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NUMERICAL SOLUTION OF FOURTH-ORDER TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS*

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Abstract In this paper, a numerical method for fourth-order time-fractional partial differential equations with variable coefficients is proposed. Our method consists of Laplace transform, the homotopy perturbation method and Stehfest's numerical inversion algorithm. We show the validity and efficiency of the proposed method (so called LHPM) by applying it to some examples and comparing the results obtained by this method with the ones found by Adomian decomposition method (ADM) and He's variational iteration method (HVIM).

Keywords Fourth-order, time-fractional differential equations, Laplace transform, homotopy perturbation method, Stehfest's numerical inversion algorithm.

MSC(2000) 26A33, 35R11.

1. Introduction

The topic of numerical study of fractional differential equations has attracted the attention of many researchers. Li and He [24] proposed an application of the fractional complex transform to fractional differential equation. In [7], different numerical methods were used to solve singularly perturbed Able Volterra integral equation associated with a fractional differential equation. Esmaeili et al. [8] developed a computational technique based on the collocation method and Muntz polynomials for the solution of fractional differential equations. In [25], the Riemann-Liouville fractional integral for repeated fractional integration was expanded in block pulse functions to yield the block pulse operational matrices for the fractional order integration. Lin and Xu [26] proposed numerical resolution for a time-fractional diffusion equation. In [16], a finite element approach was applied to solve the fractional advection-dispersion equation. Kexue and Jigen [19] used the Laplace transform method for solving fractional differential equations with constant coefficients. Merrikh-Bayat [31] developed a low-cost numerical algorithm to find the series solution for nonlinear fractional differential equations with delay.

Homotopy perturbation method [10, 11] is an important and effective mathematical tool for solving a variety of problems. It has been successfully applied

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to study limit cycle and bifurcation for nonlinear problems [12], nonlinear wave equations [13], boundary value problems [14], chemical kinetics system [1], oscillators with discontinuities [15], Riccati equation with fractional orders [20], neutron transport equation [29], nonlinear singular fourth order four-point boundary value problems [27], systems of partial differential equations [3], nonlinear ill-posed operator equations [4], stiff systems of ordinary differential equations [2], multi-order fractional differential equations [17], etc.

Laplace transform method ia a well-known technique for solving integer-order or relatively simple fractional-order differential equations [33, 35, 42]. In [28], a Laplace homotopy perturbation method was employed for solving one dimensional non-homogeneous partial differential equations with variable coefficients. Sheng et al. [34] proposed numerical inverse Laplace transform algorithms for solving complicated fractional order differential equations. Weeks [44] discussed numerical inversion of Laplace transform algorithm in terms of the Laguerre expansion and bilinear transformations [40]. For more examples and details of numerical inversion of Laplace transform method, see [30, 36–38, 41, 43] and the references therein.

A combination of Laplace transform and homotopy perturbation method (LHPM) presents an accurate methodology for solving nonhomogeneous partial differential equations with variable coefficients. The method is applied in the Laplace transform domain together with an inversion technique to retrieve the time-domain solution. In [5], the authors developed a numerical algorithm for inverting a Laplace transform when Laguerre polynomial series expansion for the inverse function is available. Stehfest's algorithm [39] has also been applied successfully for numerical inversion of Laplace transform method. In [18], the authors solved second-order time-fractional partial differential equations by applying a numerical method based on numerical Laplace inversion technique.

In this paper, we discuss a numerical method for solving fourth-order time-fractional partial differential equations with variable coefficients. Our method is based on Laplace transform, the homotopy perturbation method and Stehfest's algorithm. Using temporal Laplace transform, the problem is converted into a partial differential equation in time transform domain and is solved by means of the homotopy perturbation method. The solution in time domain is obtained via Stehfest's numerical inversion algorithm.

2. Preliminaries

This section is devoted to the background material for fractional calculus [6, 22, 23, 32], Laplace transform and homotopy perturbation method.

Definition 2.1. A function $h : R \to R^+$ is said to be in the space $C_\mu, \mu \in R$ if it can be expressed as $h(x) = x^\sigma h_1(x)$ with $\sigma > \mu$, $h_1(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m if $h^{(m)} \in C_\mu$ for $m \in N \bigcup\{0\}$.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $h \in C_\mu$ with $\mu \geq -1$ is defined as

$$\begin{aligned} &\mathcal{I}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau, \quad \alpha > 0, t > 0, \\ &\mathcal{I}^0 h(t) = h(t). \end{aligned} \tag{2.1}$$

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$ for a function $h \in C_{-1}^m$ with $m \in N \bigcup \{0\}$ is defined as

$$D^\alpha h(t) = \begin{cases} \mathcal{I}^{m-\alpha} h^{(m)}(t), & m-1 < \alpha \leq m, m \in N, \\ \dfrac{d^m h(t)}{dt^m}, & \alpha = m. \end{cases} \tag{2.2}$$

Definition 2.4. A two-parameter Mittag-Leffler function is defined by the following series

$$E_{\alpha,\beta}(t) = \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\Gamma(\alpha\kappa+\beta)}. \tag{2.3}$$

Note that $E_{1,1}(t) = e^t,\ E_{1,1}(-t) = e^{-t}$.

Definition 2.5. The Laplace transform of a function $u(x,t), t \geq 0$, denoted by $\varphi(x,s)$, is defined by

$$L\{u(x,t)\} = \varphi(x,s) = \int_0^\infty e^{-st} u(x,t) dt, \tag{2.4}$$

where s is the transform parameter and is assumed to be real and positive.

The Laplace transform of Mittag-Leffler function $E_{\alpha,\beta}(t)$ is

$$L(E_{\alpha,\beta}(t)) = \int_0^\infty e^{-st} E_{\alpha,\beta}(t) dt = \sum_{\kappa=0}^{\infty} \frac{k!}{s^{\kappa+1}\Gamma(\alpha\kappa+\beta)}. \tag{2.5}$$

The Laplace transform of $D^\alpha h(t)$ can be found as follows

$$\begin{aligned} & L(D^\alpha h(t)) = L(J^{m-\alpha} h^{(m)}(t)) \\ =& L[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} h^{(m)}(\tau) d\tau] = \frac{1}{s^{m-\alpha}} L(h^{(m)}(t)) \\ =& \frac{1}{s^{m-\alpha}} [s^m L(h(t)) - s^{m-1} h(0) - s^{m-2} h'(0) - s^{m-3} h''(0) - \cdots - h^{m-1}(0)]. \end{aligned} \tag{2.6}$$

Next, we outline the basic idea of the homotopy perturbation method [10, 11] for the convenience of the reader. For that we consider an equation of the form

$$\psi_1(u) + \psi_2(u) - g(r) = 0,\ r \in \mathcal{D}, \tag{2.7}$$

where ψ_1 is a linear operator, ψ_2 is a nonlinear operator, $g(r)$ is a known analytic function and $\mathcal{D}$ is the given domain. By the homotopy technique, we construct a homotopy $H(u,p)$ as

$$H(u,p) = \psi_1(u) - \psi_1(u_0) + p\psi_1(u_0) + p[\psi_2(u) - g(r)] = 0, \tag{2.8}$$

where $p \in [0,1]$ is an embedding parameter for (2.7). Clearly

$$H(u,0) = \psi_1(u) - \psi_1(u_0) = 0, \quad H(u,1) = \psi_1(u) + \psi_2(u) - g(r) = 0. \tag{2.9}$$

The changing process of p from zero to unity is just that of $u(r, p)$ from initial approximation $\overline{u}_0$ to the solution u of (2.7). We assume that the solution of (2.7) can be written as a power series in p, that is,

$$u = \sum_{\kappa=0}^{\infty} p^{\kappa} u_{\kappa}. \tag{2.10}$$

Substituting (2.10) in (2.8) and equating the coefficients of same powers of p, a successive procedure is used to determine u_k. Finally, setting $p = 1$ in (2.10) yields the solution for (2.7).

3. Fourth-order time-fractional differential equation-s

For $m - 1 < \alpha \leq m, m \in N$, we consider the following fourth-order time-fractional partial differential equation

$$\frac{\partial^{\alpha}\xi}{\partial t^{\alpha}}(x_1, x_2, t) = \epsilon_1(x_1, x_2)\frac{\partial^4\xi}{\partial x_1^4}(x_1, x_2, t) + \epsilon_2(x_1, x_2)\frac{\partial^4\xi}{\partial x_2^4} + h(x_1, x_2, t), \tag{3.1}$$

supplemented with initial conditions

$$\frac{\partial^k\xi}{\partial t^k}(x_1, x_2, , 0) = f_k(x_1, x_2), k = 0, 1, \cdots, m - 1, \tag{3.2}$$

where $f_k, k = 0, 1, \cdots, m - 1, h, \epsilon_1$ and ϵ_2 are known functions.

Now we explain the method for solving (3.1) supplemented with the initial conditions (3.2).

Taking Laplace transform of (3.1) via (2.6), we obtain

$$\begin{aligned} &\frac{1}{s^{m-\alpha}}[s^m\Omega(x_1, x_2, s) - s^{m-1}f_0(x_1, x_2) \\ &- s^{m-2}f_1(x_1, x_2) - s^{m-3}f_2(x_1, x_2) - \cdots - f_{m-1}(x_1, x_2)] \\ =&[\epsilon_1\frac{\partial^4}{\partial x_1^4} + \epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega(x_1, x_2, s) + \overline{h}(x_1, x_2, s), \end{aligned} \tag{3.3}$$

where $\Omega(x_1, x_2, s)$ and $\overline{h}(x_1, x_2, s)$ are the Laplace transform of $\xi(x_1, x_2, t)$ and $h(x_1, x_2, t)$ respectively. We can rewrite (3.3) as

$$\begin{aligned} s^{\alpha}\Omega(x_1, x_2, s) =&[\epsilon_1\frac{\partial^4}{\partial x_1^4} + \epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega(x_1, x_2, s) + \overline{h}(x_1, x_2, s) \\ &+ \frac{1}{s^{m-\alpha}}[s^{m-1}f_0(x_1, x_2) + s^{m-2}f_1(x_1, x_2) \\ &+ s^{m-3}f_2(x_1, x_2) + \cdots + f_{m-1}(x_1, x_2)]. \end{aligned} \tag{3.4}$$

For solving (3.4) by homotopy perturbation method, we construct a homotopy in the following form:

$$\begin{aligned} \Omega(x_1, x_2, s) =&\frac{p}{s^{\alpha}}[\epsilon_1\frac{\partial^4}{\partial x_1^4} + \epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega(x_1, x_2, s) \\ &+ \frac{1}{s^{m-2\alpha}}[s^{m-1}f_0(x_1, x_2) + s^{m-2}f_1(x_1, x_2) \\ &+ s^{m-3}f_2(x_1, x_2) + \cdots + f_{m-1}(x_1, x_2)] + \frac{1}{s^{\alpha}}\overline{h}(x_1, x_2, s). \end{aligned} \tag{3.5}$$

Let the solution of (3.5) can be expressed as

$$\overline{\Omega}(x_1,x_2,s)=\sum_{j=0}^{\infty}p^j\Omega_j(x_1,x_2,s), \tag{3.6}$$

where $\Omega_j(x_1,x_2,s), j=0,1,2,\dots,$ are unknown functions which need to be determined. Substituting (3.6) into (3.5) yields

$$\begin{aligned}\sum_{j=0}^{\infty}p^j\Omega_j(x_1,x_2,s)=&\frac{p}{s^{\alpha}}[\epsilon_1\frac{\partial^4}{\partial x_1^4}+\epsilon_2\frac{\partial^4}{\partial x_2^4}]\sum_{j=0}^{\infty}p^j\Omega_j(x_1,x_2,s)+\frac{1}{s^{\alpha}}\overline{h}(x_1,x_2,s)\\&+\frac{1}{s^{m-2\alpha}}[s^{m-1}f_0(x_1,x_2)+s^{m-2}f_1(x_1,x_2)\\&+s^{m-3}f_2(x_1,x_2)+\cdots+f_{m-1}(x_1,x_2)].\end{aligned} \tag{3.7}$$

Equating the coefficients of same powers of p, (3.7) gives

$$\begin{aligned}p^0:\quad &\Omega_0(x_1,x_2,s)=\frac{1}{s^{m-2\alpha}}[s^{m-1}f_0(x_1,x_2)+s^{m-2}f_1(x_1,x_2)\\&\qquad+s^{m-3}f_2(x_1,x_2)+\cdots+f_{m-1}(x_1,x_2)]+\frac{1}{s^{\alpha}}\overline{h}(x_1,x_2,s),\\p^1:\quad &\Omega_1(x_1,x_2,s)=\frac{1}{s^{\alpha}}[\epsilon_1\frac{\partial^4}{\partial x_1^4}+\epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega_0(x_1,x_2,s),\\p^2:\quad &\Omega_2(x_1,x_2,s)=\frac{1}{s^{\alpha}}[\epsilon_1\frac{\partial^4}{\partial x_1^4}+\epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega_1(x_1,x_2,s),\\&\vdots\\p^{n+1}:\quad &\Omega_{n+1}(x_1,x_2,s)=\frac{1}{s^{\alpha}}[\epsilon_1\frac{\partial^4}{\partial x_1^4}+\epsilon_2\frac{\partial^4}{\partial x_2^4}]\Omega_n(x_1,x_2,s).\end{aligned} \tag{3.8}$$

In the limit $p\to 1$, (3.6) becomes the approximate solution for (3.4) and is given by

$$M_n(x_1,x_2,s)=\sum_{j=0}^{n}\Omega_j(x_1,x_2,s). \tag{3.9}$$

Taking the inverse Laplace transform of (3.9), we get the approximate solution of the problem (3.1)-(3.2):

$$\xi(x_1,x_2,t)\simeq\xi_n(x_1,x_2,t)=L^{-1}(M_n(x_1,x_2,s)). \tag{3.10}$$

The original solution $\xi(x_1,x_2,t)$ can be found approximately from $M_n(x_1,x_2,s)$ by means of the Stehfest's algorithm [38] as follows

$$\xi_n(x_1,x_2,t)=S(t)\sum_{l=1}^{2\nu}\varpi_l M_n(x_1,x_2,lS(t)),$$

where ν is positive integer, $S(t)=\frac{ln(2)}{t}$ and

$$\varpi_l=(-1)^{l+\nu}\sum_{k=[\frac{l+1}{2}]}^{\min(l,\nu)}\frac{k^{\nu}(2k)!}{(\nu-k)!k!(k-1)!(l-k)!(2k-l)!},$$

$[r]$ denotes the integer part of the real number r.

4. Numerical results

In this section, we demonstrate the application LHPM approach developed in the preceding section.

Example 4.1. Consider a fourth-order time-fractional differential equations [9]

$$\frac{\partial^\alpha \xi}{\partial t^\alpha}(x,t) = -\frac{\partial^4 \xi}{\partial x^4}(x,t),\ \alpha \in (0,1),\ t>0,\ x \in [-3,3], \tag{4.1}$$

with the initial condition

$$\xi(x,0) = \exp(-x). \tag{4.2}$$

The exact solution is given by

$$\xi(x,t) = \exp(-x)(1+\sum_{k=1}^{\infty}\frac{(-t)^{k\alpha}}{\Gamma(k\alpha+1)}) = \exp(-x)E_{\alpha,1}(-t^\alpha). \tag{4.3}$$

By using LHPM, we obtain

$$\begin{aligned}
&\Omega_0(x,s) = \frac{\exp(-x)}{s},\ \Omega_1(x,s) = -\frac{1}{s^\alpha}\frac{\partial^4}{\partial x^4}\Omega_0(x,s),\ \Omega_2(x,s) = -\frac{1}{s^\alpha}\frac{\partial^4}{\partial x^4}\Omega_1(x,s),\\
&\vdots\\
&\Omega_{n+1}(x,s) = -\frac{1}{s^\alpha}\frac{\partial^4}{\partial x^4}\Omega_n(x,s),
\end{aligned} \tag{4.4}$$

and so on.

Table 1 shows the absolute errors $|\xi(x,t) - \xi_n(x,t)|$ using the LHPM with $p = 10, \alpha = 0.75, 0.85, 0.99$ for various values of x and t. As it is clear from the table, the numerical solutions are in good agreement with the exact solution.

x/t		0.2	0.4	0.8	1.0
-3	$\alpha = 0.75$	$2.24e-5$	$2.29e-4$	$3.39e-4$	$9.16e-4$
	0.85	$8.91e-4$	$4.28e-4$	$9.72e-4$	$1.06e-4$
	0.99	0.0023	$3.41e-4$	$1.77e-4$	$2.77e-4$
-1.5	$\alpha = 0.75$	$1.73e-4$	$5.91e-5$	$2.43e-5$	$2.27e-4$
	0.85	$1.78e-4$	$1.19e-4$	$1.22e-4$	$1.43e-4$
	0.99	$3.97e-5$	$1.42e-4$	$1.42e-4$	$1.56e-4$
0	$\alpha = 0.75$	$6.06e-5$	$5.18e-6$	$4.35e-5$	$2.72e-6$
	0.85	$1.28e-5$	$2.44e-5$	$5.42e-5$	$1.13e-5$
	0.99	$6.71e-5$	$1.41e-5$	$7.66e-6$	$2.02e-5$
1.5	$\alpha = 0.75$	$3.34e-6$	$7.08e-6$	$6.33e-7$	$4.68e-6$
	0.85	$9.99e-6$	$2.42e-6$	$1.36e-6$	$1.18e-5$
	0.99	$3.20e-6$	$3.85e-6$	$5.45e-6$	$1.00e-5$
3	$\alpha = 0.75$	$2.58e-6$	$1.39e-6$	$1.64e-6$	$9.96e-7$
	0.85	$1.71e-6$	$1.14e-6$	$4.02e-7$	$3.77e-6$
	0.99	$1.85e-6$	$4.62e-7$	$1.18e-6$	$1.41e-6$

Table 1. Absolute errors using the LHPM with $p = 10, \alpha = 0.75, 0.85, 0.99$ for various values of x and t for Example 4.1.

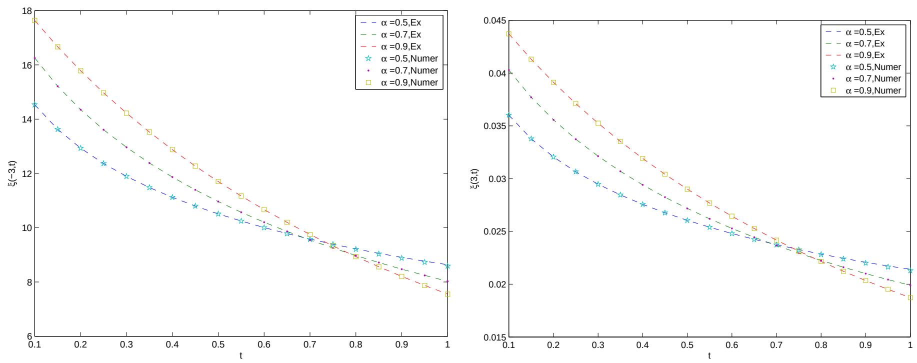

Figure 1. The exact and the numerical solutions with $x = -3, 3, n = 10, p = 10$ for various values of t and $\alpha = 0.5, 0.7, 0.9$.

In Fig. 1, we plot the exact and the numerical solutions with $x = -3, 3, n = 10, p = 10$ for various values of t and $\alpha = 0.5, 0.7, 0.9$.

In Fig. 2, we plot the numerical solution and the exact solution with $n = 10, p = 10\alpha = 0.5, 0.7, 0.9$ for various values of α and x.

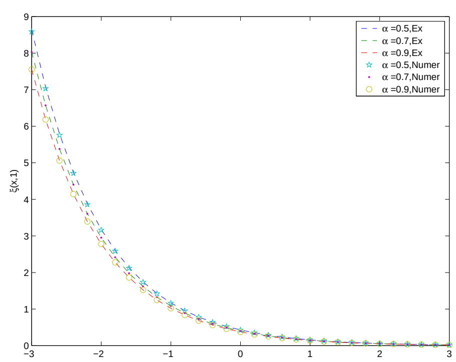

Figure 2. The numerical solution and the exact solution with $n = 10, p = 10, \alpha = 0.5, 0.7, 0.9$ for various values of α and x.

Example 4.2. Consider the time-fractional fourth-order differential equation [21]

$$\begin{aligned} &\frac{\partial^\alpha \xi}{\partial t^\alpha}(x,t) = -(\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 \xi}{\partial x^4}(x,t), \\ &\alpha \in (1,2],\ t > 0,\ x \in (\frac{1}{2}, 1), \end{aligned} \tag{4.5}$$

with the initial conditions

$$\xi(x,0) = 0, \frac{\partial \xi}{\partial t}(x,0) = 1 + \frac{x^5}{120} \tag{4.6}$$

and the boundary conditions

$$\begin{aligned}
\xi(\frac{1}{2},t) &= 1+\frac{0.5^5}{120}\sin(t,\alpha), \frac{\partial^2\xi}{\partial x^2}(\frac{1}{2},t) = \frac{1}{48}\sin(t,\alpha), \\
\xi(1,t) &= \frac{121}{120}\sin(t,\alpha), \frac{\partial^2\xi}{\partial x^2}(1,t) = \frac{1}{6}\sin(t,\alpha),
\end{aligned} \tag{4.7}$$

where the function $\sin(t,\alpha)$ is defined as $\sin(t,\alpha)=\sum_{i=0}^{\infty}\frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha+2)}$.

Using the Laplace transform and homotopy perturbation method (LHPM) on (4.1), we obtain

$$\begin{aligned}
\Omega_0(x,s) &= \frac{1}{s^2}(1+\frac{x^5}{120}), \\
\Omega_1(x,s) &= -\frac{1}{s^\alpha}(\frac{1}{x}+\frac{x^4}{120})\frac{\partial^4}{\partial x^4}\Omega_0(x,s), \\
\Omega_2(x,s) &= -\frac{1}{s^\alpha}(\frac{1}{x}+\frac{x^4}{120})\frac{\partial^4}{\partial x^4}\Omega_1(x,s), \\
&\vdots \\
\Omega_{n+1}(x,s) &= -\frac{1}{s^\alpha}(\frac{1}{x}+\frac{x^4}{120})\frac{\partial^4}{\partial x^4}\Omega_n(x,s),
\end{aligned} \tag{4.8}$$

and so on.

In Table 2, we list the results obtained by LHPM and compared with Adomian decomposition method (ADM) and He's variational iteration method (HVIM) results given by Khan et al. in [21] at $n=3, \alpha=1.5, 1, 75$ for various values of x and t. It can easily be inferred from the tabulated values that the results obtained by LHPM are better than the ones found by ADM and HVIM.

		$\alpha=1.5$			$\alpha=1.75$		
t	x	$LHPM$	ADM	$HVIM$	$LHPM$	ADM	$HVIM$
0.2	0.5	0.194734	0.194734	0.196914	0.197359	0.19736	0.197687
	0.6	0.194809	0.194809	0.196991	0.197437	0.197437	0.197763
	0.75	0.195068	0.195068	0.197527	0.197699	0.197699	0.198026
	1.0	0.196306	0.196306	0.198504	0.198952	0.198953	0.199282
0.4	0.5	0.370692	0.370692	0.377682	0.382210	0.382211	0.383217
	0.6	0.370836	0.370835	0.377828	0.382359	0.382359	0.383366
	0.75	0.371328	0.371328	0.37833	0.382867	0.382867	0.383875
	1.0	0.373683	0.373633	0.38073	0.385296	0.385296	0.33631
0.6	0.5	0.521418	0.521419	0.531411	0.546533	0.546537	0.547792
	0.6	0.521620	0.521621	0.531617	0.546747	0.546748	0.548005
	0.75	0.522311	0.522314	0.532323	0.547473	0.547475	0.548733
	1.0	0.525626	0.525627	0.5357	0.550946	0.550947	0.552214

Table 2. Comparison of the numerical solutions obtained by the present method and those obtained by ADM and HVIM with $\alpha=1.5, 1.75$ for various values of x and t.

In Fig. 3, we plot the numerical solutions with $x=0.6,\ 0.9;\ n=3,\ p=10$ for various values of t and α.

In Fig. 4, we plot the numerical solution with $n=3,\ p=10,\ t=1$ for various values of α and x.

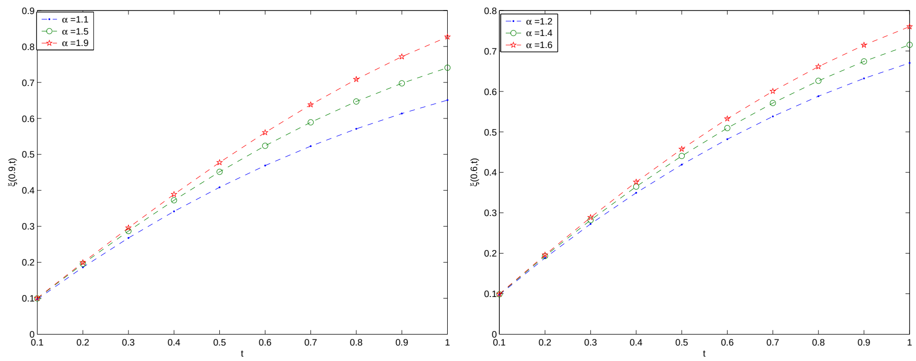

Figure 3. The numerical solutions with $x = 0.6,\ 0.9;\ n = 3,\ p = 10$ for various values of t and α.

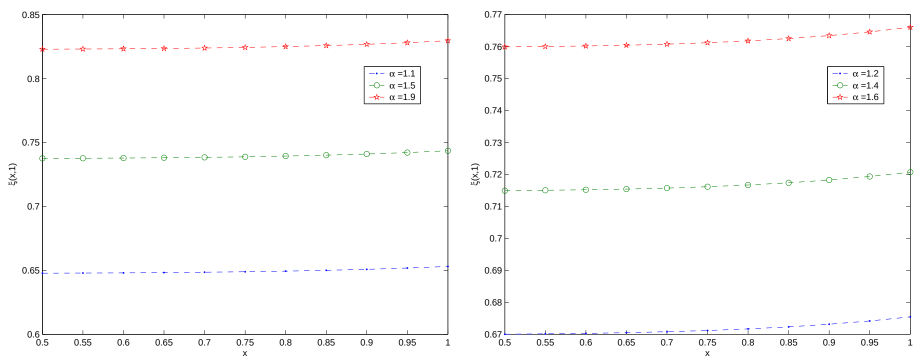

Figure 4. The numerical solution with $n = 3,\ p = 10,\ t = 1$ for various values of α and x.

5. Conclusions

In this paper, a numerical method (LHPM) consisting of the Laplace transform, homotopy perturbation method and Stehfest's numerical inversion algorithm for solving fourth-order time-fractional partial differential equations with variable coefficients is presented. The importance of the work lies in the fact that our numerical method is more accurate and efficient than Adomian decomposition method (ADM) and He's variational iteration method (HVIM) [10,14], and yields a better approximation to the exact solution. This has been shown with the aid of some examples.

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