Journal of Applied Analysis and Computation
Volume 5, Number 1, February 2015, 114–137 Website:http://jaac-online.com/
doi:10.11948/2015010

EXTINCTION FOR A QUASILINEAR PARABOLIC EQUATION WITH A NONLINEAR GRADIENT SOURCE AND ABSORPTION*

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Abstract We deal with the extinction, non-extinction and decay estimates of the non-negative nontrivial weak solutions of the initial-boundary value problem for the quasilinear parabolic equation with nonlinear gradient source and absorption.

Keywords Extinction, non-extinction, quasilinear parabolic equation, nonlinear gradient source.

MSC(2000) 35K20, 35K55.

1. Introduction

This paper is devoted to the extinction phenomenon of the following parabolic equation with nonlinear gradient source and absorption

$$\begin{cases} u_t = \operatorname{div}\left(u^\alpha \left|\nabla u\right|^{m-1} \nabla u\right) + \lambda \left|\nabla u\right|^q - \delta u^\beta, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, m, λ, q and δ are positive parameters, $0 < m+\alpha < 1$, $0 < \beta \leq 1$ and $u_0 \in L^\infty(\Omega) \cap W_0^{1,m+1}(\Omega)$ is a nonzero nonnegative function.

Problems like (1.1) arise from a variety of physical phenomena. For instance, when $\alpha = 0$, $m = 1$, the equation in problem (1.1) can be viewed as the viscosity approximation of Hamilton-Jacobi type equation from stochastic control theory (see [20]). In particular, when $\alpha = 0$, $m = 1$ and $q = 2$, the equation in problem (1.1) appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation (see [13]).

Since the equation in problem (1.1) is degenerate (or singular) at the points where $u = 0$ or $\nabla u = 0$, and hence there is no classical solution in general. We first introduce the definition of the weak for problem (1.1) as follows.

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* This work was supported by National Natural Science Foundation of China (11426099, 61402166, 11371384) and Scientific Research Fund of Hunan Provincial Education Department (14B067, 12A050).

Definition 1.1. A nonnegative measurable function $u(x,t)$ defined in $\Omega \times (0,T)$ is called a weak solution of problem (1.1) if $u^{\alpha} |\nabla u|^{m+1} \in L^1\left(0,T;L^1(\Omega)\right)$, $u_t \in L^2\left(0,T;L^2(\Omega)\right)$, $u \in C\left(0,T;L^{\infty}(\Omega)\right) \cap L^q\left(0,T;W^{1,q}(\Omega)\right)$, and the integral identity

$$\begin{aligned} &\int_{\Omega} u(x,t_2)\,\zeta(x,t_2)\,dx + \int_{t_1}^{t_2}\int_{\Omega}\left[-u\zeta_t + u^{\alpha}|\nabla u|^{m-1}\nabla u\cdot\nabla\zeta\right]dxdt \\ =&\int_{\Omega} u(x,t_1)\,\zeta(x,t_1)\,dx + \lambda\int_{t_1}^{t_2}\int_{\Omega}|\nabla u|^q\,\zeta dxdt - \delta\int_{t_1}^{t_2}\int_{\Omega}u^{\beta}\zeta dxdt \end{aligned} \tag{1.2}$$

holds for any $\zeta \in C_0^{\infty}(\Omega\times(0,T))$ and $0 < t_1 < t_2 < T$. Furthermore,

$$u(x,0) = u_0(x) \ \text{ a.e. } x \in \Omega. \tag{1.3}$$

Remark 1.1. The weak subsolution (resp. supersolution) of problem (1.1) can be defined in the similar way except that " $=$ " in (1.2) and (1.3) is replaced by " $\leq$ " (resp. " $\geq$ "), and $\zeta \in C_0^{\infty}(\Omega\times(0,T))$ is taken to be nonnegative.

Remark 1.2. The local existence result of the weak solution for problem (1.1) follows, for example, from [25]. Furthermore, from Theorem 3.9 in [24] and Subsection 1.1 in [12], we know that comparison principle is granted for problem (1.1).

In the past few decades, many mathematicians have studied the extinction behaviors of various nonlinear parabolic problems (see [1,5,7,10,11,15,18–20,26,28,30, 32] and the references therein). For instance, many authors considered the following problem

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right) + \lambda u^q - \delta u^{\beta}, & (x,t)\in\Omega\times(0,+\infty), \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,+\infty), \\ u(x,0) = u_0(x), & x\in\overline{\Omega}, \end{cases} \tag{1.4}$$

where m, q and β are positive constants, λ and δ are nonnegative constants, and $m(p-1) \in (0,1)$. When $m = 1$ and $\lambda = \delta = 0$, Yuan et al. [31] showed that the solution of problem (1.4) vanishes in finite time if and only if $p \in (1,2)$. When $m = 1$ and $\lambda = 0$, Gu [9] pointed out that the necessary and sufficient condition on the occurrence of extinction phenomenon is $p \in (1,2)$ or $\beta \in (0,1)$. When $m = 1$ and $\delta = 0$, Tian & Mu [27] proved that $q = p-1$ is the critical extinction exponent of the solution of problem (1.4). When $\delta = 0$, Jin et al. [14], Zhou & Mu [33] concluded that the critical extinction exponent of the weak solution to problem (1.4) is $q = m(p-1)$. Recently, under the restrictive condition $N > p$, Mu et al. [22] studied the extinction property of problem (1.4) with λ, $\delta \neq 0$ and $\beta \in (0,1]$. It is worth to point out that the authors of [22] did not give the precise decay estimates of the extinction solutions. Meanwhile, in the case $\beta \in (0,1)$, the question is remained whether or not the solution of problem (1.4) possesses extinction property if $q < m(p-1)$.

However, to our best knowledge, there is little literature on the study of the extinction and non-extinction properties for parabolic equations with nonlinear gradient terms. Benachour et al. discussed the following Cauchy problem with gradient absorption

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - |\nabla u|^q, & x \in \mathbf{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.5}$$

where $q > 0$ and $p \in (1, 2]$, and $u_0(x) \in \mathcal{BC}\left(\mathbf{R}^N\right) \cap L^1\left(\mathbf{R}^N\right)$ is nonnegative, here $\mathcal{BC}\left(\mathbf{R}^N\right)$ denotes the space of bounded and continuous functions in $\mathbf{R}^N$. For the special case $p = 2$, Benachour et al. [2] showed that extinction phenomenon takes place for any nonnegative and integrable solution to problem (1.5) if $q \in \left(0, \frac{N}{N+1}\right)$, and established some temporal decay estimates for the L^∞−norm of the nonnegative solutions in the case $q \geq \frac{N}{N+1}$. Later, Benachour et al. [3] investigated problem (1.5) with $p = 2$ and $q \in (0, 1)$, and pointed out that the occurrence of the extinction phenomenon depends on the asymptotic behavior of u_0 as $|x|$ tends to infinity. Roughly speaking, they proved that if the decay of initial data $u_0(x)$ is faster than that of $|x|^{-\frac{p}{1-p}}$ as $|x| \to \infty$, then extinction occurs. Otherwise, the solution of (1.5) is strictly positive for any positive initial data. In addition, they also claimed that the critical extinction exponent $p = \frac{N}{N+1}$ introduced in [2] is optimal. For $p \in (1, 2)$, based on comparison principle and gradient estimates of the solutions, Iagar & Laurençot [12] classified the behavior of the solutions for large time, obtaining either positivity as $t \to \infty$ for $q > p - \frac{N}{N+1}$, optimal decay estimates as $t \to \infty$ for $q \in \left[\frac{p}{2}, p - \frac{N}{N+1}\right]$, or extinction in finite time for $q \in \left(0, \frac{p}{2}\right)$. In addition, the authors showed that how the diffusion prevents extinction in finite time in some ranges of exponents where extinction occurs for the non-diffusive Hamilton-Jacobi equation.

Recently, Mu et al. [17, 23] considered the following fast diffusion equation

$$\begin{cases} u_t = \operatorname{div}\left(u^\alpha |\nabla u|^{m-1}\nabla u\right) + \lambda |\nabla u|^q, & (x,t) \in \Omega \times (0, +\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.6}$$

where m, q and λ are positive parameters, $0 < m + \alpha < 1$. Under the restrictive condition $N \geq m + 1$, they proved that the critical extinction exponent of problem (1.6) is $q = m + \alpha$. Xu & Fang [29] considered the special case $\alpha = 0$ of problem (1.1).

Motivated by those works above, we consider the extinction property of the weak solution for problem (1.1) by using energy estimates approach and constructing suitable subsolution.

The rest of this paper is organized as follows. In Section 2, we state three useful preliminary lemmas. Section 3 is mainly about the extinction property and decay estimate of the solution to problem (1.1) in the case $\beta = 1$. Finally, we will discuss the extinction behaviour and decay estimate of the weak solution for problem (1.1) in the case $\beta \in (0, 1)$ in Section 4.

2. Preliminary lemmas

In this section, as preliminaries, we state three well-known results, which play an important role in the study of the extinction behavior and decay estimate of the solution to problem (1.1).

Lemma 2.1 (see [4]). *Let $y(t)$ be a non-negative absolutely continuous function on $[\widehat{T_0}, +\infty)$ satisfying*

$$\begin{cases} \dfrac{dy}{dt} + \alpha y^k + \beta y \le 0, \quad t \ge \widehat{T_0}, \\ y\left(\widehat{T_0}\right) \ge 0, \end{cases}$$

where α, β are positive constants, and $k \in (0,1)$, then we have the decay estimate

$$\begin{cases} y(t) \le \left[\left(y^{1-k}\left(\widehat{T_0}\right) + \dfrac{\alpha}{\beta}\right) e^{-\beta(k-1)\left(\widehat{T_0}-t\right)} - \dfrac{\alpha}{\beta}\right]^{\frac{1}{1-k}}, & \widehat{T_0} \le t < \widehat{T_1}, \\ y(t) \equiv 0, & \widehat{T_1} \le t < +\infty, \end{cases}$$

where

$$\widehat{T_1} = \frac{1}{\beta(1-k)} \ln\left[1 + \frac{\beta}{\alpha} y^{1-k}\left(\widehat{T_0}\right)\right] + \widehat{T_0}.$$

Lemma 2.2 (see [21]). *Let $0 < k < r \le 1$, $y(t) \ge 0$ be a solution of the differential inequality*

$$\begin{cases} \dfrac{dy}{dt} + \alpha y^k + \beta y \le \gamma y^r, \quad t \ge 0, \\ y(0) = y_0 > 0, \end{cases}$$

where α, $\beta > 0$, and $0 < \gamma < \alpha y_0^{k-r}$, then there exists $\chi > \beta$ such that

$$0 \le y(t) \le y_0 e^{-\chi t} \text{ for all } t \ge 0.$$

The following lemma is about the Gagliardo-Nirenberg multiplicative embedding inequality.

Lemma 2.3 (see Theorem 2.1 in Chapter I of [6]). *Let $v \in W_0^{1,p}(\Omega)$, $p \ge 1$. For every fixed number $r \ge 1$, there exists a constant C depending only upon N, p and r such that*

$$\|v\|_{\mu,\Omega} \le C \|Dv\|_{p,\Omega}^{\theta} \|v\|_{r,\Omega}^{1-\theta}, \tag{2.1}$$

where $\theta \in [0,1]$, $\mu \ge 1$, are linked by

$$\theta = \left(\frac{1}{r} - \frac{1}{\mu}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}, \tag{2.2}$$

and their admissible range is:

(i) if $N = 1$, then $\mu \in [r, +\infty]$, and $\theta \in \left[0, \frac{p}{p+r(p-1)}\right]$;

(ii) if $1 \le p < N$, then $\theta \in [0,1]$, $\mu \in \left[r, \frac{Np}{N-p}\right]$ for $r \le \frac{Np}{N-p}$ and $\mu \in \left[\frac{Np}{N-p}, r\right]$ for $r \ge \frac{Np}{N-p}$;

(iii) if $1 < N \le p$, then $\mu \in [r, +\infty)$, and $\theta \in \left[0, \frac{Np}{Np+r(p-N)}\right)$.

3. The case $\beta = 1$

The main goal of this section is to discuss the extinction behavior of the weak solution for problem (1.1) in the case $\beta = 1$. The first result of this section shows that whether the extinction behavior occurs or not depending on the size of λ when $q = m + \alpha$.

Theorem 3.1. *Assume that $0 < m + \alpha < 1$, $\beta = 1$ and $q = m + \alpha$.*

(i) If $N \geq 2$, then the nonnegative weak solution of problem (1.1) vanishes in finite time for any nonnegative initial datum u_0 provided that λ is sufficiently small. Furthermore, we have

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \left[\left(\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_0}\right) e^{(m+\alpha-1)\delta t} - \widehat{C_0}\right]^{\frac{1}{1-m-\alpha}}, & 0 \leq t < T_0, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_0 \leq t < +\infty \end{cases}$$

for $m\left(\frac{N-m-1}{Nm+m+1} - 1\right) \leq \alpha < 1$, and

$$\begin{cases} \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \leq \Bigg[\left(\|u_0\|_{\frac{N(1-m-\alpha)}{m+1}}^{1-m-\alpha} + \widehat{C_1}\right) \\ \qquad \cdot e^{(m+\alpha-1)\delta t} - \widehat{C_1}\Bigg]^{\frac{1}{1-m-\alpha}}, & 0 \leq t < T_1, \\ \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \equiv 0, & T_1 \leq t < +\infty \end{cases}$$

for $-m < \alpha < m\left(\frac{N-m-1}{Nm+m+1} - 1\right)$, where $\widehat{C_0}$ and T_0 are given by (3.5), $\widehat{C_1}$ and T_1 are given by (3.8).

(ii) If $N = 1$, then the nonnegative weak solution of problem (1.1) vanishes in finite time for any nonnegative initial datum u_0 provided that λ is sufficiently small, and we have

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \left[\left(\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_2}\right) e^{(m+\alpha-1)\delta t} - \widehat{C_2}\right]^{\frac{1}{1-m-\alpha}}, & 0 \leq t < T_2, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_2 \leq t < +\infty, \end{cases}$$

where $\widehat{C_2}$ and T_2 are given by (3.11).

(iii) The nonnegative weak solution of problem (1.1) cannot vanish in finite time provided that λ is sufficiently large.

Proof. **(i).** Multiplying the first equation in (1.1) by u^s with $s > 0$, and integrating over Ω by parts, one has

$$\begin{aligned} &\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1}\int_\Omega \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx \\ =&\lambda\left(\frac{m+1}{m+\alpha+s}\right)^q \int_\Omega u^{\frac{s(m+1)-q(\alpha+s-1)}{m+1}}\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^q dx - \delta\int_\Omega u^{s+1}dx. \end{aligned} \tag{3.1}$$

Since $q = m + \alpha < m + 1$, Young's and Hölder's inequalities can be used to obtain

$$\begin{aligned} &\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1}\int_\Omega \left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx \\ \le&\lambda C(\epsilon_1)|\Omega|^{1-\frac{s(m+1)-q(\alpha+s-1)}{(m+1-q)(s+1)}}\left(\frac{m+1}{m+\alpha+s}\right)^q\left(\int_\Omega u^{s+1}dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(m+1-q)(s+1)}} \\ &+\lambda\epsilon_1\left(\frac{m+1}{m+\alpha+s}\right)^q\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx - \delta\int_\Omega u^{s+1}dx. \end{aligned} \tag{3.2}$$

Case a. If $m\left[\frac{N-(m+1)}{Nm+m+1}-1\right] \le \alpha < 1$. For this case, we take $s = \frac{m+\alpha}{m}$ in (3.1). Using Hölder's inequality and Sobolev embedding inequality, we can easily arrive at the following estimate

$$\begin{aligned} \int_\Omega u^{\frac{2m+\alpha}{m}}dx &\le |\Omega|^{1-\frac{2m+\alpha}{m+\alpha}\cdot\frac{N-(m+1)}{N(m+1)}}\left(\int_\Omega u^{\frac{m+\alpha}{m}\cdot\frac{N(m+1)}{N-(m+1)}}dx\right)^{\frac{2m+\alpha}{m+\alpha}\cdot\frac{N-(m+1)}{N(m+1)}} \\ &\le \kappa_1|\Omega|^{1-\frac{2m+\alpha}{m+\alpha}\cdot\frac{N-(m+1)}{N(m+1)}}\left(\int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx\right)^{\frac{2m+\alpha}{(m+1)(m+\alpha)}}, \end{aligned} \tag{3.3}$$

where κ_1 is the embedding constant, depending only on m, α and N. Let ϵ_1 be a sufficiently small constant such that $(m+\alpha)^\alpha - \lambda\epsilon_1 m^\alpha > 0$. Moreover, for such a fixed ϵ_1, one can take λ small enough to ensure that

$$C_{11} = C_{12}\left(\frac{m+\alpha}{m}\right)^\alpha - \lambda\left[\epsilon_1 C_{12} + C(\epsilon_1)|\Omega|^{\frac{m(1-m-\alpha)}{2m+\alpha}}\right]$$

is greater than zero, where

$$C_{12} = \kappa_1^{-\frac{(m+1)(m+\alpha)}{2m+\alpha}}|\Omega|^{\frac{N-(m+1)}{N}-\frac{(m+1)(m+\alpha)}{2m+\alpha}}.$$

Then from (3.2) and (3.3), it follows that

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx + C_{13}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} + C_{14}\int_\Omega u^{\frac{2m+\alpha}{m}}dx \le 0, \tag{3.4}$$

where

$$C_{13} = \frac{2m+\alpha}{m}\left(\frac{m}{m+\alpha}\right)^{m+\alpha}C_{11} \text{ and } C_{14} = \frac{\delta(2m+\alpha)}{m}.$$

Noticing that C_{13}, C_{14} are positive constants and $\frac{(m+1)(m+\alpha)}{2m+\alpha} \in (0,1)$, then from (3.4) and Lemma 2.1, one has

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \le \left[\left(\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_0}\right)e^{(m+\alpha-1)\delta t} - \widehat{C_0}\right]^{\frac{1}{1-m-\alpha}}, & 0 \le t < T_0, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_0 \le t < +\infty, \end{cases}$$

where

$$\widehat{C_0} = C_{13}C_{14}^{-1} \text{ and } T_0 = \frac{1}{\delta(1-m-\alpha)}\ln\left[1 + \widehat{C_0}^{-1}\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha}\right]. \tag{3.5}$$

Case b. If $-m < \alpha < m\left[\frac{N-(m+1)}{Nm+m+1} - 1\right]$. For this case, we choose

$$s = \frac{N\left[1-(m+\alpha)\right]-m-1}{m+1} > \frac{m+\alpha}{m}$$

in (3.1). By the choice of s and Sobolev embedding inequality, we find

$$\begin{aligned} \left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{(m+1)(s+1)}} &= \left(\int_\Omega u^{\frac{N(\alpha+m+s)}{N-(m+1)}}dx\right)^{\frac{N-(m+1)}{N(m+1)}} \\ &\le \kappa_2\left(\int_\Omega \left|\nabla u^{\frac{\alpha+m+s}{m+1}}\right|^{m+1}dx\right)^{\frac{1}{m+1}}, \end{aligned} \tag{3.6}$$

where κ_2 is the embedding constant, depending only on m, α and N. Choosing ϵ_1 sufficiently small such that

$$s\left(m+1\right)^{1-\alpha} - \lambda\epsilon_1\left(m+\alpha+s\right)^{1-\alpha}$$

is a positive number. In addition, once ϵ_1 is fixed, then one can select λ small enough to guarantee that

$$C_{15} = \frac{s}{\kappa_2^{m+1}}\left(\frac{m+1}{m+\alpha+s}\right)^{1-\alpha} - \lambda\left[\frac{\epsilon_1}{\kappa_2^{m+1}} + C\left(\epsilon_1\right)|\Omega|^{\frac{1-m-\alpha}{s+1}}\right] > 0.$$

Then from (3.1) and (3.2) and (3.6), one gets

$$\frac{d}{dt}\int_\Omega u^{s+1}dx + C_{16}\left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{s+1}} + C_{17}\int_\Omega u^{s+1}dx \le 0, \tag{3.7}$$

where

$$C_{16} = (s+1)\left(\frac{m+1}{m+\alpha+s}\right)^{m+\alpha} C_{15} \text{ and } C_{17} = \delta\left(s+1\right).$$

Noticing that C_{16}, C_{17} are positive constants and $\frac{m+\alpha+s}{s+1} \in (0,1)$, then (3.7) and Lemma 2.1 tells us

$$\begin{cases} \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \le \left[\left(\|u_0\|_{\frac{N(1-m-\alpha)}{m+1}}^{1-m-\alpha} + \widehat{C_1}\right)\cdot e^{(m+\alpha-1)\delta t} - \widehat{C_1}\right]^{\frac{1}{1-m-\alpha}}, & 0 \le t < T_1, \\ \|u\|_{\frac{N(1-m-\alpha)}{m+1}} \equiv 0, & T_1 \le t < +\infty, \end{cases}$$

where

$$\widehat{C_1} = C_{16}C_{17}^{-1} \text{ and } T_1 = \frac{1}{\delta\left(1-m-\alpha\right)}\ln\left[1 + \widehat{C_1}^{-1}\|u_0\|_{\frac{N(1-m-\alpha)}{m+1}}^{1-m-\alpha}\right]. \tag{3.8}$$

(ii). For this part, we also take $s = \frac{m+\alpha}{m}$ in (3.1). From $m > 0$ and $0 < m+\alpha < 1$, it follows that $m+1 < \frac{2m+\alpha}{m+\alpha}$. Making using of Sobolev embedding theorem, one has

$$\int_\Omega u^{\frac{2m+\alpha}{m}}dx = \int_\Omega u^{\frac{m+\alpha}{m}\cdot\frac{2m+\alpha}{m+\alpha}}dx \le \kappa_3\left(\int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx\right)^{\frac{2m+\alpha}{(m+1)(m+\alpha)}}. \tag{3.9}$$

where $\kappa_3 = \kappa_3(m,\alpha)$. Let ϵ_1 and λ be sufficiently small such that

$$C_{18} = \left[\left(\frac{m}{m+\alpha}\right)^{1-\alpha} - \lambda\epsilon_1\right]\kappa_3^{-\frac{(m+1)(m+\alpha)}{2m+\alpha}} - \lambda C(\epsilon_1)|\Omega|^{\frac{m(1-m-\alpha)}{2m+\alpha}} > 0.$$

Combining now (3.2) with (3.9), we arrive at

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx + C_{19}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} + C_{14}\int_\Omega u^{\frac{2m+\alpha}{m}}dx \le 0, \tag{3.10}$$

where

$$C_{19} = \frac{2m+\alpha}{m}\left(\frac{m}{m+\alpha}\right)^{m+\alpha} C_{18}.$$

It follows from (3.10) and Lemma 2.1 that

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \le \left[\left(\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_2}\right)e^{(m+\alpha-1)\delta t} - \widehat{C_2}\right]^{\frac{1}{1-m-\alpha}}, & 0 \le t < T_2, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_2 \le t < +\infty, \end{cases}$$

where

$$\widehat{C_2} = C_{19}C_{14}^{-1} \text{ and } T_2 = \frac{1}{\delta(1-m-\alpha)}\ln\left[1+\widehat{C_2}^{-1}\|u_0\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha}\right]. \tag{3.11}$$

(iii). Let λ_1 be the first eigenvalue and $\psi(x)$ be the corresponding eigenfunction of the following problem

$$\begin{cases} -\operatorname{div}\left(\mathcal{U}^\alpha|\nabla\mathcal{U}|^{m-1}\nabla\mathcal{U}\right) = \mu\mathcal{U}^{\alpha+1}|\mathcal{U}|^{m-1}, & x\in\Omega, \\ \mathcal{U}(x) = 0, & x\in\partial\Omega. \end{cases} \tag{3.12}$$

From Lemma 2.3 in [27] (or Lemmas 2.1 and 2.2 in [8]), we can claim that the first eigenfunction $\psi(x)$ is positive. In what follows, we assume that $\max_{x\in\Omega}\psi(x) = 1$. Define a function $f_1(t)$ as follows

$$f_1(t) = d^{\frac{1}{m+\alpha-1}}\left(1-e^{-ct}\right)^{\frac{1}{1-m-\alpha}},$$

where $d\in(1,+\infty)$, and $c\in(0,d(1-m-\alpha))$. Then it is easy to check that

$$f_1(0) = 0 \text{ and } f_1(t)\in(0,1) \text{ for } t>0. \tag{3.13}$$

Furthermore, by a series of calculation, we can verify that

$$f_1'(t) + df_1(t) - f_1^{m+\alpha}(t) < 0. \tag{3.14}$$

Let

$$\mathcal{V}(x,t) = f_1(t)\psi(x).$$

Our next goal is to show that $\mathcal{V}(x,t)$ is a weak subsolution of problem (1.1). By a straightforward computation, for any nonnegative function $\zeta(x,t)\in C_0^\infty(\Omega\times(0,T))$, we have

$$\begin{aligned}I_0 :=& \int_0^t \int_\Omega \left[\mathcal{V}_s(x,s)\zeta(x,s) + \mathcal{V}^\alpha(x,s)|\nabla\mathcal{V}(x,s)|^{m-1}\nabla\mathcal{V}\cdot\nabla\zeta(x,s)\right]dxds\\ &+ \int_0^t\int_\Omega \left[\delta\mathcal{V}(x,s)\zeta(x,s) - \lambda|\nabla\mathcal{V}(x,s)|^{m+\alpha}\zeta(x,s)\right]dxds\\ =& \int_0^t\int_\Omega f_{1s}(s)\psi(x)\zeta(x,s)\,dxds + \int_0^t\int_\Omega \delta f_1(s)\psi(x)\zeta(x,s)\,dxds\\ &+ \int_0^t\int_\Omega f_1^{\alpha+m}(s)\psi^\alpha(x)|\nabla\psi(x)|^{m-1}\nabla\psi(x)\cdot\nabla\zeta(x,s)\,dxds\\ &- \lambda\int_0^t\int_\Omega f_1^{m+\alpha}(s)|\nabla\psi(x)|^{m+\alpha}\zeta(x,s)\,dxds\\ <& \int_0^t\int_\Omega \left\{\left[f_1^{m+\alpha}(s) + (\delta-d)f_1(s)\right]\psi(x)\zeta(x,s)\right\}dxds\\ &+ \int_0^t\int_\Omega f_1^{\alpha+m}(s)\zeta(x,s)\left[\lambda_1\psi^{m+\alpha}(x) - \lambda|\nabla\psi(x)|^{m+\alpha}\right]dxds.\end{aligned}$$

Recalling that $f_1,\ \psi\in(0,1)$, then $0<m+\alpha<1$ tells us that

$$I_0 < \int_0^t\int_\Omega f_1^{m+\alpha}(s)\zeta(x,s)\left[(1+\delta+\lambda_1)\psi^{m+\alpha}(x) - \lambda|\nabla\psi(x)|^{m+\alpha}\right]dxds. \tag{3.15}$$

If

$$\lambda > \frac{(1+\delta+\lambda_1)\|\psi\|_{m+\alpha}^{m+\alpha}}{\|\nabla\psi\|_{m+\alpha}^{m+\alpha}},$$

then we can immediately claim that $I_0<0$, which implies that $\mathcal{V}(x,t)$ is a weak subsolution of problem (1.1). Then according to comparison principle, we see that $u(x,t) > \mathcal{V}(x,t) > 0$ holds for $(x,t)\in\Omega\times(0,+\infty)$, which implies that, for any nonzero nonnegative initial data u_0, the weak solution of problem (1.1) cannot vanish in finite time provided that λ is sufficiently large. The proof of Theorem 3.1 is complete. □

The following theorem shows that the extinction behavior will occur if $m+\alpha<q<\frac{m+1}{2-\alpha}$, and the initial data is sufficiently small.

Theorem 3.2. *Assume that $0<m+\alpha<1$, $\beta=1$ and $m+\alpha<q<\frac{m+1}{2-\alpha}$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that u_0 is sufficiently small. Furthermore,*

(i) if $N\geq 2$, then we have

$$\begin{cases}\|u\|_{\frac{2m+\alpha}{m}} \leq \|u_0\|_{\frac{2m+\alpha}{m}} e^{-\frac{m\chi_1}{2m+\alpha}t}, & 0\leq t<T_3,\\ \|u\|_{\frac{2m+\alpha}{m}} \leq \Bigg[\left(\|u(\cdot,T_3)\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_3}\right) & \\ \qquad\qquad \cdot e^{(m+\alpha-1)\delta(t-T_3)} - \widehat{C_3}\Bigg]^{\frac{1}{1-m-\alpha}}, & T_3\leq t<T_4,\\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_4\leq t<+\infty\end{cases}$$

for $m\left(\frac{N-m-1}{Nm+m+1}-1\right)\leq\alpha<1$, *and*

$$\begin{cases}\|u\|_{\frac{N[1-(m+\alpha)]}{m+1}}\leq\|u_0\|_{\frac{N[1-(m+\alpha)]}{m+1}}e^{-\frac{(m+1)\chi_2}{N[1-(m+\alpha)]}t}, & 0\leq t<T_5,\\ \|u\|_{\frac{N[1-(m+\alpha)]}{m+1}}\leq\left[\left(\|u(\cdot,T_5)\|^{1-m-\alpha}_{\frac{N[1-(m+\alpha)]}{m+1}}+\widehat{C_4}\right)\right.\\ \qquad\qquad\left.\cdot e^{(m+\alpha-1)\delta(t-T_5)}-\widehat{C_4}\right]^{\frac{1}{1-m-\alpha}}, & T_5\leq t<T_6,\\ \|u\|_{\frac{N[1-(m+\alpha)]}{m+1}}\equiv0, & T_6\leq t<+\infty\end{cases}$$

for $-m<\alpha<m\left(\frac{N-m-1}{Nm+m+1}-1\right)$, *where* χ_1 *and* χ_2 *are suitable positive constants, and* $\widehat{C_3}$ *and* T_4 *are given by (3.20),* $\widehat{C_4}$ *and* T_6 *are given by (3.25).*

(ii) if $N=1$, *then we have*

$$\begin{cases}\|u\|_{\frac{2m+\alpha}{m}}\leq\|u_0\|_{\frac{2m+\alpha}{m}}e^{-\frac{m\chi_3}{2m+\alpha}t}, & 0\leq t<T_7,\\ \|u\|_{\frac{2m+\alpha}{m}}\leq\left[\left(\|u(\cdot,T_7)\|^{1-m-\alpha}_{\frac{2m+\alpha}{m}}+\widehat{C_5}\right)\right.\\ \qquad\qquad\left.\cdot e^{(m+\alpha-1)\delta(t-T_7)}-\widehat{C_5}\right]^{\frac{1}{1-m-\alpha}}, & T_7\leq t<T_8,\\ \|u\|_{\frac{2m+\alpha}{m}}\equiv0, & T_8\leq t<+\infty,\end{cases}$$

where χ_3 *is an appropriate positive constant,* $\widehat{C_5}$ *and* T_8 *are given by (3.30).*

Proof. Notice that (3.2) still holds for $q\in\left(m+\alpha,\frac{m+1}{2-\alpha}\right)$.

(i). Case a. If $m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]\leq\alpha<1$. Similar to the process of the derivation of (3.4), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx+C_{20}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}}+C_{14}\int_\Omega u^{\frac{2m+\alpha}{m}}dx\\ \leq&C_{21}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}},\end{aligned}\tag{3.16}$$

where

$$C_{20}=\frac{2m+\alpha}{m}\left[\left(\frac{m}{m+\alpha}\right)^m-\lambda\epsilon_1\left(\frac{m}{m+\alpha}\right)^q\right]C_{12},$$

and

$$C_{21}=\frac{\lambda C(\epsilon_1)(2m+\alpha)}{m}\left(\frac{m}{m+\alpha}\right)^q|\Omega|^{1-\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}.$$

Let $u_0(x)$ be sufficiently small to satisfy

$$\left(\int_\Omega u_0^{\frac{2m+\alpha}{m}}dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}}\leq C_{20}C_{21}^{-1},$$

then by virtue of (3.16) and Lemma 2.2, we know that there exists a constant $\chi_1 > C_{14}$ such that, for $t \geq 0$,

$$\int_\Omega u^{\frac{2m+\alpha}{m}} dx \in \left[0, e^{-\chi_1 t} \int_\Omega u_0^{\frac{2m+\alpha}{m}} dx\right]. \tag{3.17}$$

In addition, from (3.17), one can conclude that that there exists a positive number T_3 such that, for $t \geq T_3$,

$$\begin{aligned} C_{22} &= C_{20} - C_{21} \left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} \\ &\geq C_{20} - C_{21} \left(e^{-\chi_1 T_3} \int_\Omega u_0^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} \\ &> 0. \end{aligned} \tag{3.18}$$

It follows from (3.16) and (3.18) that

$$\frac{d}{dt} \int_\Omega u^{\frac{2m+\alpha}{m}} dx + C_{22} \left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} + C_{14} \int_\Omega u^{\frac{2m+\alpha}{m}} dx \leq 0. \tag{3.19}$$

Combining (3.19) with Lemma 2.1, we get

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \Bigg[\left(\|u(\cdot, T_3)\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_3} \right) \\ \qquad\qquad \cdot e^{(m+\alpha-1)\delta(t-T_3)} - \widehat{C_3} \Bigg]^{\frac{1}{1-m-\alpha}}, & T_3 \leq t < T_4, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_4 \leq t < +\infty, \end{cases}$$

where

$$\widehat{C_3} = C_{22} C_{14}^{-1} \text{ and } T_4 = \frac{1}{\delta(1-m-\alpha)} \ln\left[1 + \widehat{C_3}^{-1} \|u(\cdot, T_3)\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha}\right] + T_3. \tag{3.20}$$

Case b. If $-m < \alpha < m\left[\frac{N-(m+1)}{Nm+m+1} - 1\right]$. By using the similar manners as the derivation of (3.7), we have

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u^{s+1} dx + C_{23} \left(\int_\Omega u^{s+1} dx\right)^{\frac{m+\alpha+s}{s+1}} + C_{17} \int_\Omega u^{s+1} dx \\ \leq &C_{24} \left(\int_\Omega u^{s+1} dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}}, \end{aligned} \tag{3.21}$$

where

$$C_{23} = \frac{s+1}{\kappa_2^{m+1}} \left[s \left(\frac{m+1}{m+\alpha+s}\right)^{m+1} - \lambda \epsilon_1 \left(\frac{m+1}{m+\alpha+s}\right)^q \right],$$

and

$$C_{24} = \lambda C(\epsilon_1)(s+1)|\Omega|^{1-\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}} \left(\frac{m+1}{m+\alpha+s}\right)^q.$$

Choosing u_0 small enough such that

$$\left(\int_\Omega u_0^{s+1}dx\right)^{\frac{(m+1)[q-(m+\alpha)]}{(s+1)(m+1-q)}} \leq C_{23}C_{24}^{-1},$$

then Lemma 2.2 tells us that there exists a constant $\chi_2 > C_{17}$ such that

$$\int_\Omega u^{s+1}dx \in \left[0, e^{-\chi_2 t}\int_\Omega u_0^{s+1}dx\right] \tag{3.22}$$

holds for all $t \geq 0$. Furthermore, from (3.22), we see that there exists a positive number T_5 such that, for $t \geq T_5$,

$$\begin{aligned} C_{25} &= C_{23} - C_{24}\left(\int_\Omega u^{s+1}dx\right)^{\frac{(m+1)[q-(m+\alpha)]}{(s+1)(m+1-q)}} \\ &\geq C_{23} - C_{24}\left(e^{-\chi_2 T_5}\int_\Omega u_0^{s+1}dx\right)^{\frac{(m+1)[q-(m+\alpha)]}{(s+1)(m+1-q)}} \\ &> 0. \end{aligned} \tag{3.23}$$

It follows from (3.21) and (3.23) that

$$\frac{d}{dt}\int_\Omega u^{s+1}dx + C_{25}\left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{s+1}} + C_{17}\int_\Omega u^{s+1}dx \leq 0. \tag{3.24}$$

Lemma 2.1 and (3.24) leads to

$$\begin{cases} \|u\|_{\frac{N[1-(m+\alpha)]}{m+1}} \leq \left[\left(\|u(\cdot,T_5)\|^{1-m-\alpha}_{\frac{N[1-(m+\alpha)]}{m+1}} + \widehat{C_4}\right)\right. \\ \qquad\qquad \left.\cdot\, e^{(m+\alpha-1)\delta(t-T_5)} - \widehat{C_4}\right]^{\frac{1}{1-m-\alpha}}, & T_5 \leq t < T_6, \\ \|u\|_{\frac{N[1-(m+\alpha)]}{m+1}} \equiv 0, & T_6 \leq t < +\infty, \end{cases}$$

where

$$\widehat{C_4} = C_{25}C_{17}^{-1} \text{ and } T_6 = \frac{1}{\delta(1-m-\alpha)}\ln\left[1 + \widehat{C_4}^{-1}\|u(\cdot,T_5)\|^{1-m-\alpha}_{\frac{2m+\alpha}{m}}\right] + T_5. \tag{3.25}$$

(ii). For this part, in view of (3.2) (with $s = \frac{m+\alpha}{m}$) and (3.9), one has

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx + C_{26}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} + C_{14}\int_\Omega u^{\frac{2m+\alpha}{m}}dx \\ \leq &C_{21}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}, \end{aligned} \tag{3.26}$$

where

$$C_{26} = \frac{2m+\alpha}{m}\kappa_3^{-\frac{(m+1)(m+\alpha)}{2m+\alpha}}\left[\left(\frac{m}{m+\alpha}\right)^m - \lambda\epsilon_1\left(\frac{m}{m+\alpha}\right)^q\right].$$

From Lemma 2.2, we see that, for any $t \geq 0$, there exists a constant $\chi_3 > C_{14}$ such that

$$\int_\Omega u^{\frac{2m+\alpha}{m}} dx \in \left[0, e^{-\chi_3 t} \int_\Omega u_0^{\frac{2m+\alpha}{m}} dx\right] \tag{3.27}$$

provided that

$$\left(\int_\Omega u_0^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} \leq C_{26} C_{21}^{-1}.$$

Moreover, from (3.27), one can claim that there exists a positive number T_7 such that, for $t \geq T_7$,

$$\begin{aligned} C_{27} &= C_{26} - C_{21} \left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} \\ &\geq C_{26} - C_{21} \left(e^{-\chi_3 T_7} \int_\Omega u_0^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m(m+1)[q-(m+\alpha)]}{(2m+\alpha)(m+1-q)}} \\ &> 0. \end{aligned} \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\frac{d}{dt} \int_\Omega u^{\frac{2m+\alpha}{m}} dx + C_{27} \left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}} + C_{14} \int_\Omega u^{\frac{2m+\alpha}{m}} dx \leq 0. \tag{3.29}$$

Combining (3.29) with Lemma 2.1, we deduce that

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \Bigg[\left(\|u(\cdot, T_7)\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha} + \widehat{C_5}\right) \\ \qquad\qquad \cdot e^{(m+\alpha-1)\delta(t-T_7)} - \widehat{C_5} \Bigg]^{\frac{1}{1-m-\alpha}}, & T_7 \leq t < T_8, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_8 \leq t < +\infty, \end{cases}$$

where

$$\widehat{C_5} = C_{27} C_{14}^{-1} \text{ and } T_8 = \frac{1}{\delta(1-m-\alpha)} \ln\left[1 + \widehat{C_5}^{-1} \|u(\cdot, T_7)\|_{\frac{2m+\alpha}{m}}^{1-m-\alpha}\right] + T_7. \tag{3.30}$$

The proof of Theorem 3.2 is complete. □

The next theorem is about the non-extinction result for the case $q < m + \alpha$.

Theorem 3.3. *Assume that $0 < m + \alpha < 1$, $\beta = 1$ and $q < m + \alpha$, then for any nonzero nonnegative initial datum u_0, the nonnegative weak solution u of problem (1.1) cannot possess extinction phenomenon provided that λ is sufficiently large.*

Proof. The proof is similar to that of part (iii) of Theorem 3.1, so we sketch it briefly here. Define a function $f_2(t)$ as follows

$$f_2(t) = d^{\frac{1}{q-m-\alpha}} \left(1 - e^{-ct}\right)^{\frac{1}{1-q}},$$

where $d \in (\max\{1, 2\delta\}, +\infty)$, and $c > 0$. It is obvious that $f_2(t)$ satisfies (3.13). Moreover, by fixing $c \in \left(0, (m+\alpha-q)\, d^{\frac{1-q}{m+\alpha-q}}\right)$, then direct computation and the inequality

$$(1-x)^a + ax < 1 \text{ for } x, a \in (0,1),$$

yield that

$$f_2'(t) + \frac{d}{2}\left[f_2(t) + f_2^{m+\alpha}(t)\right] - f_2^q(t) < 0. \tag{3.31}$$

Put

$$\mathcal{W}(x,t) = f_2(t)\,\psi(x),$$

where $\psi(x)$ is the same as that in the proof of Theorem 3.1. If

$$\lambda > \frac{(1+\lambda_1)\,\|\psi\|_{m+\alpha}^{m+\alpha}}{\|\nabla\psi\|_q^q},$$

then we can immediately show that $\mathcal{W}(x,t)$ is a weak subsolution of problem (1.1). Consequently, from comparison principle, it follows that $u(x,t) > \mathcal{W}(x,t) > 0$ for all $(x,t) \in \Omega \times (0,+\infty)$, which means that, for any nonzero nonnegative initial data u_0, extinction phenomenon in finite time cannot occur for sufficiently large λ. The proof of theorem 3.3 is complete. □

Remark 3.1. From Theorems 3.1, 3.2 and 3.3, we know that $q = m + \alpha$ is the critical extinction exponent of the weak solution of problem (1.1) with $\beta = 1$ and $m + \alpha \in (0,1)$.

4. The case $\beta \in (0,1)$

The main purpose of this section is to investigate the extinction behavior of the weak solution for problem (1.1) in the case $\beta \in (0,1)$.

Theorem 4.1. *Assume that $0 < m+\alpha < 1$, $0 < \beta < 1$ and $q = m + \alpha$.*

(i) If $N \geq 2$, then the nonnegative weak solution of problem (1.1) vanishes in finite time for any nonnegative initial datum u_0 provided that λ is sufficiently small. Furthermore, we have

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \|u_0\|_{\frac{2m+\alpha}{m}} \left[1 - \widehat{C_6}\,\|u_0\|_{\frac{2m+\alpha}{m}}^{\frac{m\Gamma_1 - 2m - \alpha}{m}}\, t\right]^{\frac{m}{2m+\alpha-m\Gamma_1}}, & 0 \leq t < T_9, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_9 \leq T < +\infty \end{cases}$$

for $m\left(\frac{N-m-1}{Nm+m+1} - 1\right) \leq \alpha < 1$, and

$$\begin{cases} \|u\|_{s+1} \leq \|u_0\|_{s+1} \left[1 - \widehat{C_7}\,\|u_0\|_{s+1}^{\Gamma_2 - s - 1}\, t\right]^{\frac{1}{s+1-\Gamma_2}}, & 0 \leq t < T_{10}, \\ \|u\|_{s+1} \equiv 0, & T_{10} \leq T < +\infty \end{cases}$$

for $-m < \alpha < m\left(\frac{N-m-1}{Nm+m+1} - 1\right)$, where $s > \frac{N[1-(m+\alpha)]-m-1}{m+1}$, and Γ_1, T_9, $\widehat{C_6}$, Γ_2, T_{10} and $\widehat{C_7}$ are given by (4.4), (4.10), (4.11), (4.12), (4.17) and (4.18), respectively.

(ii) If $N \geq 1$ and $0 < q = m + \alpha \leq \beta < 1$, then the nonnegative weak solution of problem (1.1) cannot vanish in finite time provided that λ is sufficiently large.

Proof. **(i).** Multiplying the first equation in (1.1) by u^s with $s>0$, and integrating over Ω by parts, one has

$$\begin{aligned}&\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx+s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx\\ =&\lambda\left(\frac{m+1}{m+\alpha+s}\right)^{q}\int_\Omega u^{\frac{s(m+1)-q(\alpha+s-1)}{m+1}}\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{q}dx-\delta\int_\Omega u^{s+\beta}dx.\end{aligned}\tag{4.1}$$

Since $q=m+\alpha<m+1$, Young's and Hölder's inequalities yield that

$$\begin{aligned}&\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx+C_{28}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx\\ \leq&C_{29}\left(\int_\Omega u^{s+1}dx\right)^{\frac{m+\alpha+s}{s+1}}-\delta\int_\Omega u^{s+\beta}dx,\end{aligned}\tag{4.2}$$

where

$$C_{28}=s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1}-\lambda\epsilon_2\left(\frac{m+1}{m+\alpha+s}\right)^{m+\alpha},$$

and

$$C_{29}=\lambda C\left(\epsilon_2\right)|\Omega|^{\frac{1-m-\alpha}{s+1}}\left(\frac{m+1}{m+\alpha+s}\right)^{m+\alpha}.$$

It is easy to verify that C_{28} is a positive constant provided that ϵ_2 is sufficiently small.

Case a. If $m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]\leq\alpha<1$. For this case, by taking $s=\frac{m+\alpha}{m}$ in (4.2), we arrive at

$$\begin{aligned}&\frac{m}{2m+\alpha}\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}dx+C_{28}\int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx\\ \leq&C_{29}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)(m+\alpha)}{2m+\alpha}}-\delta\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}}dx,\end{aligned}\tag{4.3}$$

For the sake of simplicity, we denote

$$\rho_1=\frac{Nm\left(m+1\right)\left(m+\alpha\right)\left(1-\beta\right)}{\left(2m+\alpha\right)\left[\left(m+1\right)\left(\alpha+m\left(\beta+1\right)\right)+mN\left(m+\alpha-\beta\right)\right]},$$

and

$$\Gamma_1=\frac{\left(m+1\right)\left(m+\alpha\right)\left[\alpha+m\left(\beta+1\right)\right]}{m\left(m+1\right)\left(m+\alpha\right)\left(1-\rho_1\right)+m\rho_1\left[\alpha+m\left(\beta+1\right)\right]}.\tag{4.4}$$

Recalling that $\beta\in(0,1)$ and $m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]\leq\alpha<1$, we can verify that $\rho_1\in(0,1)$, and

$$\Gamma_1<\frac{\alpha+m\left(\beta+1\right)}{m\left(1-\rho_1\right)},\quad\frac{m\rho_1\Gamma_1}{\left(m+1\right)\left(m+\alpha\right)}\cdot\frac{1}{1-\frac{m(1-\rho_1)\Gamma_1}{\alpha+m(1+\beta)}}=1.\tag{4.5}$$

Now, using Lemma 2.3 with $v=u^{\frac{m+\alpha}{m}}$, $\mu=\frac{2m+\alpha}{m+\alpha}$, $p=m+1$ and $r=\frac{\alpha+m(1+\beta)}{m+\alpha}$, we deduce that

$$\left(\int_\Omega u^{\frac{2m+\alpha}{m}}dx\right)^{\frac{m+\alpha}{2m+\alpha}}\leq\kappa_4\left(\int_\Omega\left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1}dx\right)^{\frac{\rho_1}{m+1}}\left(\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}}dx\right)^{\frac{(m+\alpha)(1-\rho_1)}{\alpha+m(1+\beta)}},$$

where $\kappa_4 = \kappa_4 (N, m, \alpha, \beta)$. Furthermore, we have

$$\left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m+\alpha}{2m+\alpha}\cdot\frac{m\Gamma_1}{m+\alpha}} \le \kappa_4^{\frac{m\Gamma_1}{m+\alpha}} \left(\int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx\right)^{\frac{\rho_1}{m+1}\cdot\frac{m\Gamma_1}{m+\alpha}} \times \left(\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}} dx\right)^{\frac{(m+\alpha)(1-\rho_1)}{\alpha+m(1+\beta)}\cdot\frac{m\Gamma_1}{m+\alpha}}.$$

Noticing that (4.5), and making use of Young's inequality, we obtain

$$\begin{aligned}\left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} &\le \kappa_4^{\frac{m\Gamma_1}{m+\alpha}} \left(\int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx\right)^{\frac{m\rho_1\Gamma_1}{(m+1)(m+\alpha)}} \\ &\quad\times \left(\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}} dx\right)^{\frac{m(1-\rho_1)\Gamma_1}{\alpha+m(1+\beta)}} \\ &\le \kappa_4^{\frac{m\Gamma_1}{m+\alpha}} \left(\epsilon_3 \int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx + C(\epsilon_3)\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}} dx\right),\end{aligned} \tag{4.6}$$

which implies that

$$\int_\Omega u^{\frac{\alpha+m(1+\beta)}{m}} dx \ge \frac{1}{\kappa_4^{\frac{m\Gamma_1}{m+\alpha}} C(\epsilon_3)} \left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} - \frac{\epsilon_3}{C(\epsilon_3)} \int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx. \tag{4.7}$$

Combining now (3.3), (4.3) and (4.7), one has

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}} dx + C_{30}\int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx + C_{31}\left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} \le 0, \tag{4.8}$$

where

$$C_{30} = \frac{2m+\alpha}{m}\left[C_{28} - \frac{\delta\epsilon_3}{C(\epsilon_3)} - \frac{C_{29}}{C_{12}}\right] \text{ and } C_{31} = \frac{C_{14}}{\kappa_4^{\frac{m\Gamma_1}{m+\alpha}} C(\epsilon_3)}.$$

Noticing that if λ is suitable small, then we have that $C_{28} - \frac{C_{29}}{C_{12}}$ is a positive number. Furthermore, for such a fixed λ, one can choose ϵ_3 small enough to ensure that C_{30} is positive. Then (4.8) tells us

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}} dx + C_{31}\left(\int_\Omega u^{\frac{2m+\alpha}{m}} dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} \le 0. \tag{4.9}$$

Integrating (4.9), we deduce that

$$\|u\|_{\frac{2m+\alpha}{m}} \le \|u_0\|_{\frac{2m+\alpha}{m}} \left[1 - \widehat{C_6}\,\|u_0\|_{\frac{2m+\alpha}{m}}^{\frac{m\Gamma_1-2m-\alpha}{m}} t\right]_+^{\frac{m}{2m+\alpha-m\Gamma_1}},$$

which implies that $u(x,t)$ vanishes in finite time

$$T_9 = \widehat{C_6}^{-1} \|u_0\|_{\frac{2m+\alpha}{m}}^{2m+\alpha-m\Gamma_1}, \tag{4.10}$$

where

$$\widehat{C_6} = \frac{C_{31}\left(2m+\alpha\right)}{2m+\alpha-m\Gamma_1}. \tag{4.11}$$

Case b. If $-m<\alpha<m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]$. For this case, in (4.2), we choose

$$s>\frac{N\left[1-(m+\alpha)\right]-m-1}{m+1}>\frac{m+\alpha}{m}.$$

Denote

$$\rho_2=\frac{N\left(1-\beta\right)\left(m+\alpha+s\right)}{\left(s+1\right)\left[\left(m+1\right)\left(s+\beta\right)-N\left(\beta-m-\alpha\right)\right]},$$

and

$$\Gamma_2=\frac{\left(s+1\right)\left[\left(m+1\right)\left(s+\beta\right)-N\left(\beta-m-\alpha\right)\right]}{\left(m+1\right)\left(1+\beta\right)-N\left(\beta-m-\alpha\right)}. \tag{4.12}$$

By the choice of s and recalling that $\beta\in(0,1)$ and $-m<\alpha<m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]$, we can prove that $\rho_2\in(0,1)$ and $\Gamma_2\in(s,s+1)$. Now, using Gagliardo-Nirenberg multiplicative embedding inequality and Young's inequality, and by the similar arguments of the processes of the derivation of (4.6), we obtain

$$\begin{aligned}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}} &\leq \kappa_5\left(\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx\right)^{\frac{\rho_2\Gamma_2}{m+\alpha+s}}\left(\int_\Omega u^{s+\beta}dx\right)^{\frac{\Gamma_2(1-\rho_2)}{s+\beta}}\\ &\leq \kappa_5\left(\epsilon_4\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx+C\left(\epsilon_4\right)\int_\Omega u^{s+\beta}dx\right),\end{aligned} \tag{4.13}$$

where $\kappa_5=\kappa_5\left(N,m,\alpha,\beta,s\right)$. (4.13) means that

$$\int_\Omega u^{s+\beta}dx\geq\frac{1}{\kappa_5C\left(\epsilon_4\right)}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}}-\frac{\epsilon_4}{C\left(\epsilon_4\right)}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx. \tag{4.14}$$

It follows from (3.6), (4.2) and (4.14) that

$$\frac{d}{dt}\int_\Omega u^{s+1}dx+C_{32}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx+C_{33}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}}\leq 0, \tag{4.15}$$

where

$$C_{32}=(s+1)\left[C_{28}-\frac{\epsilon_4\delta}{C\left(\epsilon_4\right)}-C_{29}\kappa_2^{m+1}\right] \text{ and } C_{33}=\frac{C_{17}}{\kappa_5C\left(\epsilon_4\right)}.$$

Let λ be small enough such that C_{29} is sufficiently small, then we have $C_{32}>0$ by choosing ϵ_4 small enough, and hence, (4.15) implies that

$$\frac{d}{dt}\int_\Omega u^{s+1}dx+C_{33}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}}\leq 0. \tag{4.16}$$

Integrating (4.16) from 0 to t, we deduce that

$$\|u\|_{s+1}\leq\|u_0\|_{s+1}\left[1-\widehat{C_7}\,\|u_0\|_{s+1}^{\Gamma_2-s-1}\,t\right]_+^{\frac{1}{s+1-\Gamma_2}},$$

which means that $u(x,t)$ vanishes in finite time

$$T_{10} = \widehat{C_7}^{-1} \|u_0\|_{s+1}^{s+1-\Gamma_2}, \tag{4.17}$$

where

$$\widehat{C_7} = \frac{C_{33}(s+1)}{s+1-\Gamma_2}. \tag{4.18}$$

(ii). The proof is similar to that of part (iii) of Theorem 3.1, so we sketch it briefly here. Define a function $f_3(t)$ as follows

$$f_3(t) = \begin{cases} d^{\frac{1}{m+\alpha-\beta}} \left(1-e^{-ct}\right)^{\frac{1}{1-m-\alpha}}, & q = m+\alpha < \beta, \\ \left[(1-m-\alpha)t\right]^{\frac{1}{1-m-\alpha}}, & q = m+\alpha = \beta, \end{cases}$$

where $d \in (1,+\infty)$, and $c \in \left(0, (\beta-m-\alpha)\, d^{\frac{1-m-\alpha}{\beta-m-\alpha}}\right)$. Then it is easy to check that $f_3(t)$ satisfies

$$f_3(0) = 0 \text{ and } f_3(t) \in (0,1) \text{ for } t > 0,$$

and

$$\begin{cases} f_3'(t) + d f_3^{\beta}(t) - f_3^{m+\alpha}(t) < 0, & q = m+\alpha < \beta, \\ f_3'(t) = f_3^{m+\alpha}(t), & q = m+\alpha = \beta. \end{cases}$$

Let

$$\mathcal{X}(x,t) = f_3(t)\,\psi(x),$$

where $\psi(x)$ is the same as that in the proof of Theorem 3.1. By a straightforward computation, we can claim that $\mathcal{X}(x,t)$ is a weak subsolution of problem (1.1) provided that

$$\lambda > \frac{(1+\delta+\lambda_1)\,\|\psi\|_{m+\alpha}^{m+\alpha}}{\|\nabla\psi\|_{m+\alpha}^{m+\alpha}}.$$

Then by comparison principle, we know that, for sufficiently large λ, the weak solution of problem (1.1) cannot vanish in finite time. The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Assume that $0 < m+\alpha < 1$, $0 < \beta < 1$, and $N \geq 2$.*

(i) If $m\left(\frac{N-m-1}{Nm+m+1} - 1\right) \leq \alpha < 1$ and $\frac{(m+1)[m(\Gamma_1-1)-\alpha]}{m\Gamma_1-\alpha(m+1)} < q < \frac{m+1}{2-\alpha}$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that u_0 is sufficiently small. Furthermore, we have

$$\begin{cases} \|u\|_{\frac{2m+\alpha}{m}} \leq \|u_0\|_{\frac{2m+\alpha}{m}} \left[1 - \widehat{C_8}\, \|u_0\|_{\frac{2m+\alpha}{m}}^{\frac{m\Gamma_1-2m-\alpha}{m}}\, t\right]^{\frac{m}{2m+\alpha-m\Gamma_1}}, & 0 \leq t < T_{11}, \\ \|u\|_{\frac{2m+\alpha}{m}} \equiv 0, & T_{11} \leq t < +\infty, \end{cases}$$

where Γ_1, T_{11} and $\widehat{C_8}$ are appropriate positive constants, given by (4.4), (4.22) and (4.23), respectively.

(ii) If $-m < \alpha < m\left[\frac{N-(m+1)}{Nm+m+1} - 1\right]$ and $\frac{(m+1)(\Gamma_2 - s)}{\Gamma_2+1-\alpha-s} < q < \frac{m+1}{2-\alpha}$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that u_0 is sufficiently small. Furthermore, we have

$$\begin{cases} \|u\|_{s+1} \le \|u_0\|_{s+1}\left[1 - \widehat{C_9}\,\|u_0\|_{s+1}^{\Gamma_2 - s-1}\, t\right]^{\frac{1}{s+1-\Gamma_2}}, & 0 \le t < T_{12}, \\ \|u\|_{s+1} \equiv 0, & T_{12} \le t < +\infty, \end{cases}$$

where $s > \frac{N[1-(m+\alpha)]-m-1}{m+1}$, and Γ_2, T_{12} and $\widehat{C_9}$ are suitable positive constants, given by (4.12), (4.27) and (4.28), respectively.

Proof. **(i).** For $m\left(\frac{N-m-1}{Nm+m+1} - 1\right) \le \alpha < 1$ and $\frac{(m+1)[m(\Gamma_1-1)-\alpha]}{m\Gamma_1 - \alpha(m+1)} < q < \frac{m+1}{2-\alpha}$. Taking $s = \frac{m+\alpha}{m}$ in (4.1), and applying Young's inequality and Hölder's inequality, we arrive at

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx + C_{14}\int_\Omega u^{\frac{\alpha+m(\beta+1)}{m}}\,dx + C_{34}\int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx \\ \le & C_{35}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}, \end{aligned} \tag{4.19}$$

where

$$C_{34} = \frac{2m+\alpha}{m}\left[\left(\frac{m}{m+\alpha}\right)^m - \lambda\epsilon_5\left(\frac{m}{m+\alpha}\right)^q\right],$$
$$C_{35} = \frac{\lambda C(\epsilon_5)(2m+\alpha)}{m}\left(\frac{m}{m+\alpha}\right)^q |\Omega|^{1-\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}},$$

and ϵ_5 is a sufficiently small positive number such that $C_{34} > 0$. Using (4.6) and (4.19), we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx + C_{36}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} + C_{37}\int_\Omega \left|\nabla u^{\frac{m+\alpha}{m}}\right|^{m+1} dx \\ \le & C_{35}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx\right)^{\frac{(m+1)[m+\alpha(1-q)]}{(2m+\alpha)(m+1-q)}}, \end{aligned} \tag{4.20}$$

where

$$C_{36} = C_{14}\left[\kappa_4 C(\epsilon_3)\right]^{-1}, \quad C_{37} = C_{34} - \epsilon_3 C_{14}\left[C(\epsilon_3)\right]^{-1},$$

and ϵ_3 is small enough such that $C_{37} > 0$. If

$$q > \frac{(m+1)\left[m(\Gamma_1 - 1) - \alpha\right]}{m\Gamma_1 - \alpha(m+1)},$$

and

$$\int_\Omega u_0^{\frac{2m+\alpha}{m}}\,dx \le \left(C_{35}^{-1}C_{36}\right)^{\frac{(2m+\alpha)(m+1-q)}{(m+1)[m+\alpha(1-q)] - m\Gamma_1(m+1-q)}},$$

then (4.20) leads to

$$\frac{d}{dt}\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx + C_{38}\left(\int_\Omega u^{\frac{2m+\alpha}{m}}\,dx\right)^{\frac{m\Gamma_1}{2m+\alpha}} \le 0, \tag{4.21}$$

where

$$C_{38} = C_{36} - C_{35}\left(\int_\Omega u_0^{\frac{2m+\alpha}{m}}dx\right)^{\frac{(m+1)[m+\alpha(1-q)]-m\Gamma_1(m+1-q)}{(2m+\alpha)(m+1-q)}}.$$

Integrating (4.21), we obtain

$$\|u\|_{\frac{2m+\alpha}{m}} \le \|u_0\|_{\frac{2m+\alpha}{m}}\left[1-\widehat{C_8}\,\|u_0\|_{\frac{2m+\alpha}{m}}^{\frac{m\Gamma_1-2m-\alpha}{m}}t\right]_+^{\frac{m}{2m+\alpha-m\Gamma_1}},$$

which tells us that $u(x,t)$ vanishes in finite time

$$T_{11} = \widehat{C_8}^{-1}\,\|u_0\|_{\frac{2m+\alpha}{m}}^{\frac{2m+\alpha-m\Gamma_1}{m}}, \tag{4.22}$$

and

$$\widehat{C_8} = \frac{(2m+\alpha)\,C_{38}}{2m+\alpha-m\Gamma_1}. \tag{4.23}$$

(ii). For $-m<\alpha<m\left[\frac{N-(m+1)}{Nm+m+1}-1\right]$ and $\frac{(m+1)(\Gamma_2-s)}{\Gamma_2+1-\alpha-s}<q<\frac{m+1}{2-\alpha}$. Using (4.1) with

$$s > \frac{N[1-(m+\alpha)]-m-1}{m+1} > \frac{m+\alpha}{m},$$

and applying Young's inequality and Hölder's inequality, one has

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{s+1}dx + C_{17}\int_\Omega u^{s+\beta}dx + C_{39}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx\\ \le& C_{40}\left(\int_\Omega u^{s+1}dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}},\end{aligned} \tag{4.24}$$

where

$$C_{39} = (s+1)\left[s\left(\frac{m+1}{m+\alpha+s}\right)^{m+1} - \lambda\epsilon_6\left(\frac{m+1}{m+\alpha+s}\right)^q\right],$$

$$C_{40} = \lambda C(\epsilon_6)(s+1)|\Omega|^{1-\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}}\left(\frac{m+1}{m+\alpha+s}\right)^q,$$

and ϵ_6 is a sufficiently small positive number such that $C_{39}>0$. Using (4.13) and (4.24), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{s+1}dx + C_{41}\int_\Omega\left|\nabla u^{\frac{m+\alpha+s}{m+1}}\right|^{m+1}dx + C_{42}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}}\\ \le& C_{40}\left(\int_\Omega u^{s+1}dx\right)^{\frac{s(m+1)-q(\alpha+s-1)}{(s+1)(m+1-q)}},\end{aligned} \tag{4.25}$$

where

$$C_{41} = C_{39} - \epsilon_4 C_{17}[C(\epsilon_4)]^{-1},\quad C_{42} = C_{17}[\kappa_5 C(\epsilon_4)]^{-1},$$

and ϵ_4 is small enough such that $C_{41}>0$. If

$$q > \frac{(m+1)(\Gamma_2-s)}{\Gamma_2+1-\alpha-s},$$

and

$$\int_\Omega u_0^{s+1}dx \le \left(C_{40}^{-1}C_{42}\right)^{\frac{(s+1)(m+1-q)}{q(\Gamma_2+1-\alpha-s)-(m+1)(\Gamma_2-s)}},$$

then from (4.25), it follows that

$$\frac{d}{dt}\int_\Omega u^{s+1}dx + C_{43}\left(\int_\Omega u^{s+1}dx\right)^{\frac{\Gamma_2}{s+1}} \le 0, \tag{4.26}$$

where

$$C_{43} = C_{42} - C_{40}\left(\int_\Omega u_0^{s+1}dx\right)^{\frac{q(\Gamma_2+1-\alpha-s)-(m+1)(\Gamma_2-s)}{(s+1)(m+1-q)}}.$$

Integrating (4.26) over $(0,t)$, we find that

$$\|u\|_{s+1} \le \|u_0\|_{s+1}\left[1-\widehat{C_9}\,\|u_0\|_{s+1}^{\Gamma_2-s-1}\,t\right]_+^{\frac{1}{s+1-\Gamma_2}},$$

which means that $u(x,t)$ vanishes in finite time

$$T_{12} = \widehat{C_9}^{-1}\,\|u_0\|_{s+1}^{s+1-\Gamma_2}, \tag{4.27}$$

and

$$\widehat{C_9} = \frac{(s+1)\,C_{43}}{s+1-\Gamma_2}. \tag{4.28}$$

The proof of Theorem 4.2 is complete. □

The final theorem is about the non-extinction result for the case $q < m+\alpha$ and $q \le \beta < 1$.

Theorem 4.3. *Assume that $0 < m+\alpha < 1$, $q < m+\alpha$,and $q \le \beta < 1$, then for any nonzero nonnegative initial datum u_0, the nonnegative weak solution u of problem (1.1) cannot possess extinction phenomenon provided that λ is sufficiently large.*

Proof. The proof is similar to that of part (iii) of Theorem 3.1, so we sketch it briefly here. Define a function $f_4(t)$ as follows

$$f_4(t) = \begin{cases} d^{\frac{1}{q-m-\alpha}}\left(1-e^{-c_1 t}\right)^{\frac{1}{1-q}}, & q=\beta<m+\alpha \text{ or } q<\beta=m+\alpha, \\ d^{\frac{1}{q-\beta}}\left(1-e^{-c_2 t}\right)^{\frac{1}{1-q}}, & q<\beta<m+\alpha, \end{cases}$$

where $d \in (1,+\infty)$, $c_1 \in \left(0,(m+\alpha-q)\,d^{\frac{1-q}{m+\alpha-q}}\right)$ and $c_2 \in \left(0,(\beta-q)\,d^{\frac{1-q}{\beta-q}}\right)$. Then it is easy to check that $f_4(t)$ satisfies

$$f_4(0)=0 \text{ and } f_4(t)\in(0,1) \text{ for } t>0,$$

and

$$\begin{cases} f_4'(t) + d f_4^{m+\alpha}(t) - f_4^q(t) < 0, & q=\beta<m+\alpha \text{ or } q<\beta=m+\alpha, \\ f_4'(t) + \frac{d}{2}\left[f_4^{m+\alpha}+f_4^{\beta}\right] - f_4^q(t) < 0, & q<\beta<m+\alpha. \end{cases}$$

Let

$$\mathcal{Y}(x,t) = f_4(t)\,\psi(x),$$

where $\psi(x)$ is the same as that in the proof of Theorem 3.1. By a straightforward computation, we can claim that $\mathcal{Y}(x,t)$ is a weak subsolution of problem (1.1) provided that

$$\lambda > \begin{cases} \frac{(1+\delta+\lambda_1)\|\psi\|_{m+\alpha}^{m+\alpha}}{\|\nabla\psi\|_q^q}, & q=\beta<m+\alpha \text{ or } q<\beta=m+\alpha, \\ \frac{(1+\delta+\lambda_1)\|\psi\|_\beta^\beta}{\|\nabla\psi\|_q^q}, & q<\beta<m+\alpha. \end{cases}$$

Consequently, by comparison principle, we know that, for sufficiently large λ, the weak solution of problem (1.1) cannot vanish in finite time. The proof of Theorem 4.3 is complete. □

Remark 4.1. In the case of $0<\beta<q\le m+\alpha<1$, we can not prove that the non-negative weak solution u of problem (1.1) does not possess extinction phenomenon provided that λ is sufficiently large.

Acknowledgements

The authors are sincerely grateful to the anonymous referees for a number of valuable remarks and comments, which greatly improved the presentation of the paper.

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