \lambda\mathrm{Re}\int |v|^2u\overline{u}_tdx + \sqrt{2}\beta\mathrm{Re}\int \overline{u}v\overline{u}_tdx, \\ 0 = \frac{\hbar^2}{8M}\frac{d}{dt}\int |(-\Delta)^{\frac{\alpha}{2}}v|^2dx + \frac{\varepsilon}{2}\frac{d}{dt}\int |v|^2dx + \frac{\lambda_v}{4}\frac{d}{dt}\int |v|^4dx + \lambda\mathrm{Re}\int |u|^2v\overline{v}_tdx \\ \quad + \frac{\beta}{\sqrt{2}}\mathrm{Re}\int u^2\overline{v}_tdx. \end{cases} \tag{2.4}$$

Summing up the two equations of the system (2.4), we have

$$\frac{\hbar^2}{4M}\frac{d}{dt}\left(\int |(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{1}{2}\int |(-\Delta)^{\frac{\alpha}{2}}v|^2dx\right) + \frac{1}{4}\frac{d}{dt}\left(\lambda_u\int |u|^4dx + \lambda_v\int |v|^4dx\right)$$
$$+ \frac{\varepsilon}{2}\frac{d}{dt}\int |v|^2dx + \frac{\lambda}{2}\frac{d}{dt}\int |u|^2|v|^2dx + \frac{\beta}{\sqrt{2}}\mathrm{Re}\frac{d}{dt}\int u^2\overline{v}dx = 0.$$

Let

$$I := \frac{\hbar^2}{4M}\left(\int |(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{1}{2}\int |(-\Delta)^{\frac{\alpha}{2}}v|^2dx\right),$$
$$II := \frac{1}{4}\left(\lambda_u\int |u|^4dx + \lambda_v\int |v|^4dx\right),$$
$$III := \frac{\lambda}{2}\int |u|^2|v|^2dx, \qquad IV := \frac{\varepsilon}{2}\int |v|^2dx, \qquad V := \frac{\beta}{\sqrt{2}}\mathrm{Re}\int u^2\overline{v}dx.$$

Then

$$E(t) = I + II + III + IV + V \equiv E(0). \tag{2.5}$$

Applying Lemma 1.1 and the Young inequality, we have

$$\|u\|_4^4 \le C\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^{\frac{1}{\theta}}\|u\|_2^{4-\frac{1}{\theta}} \le \delta\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + C_1\|u\|_2^{\frac{2(4\theta-1)}{2\theta-1}}, \tag{2.6}$$

$$\|v\|_4^4 \le C\|(-\Delta)^{\frac{\alpha}{2}}v\|_2^{\frac{1}{\theta}}\|v\|_2^{4-\frac{1}{\theta}} \le \delta\|(-\Delta)^{\frac{\alpha}{2}}v\|_2^2 + C_1\|v\|_2^{\frac{2(4\theta-1)}{2\theta-1}}. \tag{2.7}$$

Then we can bound the term II by

$$|II| \le \delta\left(\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2\right) + C\left(\|u\|_2^{\frac{2(4\theta-1)}{2\theta-1}} + \|v\|_2^{\frac{2(4\theta-1)}{2\theta-1}}\right). \tag{2.8}$$

For the term III, using the Hölder's inequality

$$\frac{\lambda}{2}\int |u|^2|v|^2dx \le \frac{\lambda}{4}\left(\|u\|_4^4 + \|v\|_4^4\right). \tag{2.9}$$

Combining the inequalities (2.6) and (2.7), the term III can be bounded by

$$|III| \le \delta\left(\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2\right) + C\left(\|u\|_2^{\frac{2(4\theta-1)}{2\theta-1}} + \|v\|_2^{\frac{2(4\theta-1)}{2\theta-1}}\right). \tag{2.10}$$

The term

$$V = \frac{\beta}{\sqrt{2}}\mathrm{Re}\int u^2\overline{v}dx \le \frac{\beta}{\sqrt{2}}\int |u|^2|v|dx \le C\|v\|_2\|u\|_4^2. \tag{2.11}$$

By applying the inequality (2.6) and Lemma 2.1 to yield

$$|V| \le \delta\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + C\|u\|_2^{\frac{2(4\theta-1)}{2\theta-1}}. \tag{2.12}$$

Using the estimates of the term II, III and V, we deduce

$$|II| + |III| + |V| \le \delta\left(\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2\right) + C\left(\|u\|_2^{\frac{2(4\theta-1)}{2\theta-1}} + \|v\|_2^{\frac{2(4\theta-1)}{2\theta-1}}\right). \tag{2.13}$$

In view of Lemma 2.1, it follows that

$$IV = \frac{\varepsilon}{2}\|v\|_2^2 \le C. \tag{2.14}$$

Combining the estimates (2.13) and (2.14)

$$I - \delta\left(\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}v\|_2^2\right) \le C,$$

a.e.

$$\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}}v\|_2^2 \le C,$$

where C is a constant depending only on $\|u_0\|_{H^\alpha_{per}}, \|v_0\|_{H^\alpha_{per}}$.

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let T be any positive number, $u_0 \in H^{2\alpha}_{per}(\Omega), v_0 \in H^{2\alpha}_{per}(\Omega)$, for $\frac{1}{2} < \alpha < 1$. Then the solution (u, v) satisfies the following estimate*

$$\sup_{0\le t\le T}\left(\|(-\Delta)^{\alpha}u\|_2^2 + \|(-\Delta)^{\alpha}v\|_2^2\right) \le C, \qquad \forall T > 0, \tag{2.15}$$

where the constant C depends only on T and $\|u_0\|_{H^{2\alpha}_{per}}, \|v_0\|_{H^{2\alpha}_{per}}$.

Proof. Differentiate (1.1) with respect to t, multiply the first equation of the system (1.1) by $\overline{u}_t$, the second equation by $\overline{v}_t$, and then integrate with respect to x , take the imaginary part to get

$$\frac{\hbar}{2}\frac{d}{dt}\|u_t\|_2^2 = \mathrm{Im}\left(\frac{d}{dt}(\lambda_u|u|^2u), \overline{u}_t\right) + \mathrm{Im}\left(\frac{d}{dt}(\lambda|v|^2u), \overline{u}_t\right) + \mathrm{Im}\left(\frac{d}{dt}(\sqrt{2}\beta\overline{u}v), \overline{u}_t\right),$$

$$\frac{\hbar}{2}\frac{d}{dt}\|v_t\|_2^2 = \mathrm{Im}\left(\frac{d}{dt}(\lambda|u|^2v), \overline{v}_t\right) + \mathrm{Im}\left(\frac{d}{dt}(\lambda_v|v|^2v), \overline{v}_t\right) + \mathrm{Im}\left(\frac{d}{dt}(\frac{\beta}{\sqrt{2}}u^2), \overline{v}_t\right).$$

But

$$\begin{aligned}
\mathrm{Im}\left(\frac{d}{dt}(\lambda_u|u|^2u), \overline{u}_t\right) &= \mathrm{Im}\int\frac{d}{dt}(\lambda_u|u|^2u)\overline{u}_t dx = \lambda_u\mathrm{Im}\int u^2\overline{u}_t^2 dx,\\
\mathrm{Im}\left(\frac{d}{dt}(\lambda|v|^2u), \overline{u}_t\right) &= \mathrm{Im}\int\frac{d}{dt}(\lambda|v|^2u)\overline{u}_t dx = \lambda\mathrm{Im}\int |v|_t^2 u\overline{u}_t dx,\\
\mathrm{Im}\left(\frac{d}{dt}(\sqrt{2}\beta\overline{u}v), \overline{u}_t\right) &= \mathrm{Im}\int\frac{d}{dt}(\sqrt{2}\beta\overline{u}v)\overline{u}_t dx\\
&= \sqrt{2}\beta\mathrm{Im}\int\overline{u}_t^2 v dx + \sqrt{2}\beta\mathrm{Im}\int\overline{u}v_t\overline{u}_t dx.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{Im}\left(\frac{d}{dt}(\lambda_v|v|^2v), \overline{v}_t\right) &= \mathrm{Im}\int\frac{d}{dt}(\lambda_v|v|^2v)\overline{v}_t dx = \lambda_v\mathrm{Im}\int v^2\overline{v}_t^2 dx,\\
\mathrm{Im}\left(\frac{d}{dt}(\lambda|u|^2v), \overline{v}_t\right) &= \mathrm{Im}\int\frac{d}{dt}(\lambda|u|^2v)\overline{v}_t dx = \lambda\mathrm{Im}\int |u|_t^2 v\overline{v}_t dx,\\
\mathrm{Im}\left(\frac{d}{dt}\left(\frac{\beta}{\sqrt{2}}u^2\right), \overline{v}_t\right) &= \mathrm{Im}\int\frac{d}{dt}\left(\frac{\beta}{\sqrt{2}}u^2\right)\overline{v}_t dx = \sqrt{2}\beta\mathrm{Im}\int uu_t\overline{v}_t dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\hbar}{2}\frac{d}{dt}\|u_t\|_2^2 &= \lambda_u\mathrm{Im}\int u^2\overline{u}_t^2 dx + \lambda\mathrm{Im}\int |v|_t^2 u\overline{u}_t dx + \sqrt{2}\beta\mathrm{Im}\int\overline{u}_t^2 v dx\\
&\quad + \sqrt{2}\beta\mathrm{Im}\int\overline{u}v_t\overline{u}_t dx,\\
\frac{\hbar}{2}\frac{d}{dt}\|v_t\|_2^2 &= \lambda_v\mathrm{Im}\int v^2\overline{v}_t^2 dx + \lambda\mathrm{Im}\int |u|_t^2 v\overline{v}_t dx + \sqrt{2}\beta\mathrm{Im}\int uu_t\overline{v}_t dx.
\end{aligned}$$

Integrating the above two equality from 0 to t, we have

$$\begin{aligned}
&\frac{\hbar}{2}(\|u_t\|_2^2 + \|v_t\|_2^2)\\
=&\int_0^t\mathrm{Im}\int\lambda_u u^2\overline{u}_t^2 dxds + \int_0^t\mathrm{Im}\int\lambda_v v^2\overline{v}_t^2 dxds + 2\int_0^t\mathrm{Im}\int\lambda uv\overline{u}_t\overline{v}_t dxds\\
&+\int_0^t\mathrm{Im}\int\sqrt{2}\beta(\overline{u}_t^2 v + \overline{u}v_t\overline{u}_t + uu_t\overline{v}_t)dxds + \frac{\hbar}{2}(\|u_t(x,0)\|_2^2 + \|v_t(x,0)\|_2^2)\\
\leq& C_1\left(\int_0^t\int|u|^2|u_t|^2 dxds + \int_0^t\int|v|^2|v_t|^2 dxds + \int_0^t\int|u||v||u_t||v_t|dxds\right)\\
&+C_2\Big(\int_0^t\int|u_t|^2|v|dxds + 2\int_0^t\int|u||u_t||v_t|dxds\Big) + \frac{\hbar}{2}(\|u_t(x,0)\|_2^2 + \|v_t(x,0)\|_2^2).
\end{aligned}$$

Applying Sobolev embedding inequality $\|u\|_\infty \le C\|u\|_{H^\alpha} \le C_1, \quad (\alpha > \frac{1}{2})$, we have

$$\begin{aligned}
&\frac{\hbar}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) \\
\le & C\left(\int_0^t \|u\|_\infty^2 \|u_t\|_2^2 ds + \int_0^t \|v\|_\infty^2 \|v_t\|_2^2 ds + \int_0^t \|u\|_\infty \|v\|_\infty \|u_t\|_2 \|v_t\|_2 ds\right) \\
&+ C_1\big(\int_0^t \|v\|_\infty \|u_t\|_2^2 ds + 2\int_0^t \|u\|_\infty \|u_t\|_2 \|v_t\|_2 ds\big) + \frac{\hbar}{2}\left(\|u_t(x,0)\|_2^2 + \|v_t(x,0)\|_2^2\right) \\
\le & C\left(\int_0^t \|u_t\|_2^2 ds + \int_0^t \|v_t\|_2^2 ds\right) + \frac{\hbar}{2}\left(\|u_t(x,0)\|_2^2 + \|v_t(x,0)\|_2^2\right)
\end{aligned}$$

In term of Gronwall inequality, we deduce that

$$\|u_t\|_2^2 + \|v_t\|_2^2 \le C. \tag{2.16}$$

Applying the system (1.1), we obtain

$$\begin{aligned}
&\|\frac{\hbar^2}{2M}(-\Delta)^\alpha u\|_2^2 \le \|\hbar u_t\|_2^2 + \|\lambda_u|u|^2 u\|_2^2 + \|\lambda|v|^2 u\|_2^2 + \|\sqrt{2}\beta\overline{u}v\|_2^2, \\
&\|\frac{\hbar^2}{4M}(-\Delta)^\alpha v\|_2^2 \le \|\hbar v_t\|_2^2 + \|\varepsilon v\|_2^2 + \|\lambda_v|v|^2 v\|_2^2 + \|\lambda|u|^2 v\|_2^2 + \|\frac{\beta}{\sqrt{2}}u^2\|_2^2.
\end{aligned}$$

Using the inequality (2.16),

$$\begin{aligned}
&\|(-\Delta)^\alpha u\|_2^2 \le C + C_1\|u\|_\infty^4 \|u\|_2^2 + C_2\|v\|_\infty^4 \|u\|_2^2 + C_3\|u\|_\infty \|v\|_\infty \|u\|_2^2 \|v\|_2^2 \le C_4, \\
&\|(-\Delta)^\alpha v\|_2^2 \le C + C_1\|v\|_\infty^4 \|v\|_2^2 + C_2\|u\|_\infty^4 \|v\|_2^2 + C_3\|u\|_\infty^2 \|u\|_2^2 \le C_4,
\end{aligned}$$

a.e.

$$\|(-\Delta)^\alpha u\|_2^2 + \|(-\Delta)^\alpha v\|_2^2 \le C,$$

where the constant C depends only on T and $\|u_0\|_{H^{2\alpha}_{per}}, \|v_0\|_{H^{2\alpha}_{per}}$.

This completes the proof of Lemma2.3. □

Lemma 2.4. *Let $\frac{1}{2} < \alpha < 1$, $u_0 \in H^{3\alpha}_{per}(\Omega), v_0 \in H^{3\alpha}_{per}(\Omega)$, and (u, v) be the solution of the system (1.1). Then*

$$\sup_{0\le t\le\infty}\left(\|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2 + \|(-\Delta)^{\frac{\alpha}{2}} v_t\|_2^2\right) \le C, \tag{2.17}$$

where the constant C depends only on $\|u_0\|_{H^{3\alpha}_{per}}, \|v_0\|_{H^{3\alpha}_{per}}$.

Proof. Differentiate (1.1) with respect to t two times, multiply the first equation of the system (1.1) by $\overline{u}_t$, the second equation by $\overline{v}_t$, and then integrate with respect to x , take the imaginary part to get

$$\begin{aligned}
&\frac{\hbar}{2}\|u_{tt}\|_2^2 = \mathrm{Im}(\frac{d^2}{dt^2}(\lambda_u|u|^2 u), \overline{u}_{tt}) + \mathrm{Im}(\frac{d^2}{dt^2}(\lambda|v|^2 u), \overline{u}_{tt}) + \mathrm{Im}(\frac{d^2}{dt^2}(\sqrt{2}\beta\overline{u}v), \overline{u}_{tt}), \\
&\frac{\hbar}{2}\|v_{tt}\|_2^2 = \mathrm{Im}(\frac{d^2}{dt^2}(\lambda_v|v|^2 v), \overline{v}_{tt}) + \mathrm{Im}(\frac{d^2}{dt^2}(\lambda|u|^2 v), \overline{v}_{tt}) + \mathrm{Im}(\frac{d^2}{dt^2}(\frac{\beta}{\sqrt{2}}u^2), \overline{v}_{tt}).
\end{aligned}$$

But applying Sobolev embedding theorem and the Young inequality, we get

$$\begin{aligned}
&\mathrm{Im}(\frac{d^2}{dt^2}(\lambda_u|u|^2u),\overline{u}_{tt}) = \mathrm{Im}\left(\lambda_u|u|^2_{tt}u + 2\lambda_u|u|^2_t u_t,\overline{u}_{tt}\right) \le C\|u_{tt}\|_2^2 + \delta\|u_t\|_4^4,\\
&\mathrm{Im}(\frac{d^2}{dt^2}(\lambda|v|^2u),\overline{u}_{tt}) = \mathrm{Im}\left(\lambda|v|^2_{tt}u + 2\lambda|v|^2_t u_t,\overline{u}_{tt}\right)\\
&\qquad\qquad \le C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + \delta(\|u_t\|_4^4 + \|v_t\|_4^4),\\
&\mathrm{Im}(\frac{d^2}{dt^2}(\sqrt{2}\beta\overline{u}v),\overline{u}_{tt}) = \mathrm{Im}(\sqrt{2}\beta(\overline{u}_{tt}v + \overline{u}_t v_t + 2\overline{u}_t v_t),\overline{u}_{tt})\\
&\qquad\qquad \le C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + \delta(\|u_t\|_4^4 + \|v_t\|_4^4),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}(\frac{d^2}{dt^2}(\lambda_v|v|^2v),\overline{v}_{tt}) = \mathrm{Im}\left(\lambda_v|v|^2_{tt}v + 2\lambda_v|v|^2_t v_t,\overline{v}_{tt}\right) \le C\|v_{tt}\|_2^2 + \delta\|v_t\|_4^4,\\
&\mathrm{Im}(\frac{d^2}{dt^2}(\lambda|u|^2v),\overline{v}_{tt}) = \mathrm{Im}\left(\lambda|u|^2_{tt}v + 2\lambda|u|^2_t v_t,\overline{v}_{tt}\right)\\
&\qquad\qquad \le C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + \delta(\|u_t\|_4^4 + \|v_t\|_4^4),\\
&\mathrm{Im}(\frac{d^2}{dt^2}(\frac{\beta}{\sqrt{2}}u^2),\overline{v}_{tt}) = \mathrm{Im}(\sqrt{2}\beta(u_{tt}u + u_t^2),\overline{v}_{tt}) \le C\|u_t\|_4^2\|v_{tt}\|_2 + \|u_{tt}\|_2\|v_{tt}\|_2.
\end{aligned}$$

Taking the above inequality to obtain

$$\frac{d}{dt}(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) \le C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + \delta(\|u_t\|_4^4 + \|v_t\|_4^4). \tag{2.18}$$

Let $\theta = \frac{1}{8\alpha} < \frac{1}{4}$. Then using Gagliardo–Nirenberg inequality and the inequality (2.16), we have

$$\|u_t\|_4 \le C\|u_t\|^{1-\theta}\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^\theta \le C_1\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^\theta, \tag{2.19}$$

$$\|v_t\|_4 \le C\|v_t\|^{1-\theta}\|(-\Delta)^{\frac{\alpha}{2}}v_t\|^\theta \le C_1\|(-\Delta)^{\frac{\alpha}{2}}v_t\|^\theta. \tag{2.20}$$

Combining the inequalities (2.18), (2.19) and (2.20), we get

$$\frac{d}{dt}(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) \le C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + \delta(\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^2 + \|(-\Delta)^{\frac{\alpha}{2}}v_t\|^2) \tag{2.21}$$

Differentiate (1.1) with respect to t, multiply the first equation of the system (1.1) by $\overline{u}_t$, the second equation by $\overline{v}_t$, and then integrate with respect to x , take the real part to get

$$\begin{aligned}
&-\mathrm{Re}\int i\hbar u_{tt}\overline{u}_t + \frac{\hbar^2}{4M}\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^2 + \mathrm{Re}\int\frac{d}{dt}(\lambda_u|u|^2u)\overline{u}_t + \mathrm{Re}\int\frac{d}{dt}(\lambda|v|^2u)\overline{u}_t\\
&+\mathrm{Re}\int\frac{d}{dt}(\sqrt{2}\beta\overline{u}v)\overline{u}_t = 0,\\
&-\mathrm{Re}\int i\hbar v_{tt}\overline{v}_t + \frac{\hbar^2}{8M}\|(-\Delta)^{\frac{\alpha}{2}}v_t\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|v_t\|_2^2 + \mathrm{Re}\int\frac{d}{dt}(\lambda_v|v|^2v)\overline{v}_t\\
&+\mathrm{Re}\int\frac{d}{dt}(\lambda|u|^2v)\overline{v}_t + \mathrm{Re}\int\frac{d}{dt}(\frac{\beta}{\sqrt{2}}u^2)\overline{v}_t = 0.
\end{aligned}$$

But applying the inequality (2.16), we have

$$\mathrm{Re}\int \frac{d}{dt}(\lambda_u|u|^2u)\overline{u}_t + \mathrm{Re}\int \frac{d}{dt}(\lambda|v|^2u)\overline{u}_t + \mathrm{Re}\int \frac{d}{dt}(\sqrt{2}\beta\overline{u}v)\overline{u}_t \leq C,$$
$$\frac{\varepsilon}{2}\frac{d}{dt}\|v_t\|_2^2 + \mathrm{Re}\int \frac{d}{dt}(\lambda_v|v|^2v)\overline{v}_t + \mathrm{Re}\int \frac{d}{dt}(\lambda|u|^2v)\overline{v}_t + \mathrm{Re}\int \frac{d}{dt}(\frac{\beta}{\sqrt{2}}u^2)\overline{v}_t \leq C.$$

Therefore

$$(\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^2 + \|(-\Delta)^{\frac{\alpha}{2}}v_t\|^2) \leq C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + C_1. \tag{2.22}$$

Using (2.21) and the above inequality, we get

$$\frac{d}{dt}(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) \leq C(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) + C_1.$$

By Gronwall inequality to obtain

$$(\|u_{tt}\|_2^2 + \|v_{tt}\|_2^2) \leq C. \tag{2.23}$$

Then combing the inequalities (2.22) and (2.23), the below inequality is true

$$(\|(-\Delta)^{\frac{\alpha}{2}}u_t\|^2 + \|(-\Delta)^{\frac{\alpha}{2}}v_t\|^2) \leq C,$$

where C is a constant depending only on $\|u_0\|_{H^{3\alpha}_{per}}, \|v_0\|_{H^{3\alpha}_{per}}$.

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let* $\frac{1}{2} < \alpha < 1$, $u_0 \in H^{4\alpha}_{per}(\Omega), v_0 \in H^{4\alpha}_{per}(\Omega)$*, the solution* (u, v) *satisfies the following estimate*

$$\sup_{0\leq t<\infty}(\|(-\Delta)^{2\alpha}u\| + \|(-\Delta)^{2\alpha}v\|) \leq C,$$

where the constant C *depends only on* $\|u_0\|_{H^{4\alpha}_{per}}, \|v_0\|_{H^{4\alpha}_{per}}$.

Proof. Using the system (1.1), we have

$$\|(-\Delta)^{2\alpha}u\|_2^2 \leq C_1\|(-\Delta)^{\alpha}u_t\|_2^2 + C_2\|(-\Delta)^{\alpha}|u|^2u\| + C_3\|(-\Delta)^{\alpha}|v|^2u\| + C_4\|(-\Delta)^{\alpha}\overline{u}v\|,$$
$$\|(-\Delta)^{2\alpha}v\|_2^2 \leq C\|(-\Delta)^{\alpha}v_t\|_2^2 + C_1\|(-\Delta)^{\alpha}v\|_2^2 + C_2\|(-\Delta)^{\alpha}|v|^2v\| + C_3\|(-\Delta)^{\alpha}|u|^2v\| + C_4\|(-\Delta)^{\alpha}u^2\|.$$

For $\frac{1}{2} < \alpha < 1$,

$$\|(-\Delta)^{2\alpha}u\|_2^2 \leq C_1\|(-\Delta)^{\alpha}u_t\|_2^2 + C_2\|\Delta(|u|^2u)\| + C_3\|\Delta(|v|^2u)\| + C_4\|\Delta(\overline{u}v)\|,$$
$$\|(-\Delta)^{2\alpha}v\|_2^2 \leq C\|(-\Delta)^{\alpha}v_t\|_2^2 + C_1\|(-\Delta)^{\alpha}v\|_2^2 + C_2\|\Delta(|v|^2v)\| + C_3\|\Delta(|u|^2v)\| + C_4\|\Delta u^2\|.$$

By Lemma 2.5 and the simple compution, we have

$$\left(\|(-\Delta)^{2\alpha}u\|_2^2 + \|(-\Delta)^{2\alpha}v\|_2^2\right) \leq C + C_1\left(\|\Delta u\| + \|\Delta v\|\right) + C_2\left(\|\nabla u\|_4^2 + \|\nabla v\|_4^2\right). \tag{2.24}$$

Let $\theta = \frac{2}{4\alpha} < 1$. Using Gagliardo–Nirenberg inequality to have

$$C\|\Delta u\| \leq C_1\|(-\Delta)^{2\alpha}u\|^{\theta}\|u\|^{1-\theta} \leq \frac{1}{4}\|(-\Delta)^{2\alpha}u\| + C_2, \tag{2.25}$$

$$C\|\Delta v\| \leq C_1\|(-\Delta)^{2\alpha}v\|^{\theta}\|v\|^{1-\theta} \leq \frac{1}{4}\|(-\Delta)^{2\alpha}v\| + C_2. \tag{2.26}$$

Define $\gamma = \frac{1}{16\alpha-4} < \frac{1}{4}$. By Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} C\|\nabla u\|_4^2 &\leq C_1\|(-\Delta)^{2\alpha}u\|^{2\gamma}\|\nabla u\|^{2(1-\gamma)} \\ &\leq C\|(-\Delta)^{2\alpha}u\|^{2\gamma}\|(-\Delta)^{\alpha}u\|^{2(1-\gamma)} \leq \frac{1}{4}\|(-\Delta)^{2\alpha}u\| + C_1, \end{aligned} \tag{2.27}$$

$$\begin{aligned} C\|\nabla v\|_4^2 &\leq C_1\|(-\Delta)^{2\alpha}v\|^{2\gamma}\|\nabla v\|^{2(1-\gamma)} \\ &\leq C\|(-\Delta)^{2\alpha}v\|^{2\gamma}\|(-\Delta)^{\alpha}v\|^{2(1-\gamma)} \leq \frac{1}{4}\|(-\Delta)^{2\alpha}v\| + C_1. \end{aligned} \tag{2.28}$$

Combining the inequalities (2.24)–(2.28), we have

$$(\|(-\Delta)^{2\alpha}u\| + \|(-\Delta)^{2\alpha}v\|) \leq C,$$

where C is a constant depending only on $\|u_0\|_{H^{4\alpha}_{per}}, \|v_0\|_{H^{4\alpha}_{per}}$.

This completes the proof of Lemma 2.5. □

3. Proof of the main results

In this section, we prove the existence of weak solution to the problem (1.1)–(1.3) by using Galerkin-Fourier method. We need the following lemmas.

Lemma 3.1. *Let B_0, B and B_1 be three Banach spaces. Assume that $B_0 \subset B \subset B_1$ and B_i, $i = 0, 1$ are reflective. Suppose also that B_0 is compactly embedded in B. Let*

$$W = \left\{ v | v \in L^{p_0}(0, T; B_0), v' = \frac{dv}{dt} \in L^{p_1}(0, T; B_1) \right\}$$

where T is finite and $1 < p_i < \infty, i = 0, 1$. W is equipped with the norm

$$\|v\|_{L^{p_0}(0,T;B_0)} + \|v'\|_{L^{p_1}(0,T;B_1)}.$$

Then W is compactly embedded in $L^{p_0}(0, T; B)$.

Lemma 3.2. *Suppose that Q is a bounded domain in $R^n_x \times R_t, g_\mu, g \in L^q(Q)(1 < q < \infty)$ and $\|g_\mu\|_{L^q(Q)} \leq C$. Furthermore, suppose that*

$$g_\mu \to g \text{ a.e. in } Q.$$

Then

$$g_\mu \rightharpoonup g \text{ weakly in } L^q(Q).$$

Lemma 3.3. *X is a Banach space. Suppose that $g \in L^p(0, T; X), \frac{\partial g}{\partial x} \in L^p(0, T; X)(1 \leq p \leq \infty)$. Then $g \in C([0, T], X)$(after possibly being redefined on a set of measure zero).*

In the following, we prove the existence of weak solution to the problem (1.1)–(1.3).

Proof of Theorem 1.1. We prove theorem 1.1. by the following three steps.

Step 1. Constructing the approximate solutions by the Galerkin-Fourier method.

Let $\{\omega_j(x)\}(j=1,2,\cdots)$ be a complete orthonormal basis of eigenfunctions for the periodic boundary problem $-\Delta u=\lambda u$ in Ω For every integer m, we are looking for an approximate solution of the system (1.1) of the form

$$u_m(t)=\sum_{j=1}^{m}\xi_{jm}(t)\omega_j,\qquad v_m(t)=\sum_{j=1}^{m}\mu_{jm}(t)\omega_j,$$

where $\xi_{jm},\ \mu_{jm}$ satisfy the following nonlinear equations

$$\begin{cases}\Big(-i\hbar u_{mt}+(\frac{\hbar^2}{2M}(-\Delta)^\alpha+\lambda_u|u_m|^2+\lambda|v_m|^2)u_m+\sqrt{2}\beta\overline{u}_m v_m,\omega_j\Big)=0,\\ \Big(-i\hbar v_{mt}+(\frac{\hbar^2}{4M}(-\Delta)^\alpha+\varepsilon+\lambda_v|v_m|^2+\lambda|u_m|^2)v_m+\frac{\beta}{\sqrt{2}}u_m^2,\omega_j\Big)=0,\end{cases}\tag{3.1}$$

the nonlinear equations (3.1) satisfy the following initial-value conditions

$$\begin{cases}u_m(0)=u_{0m},\ u_{0m}=\sum_{i=1}^{m}f_{im}\omega_i\to u_0\ \ in\ \ H^\alpha_{per}(\Omega)\ as\ m\to\infty,\\ v_m(0)=v_{0m},\ v_{0m}=\sum_{i=1}^{m}g_{im}\omega_i\to v_0\ \ in\ \ H^\alpha_{per}(\Omega)\ \ as\ \ m\to\infty.\end{cases}\tag{3.2}$$

Then (3.1) becomes the system of nonlinear ODE subject to the initial condition (3.2). According to standard existence theory for nonlinear ordinary differential equations, there exists a unique solution of (3.1) and (3.2) for a.e. $0\le t\le t_m$. By a priori estimates we obtain that $t_m=T$.

Step 2. A priori estimates.

As the proof of Lemmas 2.1 and 2.2, we have

$$(u_m,v_m)\in L^\infty(0,T;H^\alpha_{per}(\Omega))^2.\tag{3.3}$$

For $\forall(\varphi,\phi)\in H^\alpha_{per}(\Omega)\times H^\alpha_{per}(\Omega)$, we have

$$\begin{cases}\Big(-i\hbar u_{mt}+(\frac{\hbar^2}{2M}(-\Delta)^\alpha+\lambda_u|u_m|^2+\lambda|v_m|^2)u_m+\sqrt{2}\beta\overline{u}_m v_m,\varphi\Big)=0,\\ \Big(-i\hbar v_{mt}+(\frac{\hbar^2}{4M}(-\Delta)^\alpha+\varepsilon+\lambda_v|v_m|^2+\lambda|u_m|^2)v_m+\frac{\beta}{\sqrt{2}}u_m^2,\phi\Big)=0.\end{cases}\tag{3.4}$$

So

$$\begin{aligned}&|(u_{mt},\varphi)|\\ \le& C_1|((-\Delta)^\alpha u_m,\varphi)|+C_2\left|(|u_m|^2u_m,\varphi)\right|+C_3\left|(|v_m|^2u_m,\varphi)\right|+C_4\left|(\overline{u}_m v_m,\varphi)\right|\\ \le& C_1\|D^\alpha u_m\|\|D^\alpha\varphi\|+C_2\|u_m\|_4^3\|\varphi\|_4+C_3\|v_m\|_4^2\|u_m\|_4\|\varphi\|_4+C_4\|u_m\|_3\|v_m\|_3\|\varphi\|_3\end{aligned}\tag{3.5}$$

$$\begin{aligned}&|(v_{mt},\phi)|\\ \le& C_1|((-\Delta)^\alpha v_m,\phi)|+\varepsilon\left|(v_m,\phi)\right|+C_2\left|(|v_m|^2v_m,\phi)\right|+C_3|(|u_m|^2v_m,\psi)|\\ &+C_4|(u_m^2,\psi)|\\ \le& C_1\|D^\alpha v_m\|\|D^\alpha\phi\|+\varepsilon\|v_m\|_2\|\phi\|_2+C_2\|v_m\|_4^3\|\phi\|_4+C_3\|u_m\|_4^2\|v_m\|_4\|\phi\|_4\\ &+C_4\|u_m\|_4^2\|\phi\|_2\end{aligned}\tag{3.6}$$

Using the Sobolev embedding theorem, we have

$$\|\varphi\|_4 \le C\|D^\alpha \varphi\| + C_1, \quad \|\varphi\|_3 \le C_2\|D^\alpha \varphi\| + C_3, \quad \|\phi\|_4 \le C_4\|D^\alpha \phi\| + C_5.$$

So by (3.5) and (3.6), we get

$$|(u_{mt}, \varphi)| \le C\|D^\alpha \varphi\| + C_1, \quad \|(v_{mt}, \phi)| \le C_2\|D^\alpha \phi\| + C_3, \quad \forall \varphi, \phi \in H^\alpha_{per}(\Omega).$$

Therefore

$$(u_{mt}, v_{mt}) \in L^\infty(0,T; H^{-\alpha}_{per}(\Omega))^2. \tag{3.7}$$

Step 3. Passaging to the limit.

By applying (3.3) and (3.7), we deduce that there exists a subsequence u_μ from u_m, v_k from v_m such that

$$u_\mu \rightharpoonup u \text{ *-weakly in } L^\infty(0,T; H^\alpha_{per}(\Omega)), \quad u_{\mu t} \rightharpoonup u_t \text{ *-weakly in } L^\infty(0,T; H^{-\alpha}_{per}(\Omega)). \tag{3.8}$$

$$v_k \rightharpoonup v \text{ *-weakly in } L^\infty(0,T; H^\alpha_{per}(\Omega)), \quad v_{kt} \rightharpoonup v_t \text{ *-weakly in } L^\infty(0,T; H^{-\alpha}_{per}(\Omega)). \tag{3.9}$$

By (3.3), we have

$$(u_m, v_m) \text{ is bounded in } L^2(0,T; H^\alpha_{per}(\Omega))^2. \tag{3.10}$$

By (3.7), we have

$$(u_{mt}, v_{mt}) \text{ is bounded in } L^2(0,T; H^{-\alpha}_{per}(\Omega))^2. \tag{3.11}$$

Define

$$W = \left\{ v | v \in L^2(0,T; H^\alpha_{per}(\Omega)), v_t \in L^2(0,T; H^{-\alpha}_{per}(\Omega)) \right\}$$

We equip W with the norm:

$$\|v\|_W = \|v\|_{L^2(0,T;H^\alpha_{per}(\Omega))} + \|v_t\|_{L^2(0,T;H^{-\alpha}_{per}(\Omega))}.$$

Since $H^\alpha_{per}(\Omega)$ is compactly embedded in $L^2(\Omega)$ for $\frac{1}{2} < \alpha < 1$, by Lemma 3.1 we have that W is compactly embedded in $L^2(0,T; L^2(\Omega))$. By (3.10) and (3.11), $u_m \in W$. Then, there exists the subsequence u_μ, v_k (not rebelled) which satisfies

$$u_\mu \to u, \quad v_k \to v \text{ strongly in } L^2(0,T; L^2(\Omega)) \text{ and a. e.} \tag{3.12}$$

By using (3.3), (3.12) and Lemma 3.2, we have

$$|u_\mu|^2 u_\mu \rightharpoonup |u|^2 u \text{ *-weakly in } L^\infty(0,T; L^{\frac{4}{3}}(\Omega)), \tag{3.13}$$

$$|v_k|^2 v_k \rightharpoonup |v|^2 v \text{ *-weakly in } L^\infty(0,T; L^{\frac{4}{3}}(\Omega)). \tag{3.14}$$

Fixing$j$, we get

$$\begin{cases} \left(-i\hbar u_{mt} + (\frac{\hbar^2}{2M}(-\Delta)^\alpha + \lambda_u |u_m|^2 + \lambda |v_m|^2) u_m + \sqrt{2}\beta \overline{u}_m v_m, \omega_j\right) = 0, \\ \left(-i\hbar v_{mt} + (\frac{\hbar^2}{4M}(-\Delta)^\alpha + \varepsilon + \lambda_v |v_m|^2 + \lambda |u_m|^2) v_m + \frac{\beta}{\sqrt{2}} u_m^2, \omega_j\right) = 0, \end{cases} \tag{3.15}$$

By applying (3.8), (3.9), (3.13) and (3.14), we deduce that there exists a subsequence u_μ from u_m, v_k from v_m such that

$$\begin{aligned}
&((-\Delta)^{\frac{\alpha}{2}} u_\mu, \omega_j) \rightharpoonup ((-\Delta)^{\frac{\alpha}{2}} u, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&(u_{\mu t}, \omega_j) \rightharpoonup (u_t, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&((\lambda_u|u_\mu|^2 + \lambda|v_\mu|^2)u_\mu, \omega_j) \rightharpoonup ((\lambda_u|u|^2 + \lambda|v|^2)u, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&(\overline{u}_\mu v_\mu, \omega_j) \rightharpoonup (\overline{u}v, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&((-\Delta)^{\frac{\alpha}{2}} v_\mu, \omega_j) \rightharpoonup ((-\Delta)^{\frac{\alpha}{2}} v, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&(v_{\mu t}, \omega_j) \rightharpoonup (v_t, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&(v_\mu, \omega_j) \rightharpoonup (v, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&((\lambda_v|v_\mu|^2 + \lambda|u_\mu|^2)v_\mu, \omega_j) \rightharpoonup ((\lambda_v|v|^2 + \lambda|u|^2)v, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),\\
&(u_\mu^2, \omega_j) \rightharpoonup (u^2, \omega_j) \ \ \text{*-weakly in } \ L^\infty(0,T),
\end{aligned}$$

Then from (3.17), we have

$$\begin{cases} \left(-i\hbar u_t + (\frac{\hbar^2}{2M}(-\Delta)^\alpha + \lambda_u|u|^2 + \lambda|v|^2)u + \sqrt{2}\beta\overline{u}v, \omega_j\right) = 0,\\ \left(-i\hbar v_t + (\frac{\hbar^2}{4M}(-\Delta)^\alpha + \varepsilon + \lambda_v|v|^2 + \lambda|u|^2)v + \frac{\beta}{\sqrt{2}}u^2, \omega_j\right) = 0, \end{cases} \tag{3.16}$$

the above equalities hold for any fixed j. By the density of the basis $\omega_j, (j \in Z)$, we have:

$$\begin{cases} \left(-i\hbar u_t + (\frac{\hbar^2}{2M}(-\Delta)^\alpha + \lambda_u|u|^2 + \lambda|v|^2)u + \sqrt{2}\beta\overline{u}v, h\right) = 0, \ \ \forall h \in H^\alpha_{per}(\Omega),\\ \left(-i\hbar v_t + (\frac{\hbar^2}{4M}(-\Delta)^\alpha + \varepsilon + \lambda_v|v|^2 + \lambda|u|^2)v + \frac{\beta}{\sqrt{2}}u^2, g\right) = 0, \ \ \forall g \in H^\alpha_{per}(\Omega) \end{cases} \tag{3.17}$$

Hence (u, v) satisfies the system (1.1). By (3.3), (3.7) and Lemma 3.3, we obtain that

$$u_\mu \in C(0,T; H^{-\alpha}_{per}(\Omega)), \quad v_k \in C(0,T; H^{-\alpha}_{per}(\Omega)).$$

Then $u_\mu(0) \rightharpoonup u(0)$ weakly in $H^{-\alpha}_{per}(\Omega)$, $v_k(0) \rightharpoonup v(0)$ weakly in $H^{-\alpha}_{per}(\Omega)$.

But from (3.2), we have $u_\mu(0) \to u_0$ weakly in $H^\alpha_{per}(\Omega)$, $v_k(0) \to v_0$ weakly in $H^\alpha_{per}(\Omega)$. Therefore, $u(0) = u_0$, $v(0) = v_0$.

Theorem 1.4 generalizes the result of the global existence of weak solution to the nonlinear Schrödinger equations in [3]. So that Theorem 1.3. is complete.

By the *a priori* estimates from Lemma 2.1 to Lemma 2.5 and Theorem 1.3., there exists a global smooth solution (u, v) for the system (1.1)–(1.3) such that

$$(u, v) \in L^\infty\left([0,T); H^{4\alpha}_{per}(\Omega)\right)^2, \qquad (u_t, v_t) \in L^\infty\left([0,T); H^{2\alpha}_{per}(\Omega)\right)^2.$$

Finally, we prove the uniqueness of the solution to the system (1.1)–(1.3) in the following.

Let $(u_1, v_1), (u_2, v_2)$ be two solutions which satisfy the system (1.1)–(1.3), then $(s = u_1 - u_2, m = v_1 - v_2)$ satisfies

$$\begin{cases} i\hbar s_t = \frac{\hbar^2}{2M}(-\Delta)^\alpha s + \lambda_u(|u_1|^2u_1 - |u_2|^2u_2) + \lambda(|v_1|^2u_1 - |v_2|^2u_2)\\ \qquad + \sqrt{2}\beta(\overline{u}_1v_1 - \overline{u}_2v_2),\\ i\hbar m_t = \frac{\hbar^2}{4M}(-\Delta)^\alpha m + \varepsilon m + \lambda_v(|v_1|^2v_1 - |v_2|^2v_2) + \lambda(|u_1|^2v_1 - |u_2|^2v_2)\\ \qquad + \frac{\beta}{\sqrt{2}}(u_1^2 - u_2^2), \end{cases} \tag{3.18}$$

with the initial condition

$$s(0) = 0, \qquad m(0) = 0.$$

Taking the inner product of the first equation of the system (3.18) with $\overline{s}$ and the second equation with $\overline{m}$, considering the imaginary part of the resulting equations, we obtain:

$$\begin{aligned}
\frac{\hbar}{2}\frac{d}{dt}\|s\|_2^2 = &\lambda_u \mathrm{Im}\int (|u_1|^2u_1 - |u_2|^2u_2)\overline{s}dx + \lambda \mathrm{Im}\int (|v_1|^2u_1 - |v_2|^2u_2)\overline{s}dx \\
&+ \sqrt{2}\beta \mathrm{Im}\int (\overline{u}_1 v_1 - \overline{u}_2 v_2)\overline{s}dx \\
\frac{\hbar}{2}\frac{d}{dt}\|m\|_2^2 = &\lambda_v \mathrm{Im}\int (|v_1|^2v_1 - |v_2|^2v_2)\overline{m}dx + \lambda \mathrm{Im}\int (|u_1|^2v_1 - |u_2|^2v_2)\overline{m}dx \\
&+ \frac{\beta}{\sqrt{2}}\mathrm{Im}\int (u_1^2 - u_2^2)\overline{m}dx.
\end{aligned}$$

But

$$\begin{aligned}
&\lambda_u \mathrm{Im}\int (|u_1|^2u_1 - |u_2|^2u_2)\overline{s}dx \\
\le &C\int |(|u_1|^2 s\overline{s} + (|u_1|^2 - |u_2|^2)u_2\overline{s})|dx \\
\le &C\|u_1\|_\infty^2\|s\|_2^2 + C\int (|u_1|^2 - |u_2|^2)u_2\overline{s}dx \\
\le &C\|s\|_2^2.
\end{aligned}$$

And

$$\begin{aligned}
&\lambda \mathrm{Im}\int (|v_1|^2u_1 - |v_2|^2u_2)\overline{s}dx \\
\le &C\int |(|v_1|^2 s\overline{s} + (|v_1|^2 - |v_2|^2)u_2\overline{s})|dx \\
\le &C\|v_1\|_\infty^2\|s\|_2^2 + C\|u_2\|_\infty\|(|v_1|^2 - |v_2|^2)\|_2\|s\|_2 \\
\le &C(\|s\|_2^2 + \|m\|_2^2),
\end{aligned}$$

$$\begin{aligned}
&\int (\overline{u}_1 v_1 - \overline{u}_2 v_2)\overline{s}dx \\
= &\int (\overline{u}_1 v_1 - \overline{u}_1 v_2 + \overline{u}_1 v_2 - \overline{u}_2 v_2)\overline{s}dx \\
\le &C_1\int (m\overline{s} + |s|^2)dx \le C_2(\|m\|_2^2 + \|s\|_2^2),
\end{aligned}$$

$$\begin{aligned}
&\int (u_1^2 - u_2^2)\overline{m}dx \\
= &\int \left((u_1^2 - u_1u_2) + (u_1u_2 - u_2^2)\right)\overline{m}dx \\
\le &C_1\int s\overline{m}dx \le C_2(\|s\|_2^2 + \|m\|_2^2).
\end{aligned}$$

By the above inequalities, one can easily obtain

$$\frac{d}{dt}(\|s\|_2^2 + \|m\|_2^2) \leq C(\|s\|_2^2 + \|m\|_2^2).$$

Applying the Gronwall inequality, we get $s = 0, m = 0$. Thus the uniqueness is obtained.

So we complete Theorem 1.4.

Remark 3.1. All the above estimates are unconcerned with the period L and only depend on the norm of initial data. Therefore, by using the *a priori* estimates of the solution to the system (1.1)–(1.3) for L, as in Ref. [18], we derive the global smooth solution as $L \to \infty$. So that, Theorem 1.5. is obtained.

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