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A NEW DEFY FOR ITERATION METHODS

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Abstract The work presents an adaptation of iteration method for solving a class of thirst order partial nonlinear differential equation with mixed derivatives. The class of partial differential equations present here is not solvable with neither the method of Green function, the most usual iteration methods for instance variational iteration method, homotopy perturbation method and Adomian decomposition method, nor integral transform for instance Laplace, Sumudu, Fourier and Mellin transform. We presented the stability and convergence of the used method for solving this class of nonlinear chaotic equations. Using the proposed method, we obtained exact solutions to this kind of equations.

Keywords Extension of iteration method, nonlinear equation, Stability, convergence.

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1. Introduction

In the last century, mathematics tools were used to model real world problems, which occur in all branches of sciences. Many chaotic problems are usually described via ordinary, partial, or fractional differential equations. It is important to point out that, the use of this differential equation is one hand to predict the future behavior of the phenomenon under investigation. This investigation can be done numerically or analytically. Doing the investigation analytically is usually a very difficult task. In the recent decade, many scholars have focused their attention in developing analytical methods to find the solution of these equations describing real world problem. All of these methods have their strength and their limitations while dealing with nonlinear chaotic model. In the recent decade, it was observed by many scholars in the area of partial and ordinary differential equations that to find the exact solution of this class of equation is difficult task. Nevertheless, scholars in this area have developed numerous iterative methods to deal with this class of equations. Their methods sometime are used to find exact or approximate solutions. One of the commonly used iteration methods is the Homotopy perturbation method [8, 11, 14–16]. Atangana has modified this method recently, while solving the conventional groundwater flow equation [1, 7]. This modified version was found to be very helpful and powerful tool to deal with nonlinear, linear ordinary and partial differential equations including those with non-integer order generally called fractional differential equations [2, 4, 10, 12, 13]. However, recently, a class of third orders differential equation namely Mboctara equations were presented and examined in the work by [3]. The so-called Mboctara equations were analytically

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solved via the triple Laplace transform. Since then, no other method has been used to handle this class of partial differential equation, in particular no iteration methods has been used in this regards due to the form of the equation. The Mboctara equation was given in the following form

$$\partial_{xyt}u(x,y,t) + u(x,y,t) = f(x,y,t). \tag{1.1}$$

In this work, we are not much interesting in the above equation, since it is linear and also it was solved via the triple Laplace transform. We are instead interested in the general and nonlinear form of this class of equation. It is perhaps important to recall that, when dealing with nonlinear equations, neither the Laplace nor the Fourier transform are suitable for getting to the bottom of their solution. On the other hand, no iteration method has been used to give approximate or exact solution to the Mboctara equation. During our investigation in this work, we will be using the homotopy decomposition method to solve the proposed equation with the general form

$$\partial_{xyt}u(x,y,t) + R(u(x,y,t)) + L(u(x,y,t)) = f(x,y,t). \tag{1.2}$$

R is the non-linear operator and L the linear operator. The aim of this paper is to modify existing iteration methods to analyze the above equation. It is important to notice that all papers that have been published in which iteration methods were used to find analytical solutions of nonlinear were in the form of

$$\partial_t^m u(x,y,t) + R(u(x,y,t)) + L(u(x,y,t)) = f(x,y,t),\ m \geq 1. \tag{1.3}$$

The above version is not as challenging as equation (1.2), because to solve the above, one needs just to apply the inverse operator of ∂_t^m and further apply the homotopy method. We shall recall that the nonlinear equation under investigation here is given as:

$$\partial^3_{xyt}M(x,y,t) + \partial_x M(x,y,t)\partial_y M(x,y,t)\partial_t M(x,y,t) + R(M(x,y,t)) = f(x,y,t). \tag{1.4}$$

$R(M(x,y,t))$ is a nonlinear or linear operator with mixed derivatives. The above equation is perhaps amount the most difficult equations for which it is not easier to find solution analytically; because it is not possible to use any integral transform for instance: the Laplace, Fourier, Mellin and Sumudu transforms. On the other hand, one cannot use in this case the Green function technique due to the form of this equation. Not to mention that even some advance and powerful numerical scheme cannot handle this equation easily. In the next section we shall present the general methodology of the used method.

2. Methodology of the HDM for the nonlinear equation

We shall consider the more general form of the nonlinear equation given as

$$\partial^{n+m+l}_{x^n y^m t^l} M(x,y,t) = L(M(x,y,t)) + R(M(x,y,t)) + f(x,y,t). \tag{2.1}$$

Here L is the linear operator with mixed derivatives and R nonlinear operator with mixed derivatives, f is a known operator. The first think to do is to convert the above equation to an integral equation by applying on both sides the inverse operator of $\partial^{n+m+l}_{x^n y^m t^l}$ in order to obtain the following

$$
\begin{aligned}
&M(x,y,t) - I(x,y,t)\\
=&\frac{1}{\Gamma(n-1)\Gamma(m-1)\Gamma(l-1)}\int_0^x\int_0^y\int_0^t (t-w)^{l-1}(x-j)^{n-1}L(M(w,h,j))\\
&+R(M(w,h,j)) + f(w,h,j)dwdhdj, \qquad (2.2)\\
&I(x,y,t) = \sum_{w=0}^{n-1}\sum_{h=0}^{m-1}\sum_{j=0}^{l-1}\frac{f_{w,h,j}(x,y,t)}{w!h!j!}x^w y^h t^j.
\end{aligned}
$$

Here we shall recall that $I(x,y,t)$ is the contribution of all initial conditions. The next conventional step is to assume that, the solution of the above equation can be obtained in series for in which an embedding parameter p is introduced, this idea is proper to the Poincare series, and then we suppose that,

$$M(x,y,t) = \lim_{p\longrightarrow 1}\sum_{k=0}^{\infty} p^k M_k(x,y,t). \qquad (2.3)$$

With the primary intention of the decomposition method, the above expression is directly replaced in equation (2.1) as follows

$$
\begin{aligned}
&\sum_{k=0}^{\infty} p^{k-1}M_k(x,y,t) - p^{-1}I(x,y,t)\\
=&\frac{1}{\Gamma(n-1)\Gamma(m-1)\Gamma(l-1)}\int_0^x\int_0^y\int_0^t (t-w)^{l-1}(y-k)^{m-1}(x-j)^{n-1} \qquad (2.4)\\
&L\left(\sum_{k=0}^{\infty} p^k M_k(x,y,t)\right) + R\left(\sum_{k=0}^{\infty} p^k M_k(x,y,t)\right) + f(w,h,j)dwdhdj.
\end{aligned}
$$

The general idea at this stage is to put together expression with identical coefficient of the embedding parameter p to thus obtain a set of integral equations that need to be computed to obtain an approximate or exact solution to this class of equations. It is perhaps important to notify that, during the process of comparison the bigger challenge is to deal with the nonlinear part of the Mboctara equation, but to handle this difficulty we borrow the idea of He's polynomial that can be found in several papers using the homotopy perturbation method.

2.1. Convergence analysis

One of the important parts of any iteration method is to prove the uniqueness and the convergence of the method; we are going to show the analysis underpinning the convergence and the uniqueness of the proposed method for the general solution for $p = 1$.

Theorem 2.1. *Assuming that X and Y are Banach spaces and $V : X \longrightarrow Y$ is contraction nonlinear mapping. If the progression engender by the three dimensional homotopy decomposition method is regarded as*

$$M_n(x,y,t) = V(M_{n-1}(x,y,t)) = \sum_{k=0}^{n-1} M_k(x,y,t), \quad n = 1,2,3\cdots. \qquad (2.5)$$

Then, the following statements hold

(a) $\|M_n(x,y,t) - M(x,y,t)\| \le \rho^n \|I(x,y,t) - M(x,y,t)\|$, *with* $0 < \rho < 1$.

(b) *For any* n *greater than 0,* $M(x,y,t)$ *is always in the neighborhood of the exact solution* $M(x,y,t)$.

(c) $\lim_{n\to\infty} M(x,y,t) = M(x,y,t)$.

Proof. **a)** The proof of (a) shall be achieved via induction on the natural number n. However, when $n = 1$, we have the following.

$$\|M_1(x,y,t) - M(x,y,t)\| = \|V(M_0(x,y,t)) - M(x,y,t)\|.$$

However, by hypothesis, we have that V has a fixed point, which is the exact solution. Because if $M(x,y,t)$ is the exact solution, then,

$$\begin{aligned} M(x,y,t) = M_\infty(x,y,t) &= V(\sum_{k=0}^{\infty-1} M_k(x,y,t)) \\ &= V(\sum_{k=0}^{\infty} M_k(x,y,t)) = \sum_{k=0}^{\infty} M_k(x,y,t), \end{aligned}$$

since $\infty - 1$ is the same as ∞, therefore we have that

$$M(x,y,t) = V(M(x,y,t)).$$

Then,

$$\|M_1(x,y,t) - M(x,y,t)\| = \|V(M_0(x,y,t)) - V(M(x,y,t))\|.$$

Since V is a contractive nonlinear mapping, we shall have the following inequality

$$\begin{aligned} \|V(M_0(x,y,t)) - V(M(x,y,t))\| &= \|V(M_{n-1}(x,y,t)) - M(x,y,t)\| \\ &= V(M_{n-1}(x,y,t)) - V(M(x,y,t))\|. \end{aligned}$$

Using the fact that V is a nonlinear contractive mapping we have the following

$$V(M_{n-1}(x,y,t)) - V(M(x,y,t))\| < \rho\|M_{n-1}(x,y,t)) - V(M(x,y,t))\|.$$

Furthermore using the induction hypothesis, we arrive at

$$\rho\|M_{n-1}(x,y,t) - V(M(x,y,t))\| < \rho\rho^{n-1}\|M_0(x,y,t) - M(x,y,t)\|.$$

And the proof is completed.

b) Again we shall proof this by employing induction technique on m. Now for $m = 0$, we have that

$$M_0(x,y,t) = I(x,y,t) = \sum_{w=0}^{n-1}\sum_{h=0}^{m-1}\sum_{j=0}^{l-1} \frac{f_{w,h,j}}{w!h!j!} x^w y^h t^j.$$

According to the idea of the homotopy decomposition method, the above is the contribution of the initial conditions. More importantly, the above is nothing more than Taylor series of the exact solution of order nml, thus this leads us to the situation that, we can find a positive real number r such that,

$$\|M_0(x,y,t) - M(x,y,t)\| < r.$$

This is true, because the contribution of the initial conditions is in the same neighborhood of the exact solution. Then the property is verified for $m = 0$, let us assume that, the property is also true for $m-1$, that is we assume that, we can find a positive real number r such that

$$\|M_{m-1}(x,y,t) - M(x,y,t)\| < r.$$

We now want to show that the property is also true for m. In fact

$$\|M_m(x,y,t) - M(x,y,t)\| = \|V(M_{m-1}(x,y,t)) - V(M(x,y,t))\|,$$

using the fact that V is a nonlinear contractive mapping leads us to obtain

$$\|V(M_{m-1}(x,y,t)) - V(M(x,y,t))\| < \rho\|M_{m-1}(x,y,t) - M(x,y,t)\| < \rho r,$$

since $\rho < 1$, we finally have

$$\|M_m(x,y,t) - M(x,y,t)\| < r,$$

and this completes the proof.

c) The proof of (c) is directly achieved using the a) as follow

$$\lim_{n\to\infty} \|M_n(x,y,t) - M(x,y,t)\| \leq \lim_{n\to\infty} \rho^n \|I(x,y,t) - M(x,y,t)\| = 0,$$

then,

$$lim_{n\to\infty} M_n(x,y,t) = M(x,y,t).$$

In order to show the efficiency and applicability of this method for handling the nonlinear Mboctara equation, we shall present some applications in the next section. □

3. Application

We shall in this section expose the effectiveness and the possible extension of iteration method to handle this kind of nonlinear third order partial differential equation. To achieve this, we shall present some example of equation together with their exact solution: We shall start with the Example 3.1 Mboctara equation.

Example 3.1. We shall consider in this in this example the following Mboctara equation

$$\partial^3_{xyt} M(x,y,t) + \partial_x M(x,y,t)\partial_y M(x,y,t)\partial_t M(x,y,t) + M(x,y,t) = -\exp(3x+3y-3t). \tag{3.1}$$

With initial conditions:

$$\begin{array}{lll} M(0,0,0) = 1, & M(x,y,0) = exp(x+y), & M(x,0,t) = exp(x-t), \\ M(0,0,t) = exp(t), & M(0,y,0) = exp(y), & M(x,0,0) = exp(x). \end{array}$$

Applying the first step of the HDM to the above equation, we obtain the following integrals

$$\begin{aligned}
M_0(x,y,t) =& I(x,y,t), \qquad (3.2)\\
M_1(x,y,t) =& -\int_0^x\int_0^y\int_0^t (\partial_\alpha M_0(\alpha,\beta,\gamma)\partial_\beta M_0(\alpha,\beta,\gamma)\partial_\gamma M_0(\alpha,\beta,\gamma)\\
&+ M_0(\alpha,\beta,\gamma) - \exp(3\alpha+3\beta-3\gamma)d\alpha d\beta d\gamma,\\
M_2(x,y,t) =& -\int_0^x\int_0^y\int_0^t (H_2(M_1,M_0) + M_1(\alpha,\beta,\gamma)d\alpha d\beta d\gamma,\\
M_3(x,y,t) =& -\int_0^x\int_0^y\int_0^t (H_3(M_2,M_1,M_0) + M_2(\alpha,\beta,\gamma)d\alpha d\beta d\gamma,\\
M_4(x,y,t) =& -\int_0^x\int_0^y\int_0^t (H_4(M_3,M_2,M_1,M_0) + M_3(\alpha,\beta,\gamma)d\alpha d\beta d\gamma,\\
M_5(x,y,t) =& -\int_0^x\int_0^y\int_0^t (H_5(M_5,M_3,M_2,M_1,M_0) + M_4(\alpha,\beta,\gamma)d\alpha d\beta d\gamma.
\end{aligned}$$

Therefore in general one will use the following iteration formula to compute the rest of the components

$$M_5(x,y,t) = -\int_0^x\int_0^y\int_0^t (H_5(M_5,M_3,M_2,M_1,M_0) + M_4(\alpha,\beta,\gamma)d\alpha d\beta d\gamma. \qquad (3.3)$$

Where of course $H_n(M_{n-1},,M_3,M_2,M_1,M_0)$ is the so called the He's polynomial. Now Replacing the initial conditions and also integrating the above equations, we obtain the following solutions

$$\begin{aligned}
M_0(x,y,t) &= I(x,y,t), \qquad (3.4)\\
M_1(x,y,t) &= -I(x,y,t)+1,\\
M_2(x,y,t) &= xy(-t),\\
M_3(x,y,t) &= \frac{(x^2y^2(-t)^2)}{8},\\
M_4(x,y,t) &= \frac{(x^3y^3(-t)^3)}{827},\\
M_5(x,y,t) &= \frac{(x^4y^4(-t)^4)}{(8274^3)},\\
M_n(x,y,t) &= \frac{(x^{n-1}y^{m-1}(-t)^{l-1})}{(n-1)!(m-1)!(l-1)!}.
\end{aligned}$$

And the approximate solution of the nonlinear equation is given in series form as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\frac{x^{n-1}y^{m-1}(-t)^{l-1}}{(n-1)!(m-1)!(l-1)!} = \exp(x+y-t). \qquad (3.5)$$

It is very clear that, the above solution is the exact solution to the nonlinear Mboctara (1.1). Since one example is not enough to show the efficiency of a method, we shall present other examples more complicated.

Example 3.2. We shall get to the bottom of the following equation by using the extended iteration method.

$$\begin{aligned}&\partial^3_{xyt}M(x,y,t)+\partial_x M(x,y,t)\partial_y M(x,y,t)\partial_t M(x,y,t)\frac{-t^2}{xy}(\partial_t M(x,y,t))^3\\=&[1-(xyt)^2]\cos(xyt)-3xyt\sin(xyt),\\&M(0,0,0)=M(x,y,0)=M(x,0,t)=M(0,0,t)=M(0,y,0)=M(x,0,0)=0.\end{aligned}\tag{3.6}$$

Applying the first step of the HDM to the above equation, we obtain the following integrals

$$\begin{aligned}M_0(x,y,t)=&0,\\M_1(x,y,t)=&-\int_0^x\int_0^y\int_0^t(\partial_\alpha M_0(\alpha,\beta,\gamma)\partial_\beta M_0(\alpha,\beta,\gamma)\partial_\gamma M_0(\alpha,\beta,\gamma)\\&+\frac{\gamma^2}{\alpha\beta}(\partial_\gamma M_0(\alpha,\beta,\gamma))^3-[1-(\alpha\beta\gamma)^2]\cos(\alpha\beta\gamma)\\&-3\alpha\beta\gamma\sin(\alpha\beta\gamma))d\alpha d\beta d\gamma,\\M_2(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_2(M_1,M_0)+K_2(M_1,M_0))d\alpha d\beta d\gamma,\\M_3(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_3(M_2,M_1,M_0)+K_3(M_2,M_1,M_0))d\alpha d\beta d\gamma,\\M_4(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_4(M_3,M_2,M_1,M_0)+K_4(M_3,M_2,M_1,M_0))d\alpha d\beta d\gamma,\\M_5(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_5(M_5,M_3,M_2,M_1,M_0)\\&+K_5(M_4,M_3,M_2,M_1,M_0))d\alpha d\beta d\gamma.\end{aligned}\tag{3.7}$$

Therefore in general one will use the following iteration formula to compute the rest of the components

$$\begin{aligned}M_n(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_n(M_{n-1},\dots,M_3,M_2,M_1,M_0)\\&+K_{n-1}(M_{n-2},\dots,M_3,M_2,M_1,M_0))d\alpha d\beta d\gamma.\end{aligned}\tag{3.8}$$

It is perhaps important to accommodate readers that are not used to the iterations method that the He's polynomial used here are generally presented as Applying the first step of the HDM to the above equation, we obtain the following integrals

$$\begin{aligned}H_n(M_{n-1},\dots,M_3,M_2,M_1,M_0)&=\sum_{j=0}^{n-1}\sum_{k=0}^{j}\partial_x M_j\partial_y M_{j-k}\partial_t M_{n-j-k},\\K_n(M_{n-1},\dots,M_3,M_2,M_1,M_0)&=\frac{t^2}{xy}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\partial_t M_j\partial_t M_{j-k}\partial_t M_{n-j-k}.\end{aligned}\tag{3.9}$$

Replacing the initial conditions and also integrating the above equations, we obtain the following solutions

$$M_0(x,y,t)=0,\quad M_1(x,y,t)=\sin(xyt),\tag{3.10}$$

$$M_2(x,y,t)=0,\quad M_3(x,y,t)=0,$$
$$M_4(x,y,t)=0,\quad M_5(x,y,t)=0,$$
$$M_n(x,y,t)=0.$$

And the approximate solution of the nonlinear equation is given in series form as

$$M(x,y,t)=\sum_{k=0}^{\infty}M_k(x,y,t)=\sin(xyt). \tag{3.11}$$

It is very clear that, the above solution is the exact solution to the nonlinear Mboctara (3.3). One can see that even with strong linearity this method still get to the bottom of the exact solution of nonlinear Mboctara equation analytically.

Example 3.3. We shall get to the bottom of the following nonlinear equation by using the extended iteration method.

$$\begin{aligned}&\partial^3_{(xyt)}M(x,y,t)+\frac{1}{(xyt)^2}\partial_x M(x,y,t)\partial_y M(x,y,t)\partial_t M(x,y,t)\\&-\frac{1}{yt}(\cosh(xyt))^2\partial_x M(x,y,t)=(1+t^2x^2y^2)\cosh(xyt)+3txy\sinh(xyt).\end{aligned} \tag{3.12}$$

With the initial conditions

$$M(0,0,0)=M(x,y,0)=M(x,0,t)=M(0,0,t)=M(0,y,0)=M(x,0,0)=0.$$

Applying the first step of the HDM to the above equation, we obtain the following integrals

$$\begin{aligned}M_0(x,y,t)=&0, \qquad (3.13)\\ M_1(x,y,t)=&-\int_0^x\int_0^y\int_0^t\Big(\frac{1}{(\alpha\beta\gamma)^2}\partial_\alpha M_0(\alpha,\beta,\gamma)\partial_\beta M_0(\alpha,\beta,\gamma)\partial_\gamma M_0(\alpha,\beta,\gamma)\\&-\sinh(\alpha,\beta,\gamma))^2M_0[1+(\alpha\beta\gamma)^2]\cosh(\alpha\beta\gamma)+3\alpha\beta\gamma\sinh(\alpha\beta\gamma))d\alpha d\beta d\gamma,\\ M_2(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_2(M_1,M_0)-\sinh(\alpha\beta\gamma)^2M_1)d\alpha d\beta d\gamma,\\ M_3(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_3(M_2,M_1,M_0)-\sinh(\alpha\beta\gamma)^2M_2)d\alpha d\beta d\gamma,\\ M_4(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_4(M_3,M_2,M_1,M_0)-\sinh(\alpha\beta\gamma)^2M_3)d\alpha d\beta d\gamma,\\ M_5(x,y,t)=&-\int_0^x\int_0^y\int_0^t(H_5(M_5,M_3,M_2,M_1,M_0)-\sinh(\alpha\beta\gamma)^2M_4)d\alpha d\beta d\gamma.\end{aligned}$$

Therefore in general one will use the following iteration formula to compute the rest of the components

$$\begin{aligned}M_n(x,y,t)=-\int_0^x\int_0^y\int_0^t&(H_{n-1}(M_{n-1},\ldots,M_3,M_2,M_1,M_0)\\&-\sinh(\alpha\beta\gamma)^2M_{n-1})d\alpha d\beta d\gamma,\end{aligned} \tag{3.14}$$

the He's polynomial used here are generally presented as

$$H_n(M_{n-1},\ldots,M_3,M_2,M_1,M_0)=\frac{1}{(xyt)^2}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\partial_x M_j\partial_y M_{j-k}\partial_t M_{n-j-k}. \tag{3.15}$$

Replacing the initial conditions and also integrating the above equations, we obtain the following solutions

$$\begin{aligned} &M_0(x,y,t)=0, \quad M_1(x,y,t)=\sinh(xyt),\\ &M_2(x,y,t)=0, \quad M_3(x,y,t)=0,\\ &M_4(x,y,t)=0, \quad M_5(x,y,t)=0,\\ &M_n(x,y,t)=0. \end{aligned} \tag{3.16}$$

And the approximate solution of the nonlinear equation is given in series form as

$$M(x,y,t)=\sum_{k=0}^{\infty} M_k(x,y,t)=\sinh(xyt). \tag{3.17}$$

The above solution is exact solution of (3.12).

Example 3.4. We shall get to the bottom of the following equation by using the extended iteration method.

$$\begin{aligned} &\partial^3_{(xyt)}M(x,y,t)-\partial_x M(x,y,t)\partial_y M(x,y,t)\partial_t M(x,y,t)+M^2(x,y,t)cosh(y)\\ &=\cosh(y)(\cos(x)+2t^2\sin(x/2)^2\sin(x)^2\sinh(y)^2), \end{aligned} \tag{3.18}$$

with the initial conditions

$$M(0,0,0)=M(x,y,0)=M(x,0,t)=M(0,0,t)=M(0,y,0)=M(x,0,0)=0.$$

Applying the first step of the HDM to the above equation, we obtain the following integrals

$$\begin{aligned} M_0(x,y,t)=&0, \\ M_1(x,y,t)=&\int_0^x\int_0^y\int_0^t(-\partial_\alpha M_0(\alpha,\beta,\gamma)\partial_\beta M_0(\alpha,\beta,\gamma)\partial_\gamma M_0(\alpha,\beta,\gamma)\\ &+\cosh(\beta)M_0^2(\alpha,\beta,\gamma)\cosh(\alpha)\cosh(\beta))d\alpha d\beta d\gamma,\\ M_2(x,y,t)=&\int_0^x\int_0^y\int_0^t(-(H_2(M_1,M_0)+2\cosh(\beta)M_1M_0))d\alpha d\beta d\gamma,\\ M_3(x,y,t)=&\int_0^x\int_0^y\int_0^t(-H_3(M_2,M_1,M_0)+\cosh(\beta)(2M_2M_0+M_1^2))d\alpha d\beta d\gamma,\\ M_4(x,y,t)=&\int_0^x\int_0^y\int_0^t(-H_4(M_3,M_2,M_1,M_0)\\ &+\cosh(\beta)(2M_2M_1+2M_3M_0))d\alpha d\beta d\gamma,\\ M_5(x,y,t)=&\int_0^x\int_0^y\int_0^t(-H_5(M_5,M_3,M_2,M_1,M_0)\\ &+\cosh(\beta)(2M_0M_4+2M_3M_1+M_2^2))^2M_4)d\alpha d\beta d\gamma. \end{aligned} \tag{3.19}$$

Therefore in general one will use the following iteration formula to compute the rest of the components

$$\begin{aligned} M_5(x,y,t)=&\int_0^x\int_0^y\int_0^t(-H_n(M_{n-1},\ldots,M_3,M_2,M_1,M_0)\\ &+\cosh(\beta)\sum_{j=0}^{n-1}M_jM_{n-j-1}d\alpha d\beta d\gamma. \end{aligned} \tag{3.20}$$

Replacing the initial conditions and also integrating the above equations, we obtain the following solutions

$$\begin{aligned} &M_0(x,y,t) = 0, \quad M_1(x,y,t) = t\sin(x)\sinh(y), \\ &M_2(x,y,t) = 0, \quad M_3(x,y,t) = 0, \\ &M_4(x,y,t) = 0, \quad M_5(x,y,t) = 0, \\ &M_n(x,y,t) = 0. \end{aligned} \tag{3.21}$$

And the approximate solution of the nonlinear equation is given in series form as

$$M(x,y,t) = \sum_{k=0}^{\infty} M_k(x,y,t) = tsin(x)\sinh(y). \tag{3.22}$$

The above solution is exact solution of equation (3.18).

4. Conclusion

In the literature nowadays, lot of papers using iteration methods for solving linear, nonlinear partial and ordinary differential equations focused their attention on partial differential equation where there is a single partial derivative. It is very simple in this case to isolate the single partial derivative part and apply the inverse operator in order to use the homotopy technique. For those using the variational iteration method, they obtain a one-dimensional Lagrange multiplier, which is easier to handle. The class of equation we introduced in this work cannot be handled by any integral transform for instance; Laplace, Mellin, Sumudu and Fourier transform methods. Methods like Green function cannot be used in this case to get to the bottom of the solution analytically. To get to the bottom of the solution of this class of equation, we have extended the idea of homotopy decomposition method in three dimensional spaces. Just to check the efficiency of this extended method, we apply it in solving 4 examples, where we obtain the exact solutions at the second iteration. With this examples we can concluded that, the three dimensional homotopy decomposition methods is efficient and easier to handle this class of equation that other methods were unable to tackle.

Conflict of interest

The author confirms that there is no conflict of interest in this paper.

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References

[1] A. Atangana and J.F. Botha, *Analytical solution of the groundwater flow equation obtained via homotopy decomposition method*, Journal of Earth Science & Climatic Change, 3(2012), 1-15.

[2] A. Atangana, *Drawdown in prolate spheroidal-spherical coordinates obtained via Green's function and perturbation methods*, Communications in Nonlinear Science and Numerical Simulation, 19(2014), 1259-1269.

[3] A. Atangana, *A note on the triple Laplace transform and its applications to some kind of third-order differential equation*, Abstract and Applied Analysis, 2013(2013), 1-10, Article ID 769102.

[4] M. Alquran and A. Qawasmeh, *Soliton solutions of shallow water wave equations by means of G/G expansion method*, Journal of Applied Analysis and Computation, 4(2014), 221-229.

[5] Y.A. Brychkov, H.-J. Glaeske, A.P. Prudnikov, and V.K. Tuan, *Multidimensional Integral Transformations*, Gordon and Breach Science Publishers, Philadelphia, Pa, USA, 1992.

[6] D.G. Duffy, *Transform Methods for Solving Partial Differential Equations*, CR-C Press, New York, NY, USA, 2004.

[7] S.A. El-Wakil, E.M. Abulwafa, M.A. Zahran, and A.A. Mahmoud, *Time-fractional KdV equation: formulation and solution using variational methods*, Nonlinear Dynamics, 65(2011), 55-63.

[8] H. Jafari and V. Daftardar-Gejji, *Solving a system of nonlinear fractional differential equations using Adomian decomposition*, Journal of Computational and Applied Mathematics, 196(2006), 644-651.

[9] A. Kiliman and H.E. Gadain, *On the applications of Laplace and Sumudu transforms*, Journal of the Franklin Institute, 347(2010), 848-862.

[10] M. Merdan and S.T. Mohyud-Din, *A new method for time-fractionel coupled-KDV equations with modified Riemann-Liouville derivative*, Studies in Nonlinear Science, 2(2011), 77-86.

[11] A.M. Noor and S.T. Mohyud-Din, *Homotopy perturbation method for solving fourth-order boundary value problems*, Mathematical Problems in Engineering, 2007(2007), 1-15 Article ID 98602.

[12] Y. Tan and S. Abbasbandy, *Homotopy analysis method for quadratic Riccati differential equation*, Communications in Nonlinear Science and Numerical Simulation, 13(2008), 539-546.

[13] Z. Wang and R. Xu, *Stability and traveling waves of an epidemic model with relapse and spatial diffusion*, Journal of Applied Analysis and Computation, 4(2014), 307-322.

[14] X.J. Yang and D. Baleanu, *Fractal heat conduction problem solved by local fractional variation iteration method*, Thermal Science, 17(2013), 625-628.

[15] X.J. Yang, *Local fractional integral transforms*, Progress in Nonlinear Science, 4(2011), 12-25.

[16] X.J. Yang, H.M. Srivastava, J.H. He, and D. Baleanu, *Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives*, Physics Letters A, 377(2013), 1696-1700.