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CHAOTIC BEHAVIOR IN A NEW FRACTIONAL-ORDER LOVE TRIANGLE SYSTEM WITH COMPETITION

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Abstract In the present work, we first modify the Sprott's nonlinear love triangle model by introducing the competition term and find that the new system also exhibits chaotic behavior. Then, to make the model more realistic, we go further to construct its corresponding fractional-order system and get the necessary condition for the existence of chaotic attractors. Finally, based on an improved version of Adams Bashforth Moulton numerical algorithm, we validate the chaotic attractors of this new fractional-order love triangle system by computer simulations.

Keywords Love triangle model, differential equations, competition, fractional-order, chaotic attractors, stimulation.

MSC(2000) 91D10, 93A30, 34A08, 74H65.

1. Introduction

Dynamical phenomena in physics [7, 28], biology [13, 14], economy [2] and all other sciences [1, 8, 27, 29] have been extensively and thoroughly studied by means of differential equations or fractional-order differential equations. However, one of the most important problems concerning our lives, namely, the dynamics of love has not yet been properly tackled in the way. In a one-page influential work [24] and later in a book [25], Strogatz applied a system of linear differential equations to study Shakespearean model of love affair of Romeo and Juliet. The same model was analyzed earlier by Rapoport [17] and also by Radzicki [18]. Recently, the mathematical models capturing the dynamics of love between two people have gained attention among many researchers [4, 12, 19–21, 23, 26], who have provided extensions to Strogatzs seminal model. For example, Rinaldi [19,20] proposed a modified version of the Strogatz's linear model, taking into account the appeal that each partner presents to the others in absent of other feelings. Gottman etc. [4] proposed dynamical models of the verbal interaction of married couples. Sprott [23] presented a sequence of dynamical models involving coupled ordinary differential equations describing the time variation of the love or hate displayed by individuals in a romantic relationship. In [23], Sportt started the analysis with nonlinear system of two individuals and advanced to love triangles and finally made the inclusion of the effect of nonlinearities and demonstrated the origin of chaos.

However, for love triangles, although Sportt's model [23] is encouraging, it is relatively simple and a little far from the realistic case. Therefore, to make the

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model more accurate, we first modify his model by introducing the competition term. Besides, we have learned that the integer-order differential equation is only the special example of fractional-order differential equations and many dynamical systems can be only properly described by using the fractional-order system theory [6, 9–11]. Moreover, there are certain limits of the integer-order models because people's love is a state which is always influenced by history information. Due to the fact that the fractional-order dynamical models possess memory, it offers a broader insight into the modeling of dynamical systems in psychology and life sciences. Taking these factors into consideration, we go further to construct the corresponding fractional-order system.

This paper is organized as follows. In Section 2, we start with modifying the Sprott's nonlinear love triangle system by introducing the competition term, and we will show that the new system also exhibits chaos. In Section 3, we go further to construct its corresponding fractional-order system and get the necessary condition for the existence of chaotic attractors. Also, the chaotic attractors for this fractional-order love triangle system are obtained by computer simulations. And finally, some concluding remarks are made in Section 4.

2. Love triangle system with competition

In [25], Strogatz considered a love affair between Romeo and Juliet, where $R(t)$ is Romeo's love for Juliet at time t and $J(t)$ is Juliet's love for Romeo at the same instant. The simplest linear model is given by

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= aR + bJ, \\ \frac{\mathrm{d}J}{\mathrm{d}t} &= cR + dJ, \end{aligned} \tag{2.1}$$

where a and b specify Romeo's romantic style, and c and d specify Juliet's style. The parameter a describes the extent to which Romeo is encouraged by his own feelings, and b is the extent to which he is encouraged by Juliet's feelings. The parameters d and c have the equivalent significance from the perspective of Juliet. These four parameters can be positive or negative, Romeo can exhibit of four romantic styles depending on the signs of a and b, which are given below:

(1) Eager beaver: $a > 0, b > 0$ (Romeo is encouraged by his own feelings as well as Juliet's.)

(2) Narcissistic nerd: $a > 0, b < 0$ (Romeo wants more of what he feels but retreats from Juliet's feelings.)

(3) Cautious (or secure) lover: $a < 0, b > 0$ (Romeo retreats from his own feelings but is encouraged by Juliet's.)

(4) Hermit: $a < 0, b < 0$ (Romeo retreats from his own feelings as well as Juliet's.) The same format can be generated from Juliet involving the parameters c and d.

However, the model proposed by Strogatz [25] is definitely unrealistic because it does not taken into account the appeal of the two individuals. Thus, Strogatz's model does not explain why two persons who are initially completely indifferent to each other can develop a love affair. We also note that his models only considered the linear relationships of two persons. But, the relationships of two persons involved should be nonlinear. Thus, Strogatz's approach has never been followed on social psychology. In [23], Sprott presented a new nonlinear model

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= aR + bJ\left(1 - |J|\right), \\ \frac{\mathrm{d}J}{\mathrm{d}t} &= cR\left(1 - |R|\right) + dJ, \end{aligned} \tag{2.2}$$

where the logistic function $bJ\left(1 - |J|\right)$ is called "repair nonlinearity" by Gottman etc. [4], which amounts to measuring J in units such that $J = 1$ corresponds to the value at which her love become counterproductive. Qualitatively similar results follow from the function $bJ\left(1 - J^2\right)$, which is the case considered by Rinaldi [20]. This model is more realistic, but its dynamics is limited. It is obvious that the system does not admit limit cycles, and there is no chaos since the system is only two-dimensional.

Richer dynamics can occur if a third party is added. Suppose Romeo has a mistress, Guinevere. In the simplest case, Juliet and Guinevere would not know about one another and Romeo would adopt the same romantic style towards them both. The nonlinear system is given by Sprott in [23] as

$$\begin{aligned} \frac{\mathrm{d}R_J}{\mathrm{d}t} &= aR + bf_1\left(J - G\right), \\ \frac{\mathrm{d}J}{\mathrm{d}t} &= cf_1\left(R_J\right) + dJ, \\ \frac{\mathrm{d}R_G}{\mathrm{d}t} &= aR + bf_1\left(G - J\right), \\ \frac{\mathrm{d}J}{\mathrm{d}t} &= ef_1\left(R_G\right) + fJ, \end{aligned} \tag{2.3}$$

where $f_1\left(x\right) = x\left(1 - |x|\right)$, R_J is Romeo's feelings for Juliet and R_G is Romeo's feelings for Guinevere. This system can exhibit chaos with strange attractors, an example of which is in [23, Fig. 4].

However, the more realistic and complex case is that Juliet knows the existence of Guinevere, and so does Guinevere. Then they may compete with each other. We know that Juliet's love for Romeo will be affect by Guinevere's. If Guinevere show more love for Romeo, then Juliet's love for Romeo will be reduced respectively. The same is true for Guinevere. Therefore, here we use JG as the competition term and take $f_1(x) = x(1 - x^2)$, just as Rinaldi [20] did, then the model becomes

$$\begin{aligned} \frac{\mathrm{d}R_J}{\mathrm{d}t} &= aR_J + b\left(J - G\right)\left(1 - \left(J - G\right)^2\right), \\ \frac{\mathrm{d}J}{\mathrm{d}t} &= cR_J\left(1 - R_J^2\right) + dJ - mJG, \\ \frac{\mathrm{d}R_G}{\mathrm{d}t} &= aR_G + b\left(G - J\right)\left(1 - \left(G - J\right)^2\right), \\ \frac{\mathrm{d}G}{\mathrm{d}t} &= eR_G\left(1 - R_G^2\right) + fG - nJG, \end{aligned} \tag{2.4}$$

where m, n are positive competition coefficients. By adding competition item, we will show that the system will also produce chaos according to different competition coefficients. As an example, here we suppose that Romeo is a cautious lover

($a=-3, b=4$) and Juliet ($c=-7, d=2$) and Guinevereis ($e=2, f=-1$) are narcissistic nerds (of course, we could consider other various situations), then we can obtain the chaotic attractors of system (2.4) corresponding to different competition coefficients m and n, as shown in Figure 1 to Figure 3. Also, in Figure 4, by changing the initial conditions, we show how it will affect the chaotic evolution of Romeo's love for Juliet.

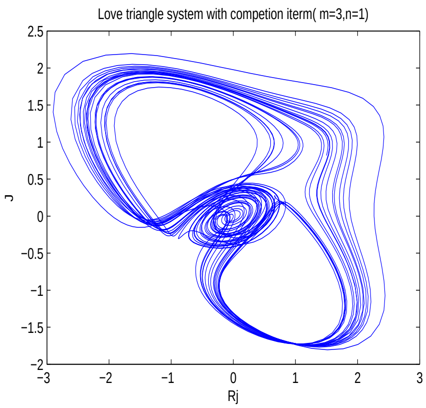

Figure 1. Strange attractor from the nonlinear love triangle in (2.4) with $m=3$ and $n=1$.

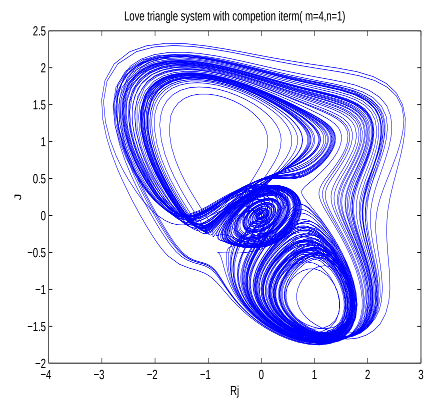

Figure 2. Strange attractor from the nonlinear love triangle in (2.4) with $m=4$ and $n=1$.

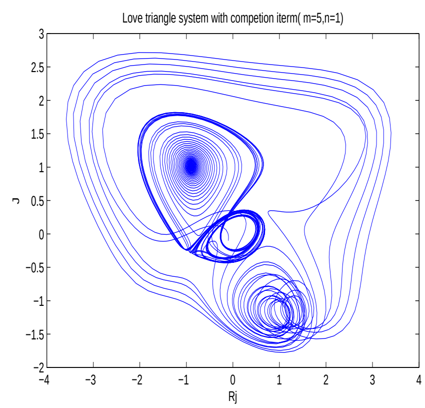

Figure 3. Strange attractor from the nonlinear love triangle in (2.4) with $m=5$ and $n=1$.

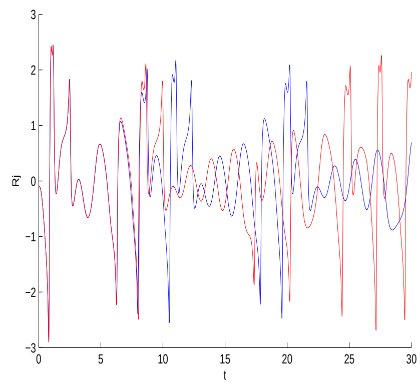

Figure 4. Chaotic evolution of Romeos love for Juliet from (2.4) showing the effect of changing the initial conditions by 1%.

3. Fractional-order love triangle system with competition

It has been found that the behaviors of many physical systems and other systems can be properly described by using the fractional-order system theory, such as such as heat conduction [9], electromagnetic waves [6], quantum evolution of complex systems [10] and quantitative finance [11]. The reason is that the integer-order differential equation is only the special example of fractional-order differential equations. What's more, the integer-order model has its limitation in that peoples love is a state which is always influenced by history information. Based on the fact that the fractional-order dynamical models possess memory, it could present a broader insight into the modeling of dynamical systems. Therefore, to make our model more realistic, we go further to construct the fractional-order love triangle system. In the rest of the paper, we employ Caputo definition of fractional derivative. The Caputo derivative of order q is given by [16] as

$$ {}_aD_t^q = \frac{1}{\Gamma(m-q)} \int_a^t \frac{f^{(m)}}{(t-x)^{q+1-m}} \mathrm{dt}, \tag{3.1} $$

where $m = [q]$, a, t are the integral lower limit and upper limit, q is the differential order.

Using this definition, we can easily get the fractional-order of system (2.4):

$$ \begin{aligned} \frac{\mathrm{d}^{q_1} R_J}{\mathrm{d}t^{q_1}} &= aR_J + b(J-G)\left(1-(J-G)^2\right), \\ \frac{\mathrm{d}^{q_2} J}{\mathrm{d}t^{q_2}} &= cR_J\left(1-R_J^2\right) + dJ - mJG, \\ \frac{\mathrm{d}^{q_3} R_G}{\mathrm{d}t^{q_3}} &= aR_G + b(G-J)\left(1-(G-J)^2\right), \\ \frac{\mathrm{d}^{q_4} G}{\mathrm{d}t^{q_4}} &= eR_G\left(1-R_G^2\right) + fG - nJG, \end{aligned} \tag{3.2} $$

where $0 < q_i < 1$ and $i = 1, 2, 3, 4$. The fractional-order q_i will be determined later. We want to point out here that the order q_i, according to [22], has its practical physical meanings which represents the impact factor of memory of an individual. As above, we also take $a = -3$, $b = 4$, $c = -7$, $d = 2$, $m = -3.5$, $e = 2$, $f = -1$, $n = -1$.

3.1. The necessary condition for the existence of chaotic attractors

Now, we discuss the necessary condition for the existence of chaotic attractors in fractional-order love triangle system with competition (3.2). Set $\frac{\mathrm{d}^{q_1} R_J}{\mathrm{d}t^{q_1}} = 0$, $\frac{\mathrm{d}^{q_2} J}{\mathrm{d}t^{q_2}} = 0$, $\frac{\mathrm{d}^{q_3} R_G}{\mathrm{d}t^{q_3}} = 0$, $\frac{\mathrm{d}^{q_4} G}{\mathrm{d}t^{q_4}} = 0$, we get the following equilibrium points of system (3.2):

$$ \begin{aligned} E_0 &= (0, 0, 0, 0), \\ E_1 &= (-1.16933, 1.16933, 0.82413, -0.470931), \\ E_2 &= (-0.0676541, 0.0676541, -0.619559, 0.354034), \\ E_3 &= (0.268126, -0.268126, 0.558761, -0.319292), \\ E_4 &= (1.02162, -1.02162, -0.806158, 0.460662). \end{aligned} \tag{3.3} $$

At the equilibrium point E_0, the Jacobian matrix is

$$ J(E_0) = \begin{pmatrix} -3 & 4 & 0 & -4 \\ -7 & 2 & 0 & 0 \\ 0 & -4 & -3 & 4 \\ 0 & 0 & 2 & -1 \end{pmatrix}. \tag{3.4} $$

We can also get $J(E_1)$, $J(E_2)$, $J(E_3)$, $J(E_4)$, then we can calculate their eigenvalues. The results are showing in the following table:

$J(E_0)$	$\lambda_1 = -3.000$ $\lambda_3 = 0.1933 + 3.9856i$	$\lambda_2 = -2.3866$ $\lambda_4 = 0.1933 - 3.9856i$
$J(E_1)$	$\lambda_1 = 18.0408$ $\lambda_3 = -3.0000$	$\lambda_2 = -18.4956$ $\lambda_4 = -0.7211$
$J(E_2)$	$\lambda_1 = 6.3940$ $\lambda_3 = -3.0000$	$\lambda_2 = -7.1837$ $\lambda_4 = -1.8299$
$J(E_3)$	$\lambda_1 = 4.2774$ $\lambda_3 = -2.8593 + 1.0984i$	$\lambda_2 = -3.0000$ $\lambda_4 = -2.8593 - 1.0984i$
$J(E_4)$	$\lambda_1 = -3.0000$ $\lambda_3 = -0.2643 + 12.5393i$	$\lambda_2 = -2.2776$ $\lambda_4 = -0.2643 - 12.5393i$

Now we can yield that equilibrium point E_1, E_2, E_3 and E_4 are saddle points of index 1. E_0 is a saddle point of index 2.

Suppose λ is the unstable eigenvalue of the saddle points of index 2, the necessary condition for the fractional-order system (3.2) to remain chaotic is keeping the eigenvalue λ in the unstable region. According to [15] (see also [30]), if the eigenvalue λ is in the unstable region, then the following condition is satisfied:

$$|\arg \lambda_i| > q\pi/2, \tag{3.5}$$

where $|\arg \lambda_i|$ denotes the argument of the eigenvalue λ. That is

$$q > \frac{2}{\pi}|\arg \lambda_i|. \tag{3.6}$$

Therefore, the necessary condition for the existence of chaotic attractors in the fractional-order love triangle system (3.2) is

$$q > \frac{2}{\pi}|\arg \lambda_i| = \frac{2}{\pi}\arctan\frac{3.9856}{0.1933} = 0.9691.$$

3.2. Some numerical simulations

Now, we take some cases for numerical simulation. The method for numerical simulation of this paper is an improved version of Adams Bashforth Moulton algorithm [3]. To give the approximate solution of nonlinear fractional-order differential equations by means of this algorithm, we consider the following differential equation

$$D^q y(t) = r(t, y(t)), 0 \le t \le T, y^k(0) = y_0^k, k = 0, 1, ..., m-1. \tag{3.7}$$

This differential equation is equivalent to Volterra integral equation [5]

$$y(t) = \sum_{k=0}^{[q]-1} y_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} r(s, y(s))\mathrm{d}s. \tag{3.8}$$

Now, set

$$h = \frac{T}{h}, t_n = nh \ \ (n = 0, 1, ..., N). \tag{3.9}$$

Then Eq. (3.8) can be discretized as follows

$$\begin{aligned} y_h(t_{n+1}) = &\sum_{k=0}^{[q]-1} y_0^k \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} r(t_{n+1}, y_h^p(t_{n+1})) \\ &+ \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} r(t_j, y_h(t_j)), \end{aligned} \tag{3.10}$$

where predicted value $y_h(t_{n+1})$ is determined by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} r(t_j, y_h(t_j)), \tag{3.11}$$

in which

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n-1 \\ 1, & j = n \end{cases} \tag{3.12}$$

and

$$b_{j,n+1} = \frac{h^q}{q} \left[(n-j+1)^q - (n-j)^q \right], \ 0 \le j \le n. \tag{3.13}$$

The estimation error of this approximation is described as follows:

$$\max_{0 \le j \le n} |y(t_j) - y_h(t_j)| = O(h^p), \tag{3.14}$$

where $p = \min\{2, 1+a\}$.

It is noteworthy that the approximation of fractional dynamics, that is the improved version of Adams Bashforth Moulton numerical algorithm, depends on the history of the system. This is our motivation for using fractional modeling to describe romantic relationships, since the time evolution of a romantic relationship is impacted by its history.

Numerical solution of a fractional-order system can be determined by applying the above mentioned method. Let us consider the following fractional-order system

$$\begin{aligned} D^{q_1} x = f_1(x, y, z, s), \quad D^{q_2} y = f_2(x, y, z, s), \\ D^{q_3} z = f_3(x, y, z, s), \quad D^{q_4} s = f_4(x, y, z, s), \end{aligned} \tag{3.15}$$

with $0 \le q_i \le 1, i = 1, 2, 3, 4$ and initial conditions (x_0, y_0, z_0, s_0). Applying the above method, the system (3.15) can be discretized as follows:

$$\begin{aligned} x_{n+1} =& x_0 + \frac{q_1}{\Gamma(q_1+2)} \left[f_1\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, s_{n+1}^p\right) \right] \\ &+ \sum_{j=0}^{n} \gamma_{1,j,n+1} f_1(x_j, y_j, z_j, s_j), \\ y_{n+1} =& y_0 + \frac{q_2}{\Gamma(q_2+2)} \left[f_2\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, s_{n+1}^p\right) \right] \\ &+ \sum_{j=0}^{n} \gamma_{2,j,n+1} f_2(x_j, y_j, z_j, s_j), \\ z_{n+1} =& z_0 + \frac{q_3}{\Gamma(q_3+2)} \left[f_3\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, s_{n+1}^p\right) \right] \\ &+ \sum_{j=0}^{n} \gamma_{3,j,n+1} f_3(x_j, y_j, z_j, s_j), \\ s_{n+1} =& s_0 + \frac{q_4}{\Gamma(q_4+2)} \left[f_4\left(x_{n+1}^p, y_{n+1}^p, z_{n+1}^p, s_{n+1}^p\right) \right] \\ &+ \sum_{j=0}^{n} \gamma_{4,j,n+1} f_4(x_j, y_j, z_j, s_j), \end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
x_{n+1}^{p} &= x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} \beta_{1,j,n+1} f_1\left(x_j, y_j, z_j, s_j\right), \\
y_{n+1}^{p} &= y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} \beta_{2,j,n+1} f_2\left(x_j, y_j, z_j, s_j\right), \\
z_{n+1}^{p} &= z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^{n} \beta_{3,j,n+1} f_3\left(x_j, y_j, z_j, s_j\right), \\
s_{n+1}^{p} &= s_0 + \frac{1}{\Gamma(q_4)} \sum_{j=0}^{n} \beta_{4,j,n+1} f_4\left(x_j, y_j, z_j, s_j\right),
\end{aligned} \tag{3.17}
$$

$$
\gamma_{j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j=0 \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1 \le j \le n-1 \\ 1, & j=n, \end{cases} \tag{3.18}
$$

and

$$
\beta_{j,n+1} = \frac{h^{q_i}}{q_i}\left[(n-j+1)^{q_i} - (n-j)^{q_i}\right], 0 \le j \le n \;\; (i=1,2,3,4). \tag{3.19}
$$

Based on the above algorithm, we can prove our results of the fractional-order love triangle system with competition (3.2) by computer stimulations. According to what have been discussed above, we know that the system exhibit chaos when $q \geq 0.9691$, as showing is in Figure 5 to Figure 7. And when $q < 0.9691$, we find the cases that the system (3.2) does not show chaotic behavior, as showing in Figure 8 to Figure 10.

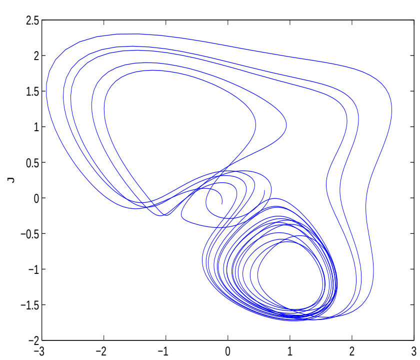

Figure 5. Chaotic attractor of fractional-order love triangle system (3.2) with $q_1 = q_2 = q_3 = q_4 = 0.97 > 0.9691$.

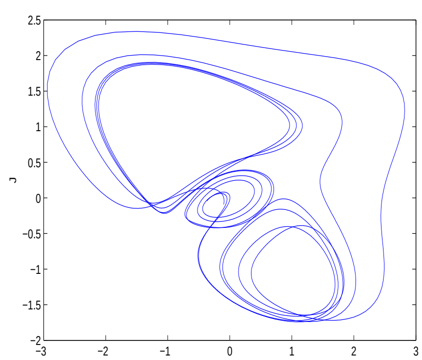

Figure 6. Chaotic attractor of fractional-order love triangle system (3.2) with $q_1 = q_2 = q_3 = q_4 = 0.98 > 0.9691$.

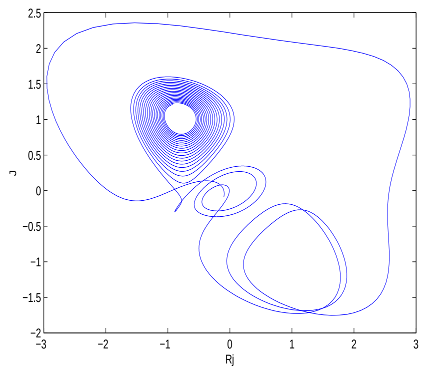

Figure 7. Chaotic attractor of fractional-order love triangle system (3.2) with $q_1 = q_2 = q_3 = q_4 = 0.985 > 0.9691$.

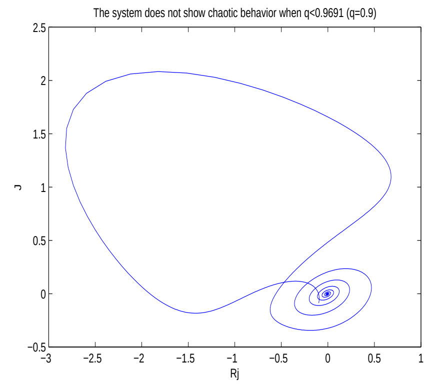

Figure 8. The fractional-order love triangle system (3.2) does not show chaotic with $q_1 = q_2 = q_3 = q_4 = 0.9 < 0.9691$.

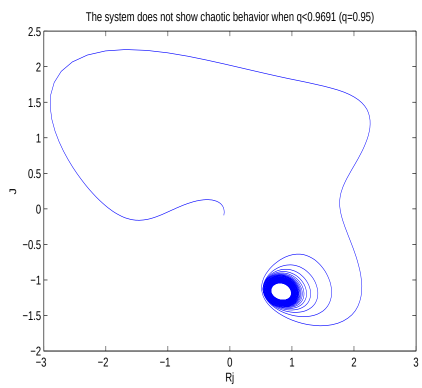

Figure 9. The fractional-order love triangle system (3.2) does not show chaotic with $q_1 = q_2 = q_3 = q_4 = 0.95 < 0.9691$.

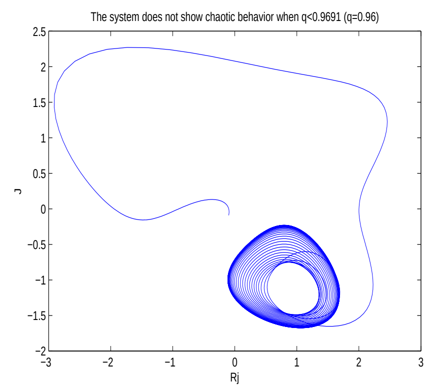

Figure 10. The fractional-order love triangle system (3.2) does not show chaotic with $q_1 = q_2 = q_3 = q_4 = 0.96 < 0.9691$.

4. Conclusions

Just like Strogatz and Sprott, the aim of this paper is to create interests and spark research efforts in the field of "psychology and life sciences". In the present work, by introducing the competition term, we first modify the nonlinear love triangle system and study its chaotic dynamical behaviors. We find that the dynamical behavior of the love triangle system is strongly related to the competition coefficients. Moreover, to make the model more realistic, we construct its corresponding fractional-order system and get the necessary condition for the existence of the chaotic attractors, which offer more insights towards the understanding of the dynamical behaviors of these systems. In the end, the chaotic attractors for this fractional-order love triangle system are also obtained by computer simulations.

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