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SECOND ORDER NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMMING UNDER GENERALIZED UNIVEX FUNCTIONS

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Abstract In this paper, a new class of second order (d, ρ, η, θ)-type 1 univex function is introduced. The Wolfe type second order dual problem (SFD) of the nondifferentiable multiobjective fractional programming problem (MFP) is considered, where the objective and constraint functions involved are directionally differentiable. Also the duality results under second order(d, ρ, η, θ)-type 1 univex functions are established.

Keywords Second order directional derivative, generalized second order (d, ρ, η, θ)-type 1 univex function, second order fractional duality, efficient solution.

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1. Introduction

Fractional programming problem arises in many types of optimization problems such as portfolio selection, production, information theory and numerous decision making problems in management science. Many economic, noneconomic and indirect application of fractional programming problem have also been given by Bector[5], Bector and Chandra[6], Bector et al.[7], Schaible [32-34].

The central concept in optimization is known as the duality theory which asserts that, given a (primal) minimization problem the infimum value of the primal problem cannot be smaller than the supremum value of the associated (dual) maximization problem and the optimal values of primal and dual problems are equal.Duality in fractional programming is an important class of duality theory and several contribution have been made in past[1,5-7,16,28,32,33].

Second order duality provides tighter bounds for the value of the objective function when approximations are used. For more detail, one can consult ([27],page-93). Another advantage of second order duality when applicable is that,if a feasible point in the primal is given and first order duality does not apply, then we can use second order duality to provide lower bound of the value of the primal (see [24]). In 1960, Dorn[13] studied the duality results for the problem of minimizing a convex differentiable function subject to linear constraints. various classes of functions have been defined for the purpose of weakening the limitation of convexity in mathematical

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programming. Over the years many generalization of these results to nondifferentiable convex problem(see [35]) and differentiable non-convex problem (see [17])have append in the literature. Hanson [17] introduced the concept of invexity and had shown that Khun-Tucker condition are sufficient condition for optimality. Hanson and Mond [19] introduced two new classes of functions called Type I and Type II functions for the scalar optimization problems, which were further generalized to pseudo-Type I and quasi-Type I by Rueda and Hanson [31]. Both classes are related to but more general than invex functions. Zhao [40] gave Karush-Kuhn-Tucker type sufficiency and duality in nondifferentiable scalar optimization assuming Clarke [11] generalized subgradients under Type I function.

Kaul et al. [21] extended the concept of Type I function from single objective to a multiobjective programming problem by defining the Type I and its various generalization. They investigated necessary and sufficient optimality conditions and derived Wolfe type and Mond-Weir type duality results. Suneja and Srivastava [36] introduced generalized d-Type I functions in terms of directional derivative for a multiobjective programming problem and discussed Wolfe type and Mond-Weir type duality results. In [3], Aghezzaf and Hachimi introduced classes of generalized Type I vector valued function for a differentiable multiobjective programming problem and established duality results. Kuk and Tanino [22] derived optimality conditions and duality theorems for nonsmooth multiobjective programming problems involving generalized Type I vector valued functions.

Multiobjective fractional programming duality has been of much interest in the recent past. Schaible [32] and Bector et al [7] derived Fritz John and Karush-Kuhn Tucker necessary and sufficient optimality condition for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems. Mishra and Rautela [25] formulated a general dual and proved the duality results under generalized semi locally type 1 univex and related function. Bector et al [8] introduce univex function. Mishra et al [26] introduced type 1 univex, pseudo type 1 univex, quasi type 1 univex function and obtained optimality results for mathematical programs under generalized type 1 univex function. Antczak [4] used directional derivative in association with a hypothesis of an invex kind following Ye [39]. Nahak and Mohapatra [29] obtained duality results for multiobjective programming problems under (d, ρ, η, θ)-invexity assumption. Recently, Tripathy and Devi [37] introduced a new generalized class of (d, ρ, η, θ)-type 1 univex function and introduced Wolfe type duality problem (MFXD) of the nondifferentiable multiobjective fractional programming problem (MFP) and established the duality results under generalized class of (d, ρ, η, θ)-type 1 univex function.

In this paper, motivated by Tripathy and Devi[37],Huck and Tanino [22] and Ahmad et al.[2], we introduce a new generalized class of second order (d, ρ, η, θ) type 1 univex functions. We introduce Wolfe type duality problem (SFD) of the nondifferentiable multiobjective fractional programming problem (MFP) and established the duality results under generalized class of second order (d, ρ, η, θ)-type 1 univex function.

2. Notations and Definitions

Let R^n be n-dimensional Euclidean space and R^n_+ be the nonnegative orthant. For vectors x and y in R^n , we denote $x < y \Leftrightarrow x_i < y_i$ for $i = 1, 2, .., n$; $x \leq y \Leftrightarrow x_i \leq y_i$ for $i = 1, 2, .., n$. Let $D \subseteq R^n$ be an invex set and x_0 be an arbitrary point on D. Suppose $\eta : D \times D \to D$, $f = (f_1, f_2, ..., f_n) : D \times D \to R^n$, $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$.

Definition 2.1. (Ben-Israel and Mond [9]) A function $f_i : D\times D \to R$ is called preinvex (with respect to η) at $x_0 \in D$, if there exist a vector function $\eta : D\times D \to D$ such that for all $x \in D$,

$$\lambda f_i(x) + (1-\lambda) f_i(x_0) \geq f_i(x_0 + \lambda \eta(x, x_0)),\ \forall \lambda \in [0,1].$$

Definition 2.2. (Bector et al [8]) A differentiable function $f_i : D \times D \to R$ is said to be univex at at $x_0 \in D$ if there exist $\psi : R \to R$, $b : D \times D \to R$ such that for all $x \in D$,

$$b(x, x_0)\psi f_i(x) - f_i(x_0) \geq \eta(x, x_0)^T \nabla f(x_0).$$

Definition 2.3. The function $f_i : D \times D \to R$ is called directionally differentiable at $x_0 \in D$ in the direction $\eta(x, x_0)$ if there exist a vector function $\eta : D \times D \to D$ such that for all $x \in D$,

$$f_i'(x; \eta(x, x_0)) = \lim_{\lambda \to 0^+} \frac{f_i(x_0 + \lambda\eta(x, x_0)) - f_i(x_0)}{\lambda} \text{ exist.}$$

Definition 2.4. The function $f_i : D \times D \to R$ is called second order directionally differentiable at $x_0 \in D$ in the direction $\eta(x, x_0)$ if there exist a vector function $\eta : D \times D \to D$ such that for all $x \in D$,

$$f_i''(x, \eta(x, x_0)) = \lim_{\lambda \to 0^+} \frac{f_i(x_0 + \lambda\eta(x, x_0)) - f_i(x_0) - \lambda f'(x_0; \eta(x, x_0))}{\lambda^2} \text{ exist.}$$

For the following definitions assume that $f_i : D \times D \to R$ and $h_i : D \times D \to R$ are second order directionally differentiable at $x_0 \in D$.

Definition 2.5. (f_i, h_i) is said to be second order (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned} &b_0(x, x_0)\psi_0\{f_i(x) - f_i(x_0) - \frac{1}{2}p^T f_i''(x; \eta(x, x_0)p\} \\ \geq &f_i'(x; \eta(x, x_0)) + f_i''(x, \eta(x, x_0))p + \rho_0\|\theta(x, x_0)\|^2 \end{aligned}$$

and

$$\begin{aligned} &-b_1(x, x_0)\psi_1\{h_i)(x_0) - \frac{1}{2}q^T h_i''(x; \eta(x, x_0)q\} \\ \geq &h_i'(x; \eta(x, x_0)) + h_i''(x, \eta(x, x_0))q + \rho_1\|\theta(x, x_0)\|^2. \end{aligned}$$

Definition 2.6. (f_i, h_i) is said to be second order quasi (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned} &b_0(x, x_0)\psi_0\{f_i(x) - f_i(x_0) - \frac{1}{2}p^T f_i''(x; \eta(x, x_0)p\} \leq 0 \\ \Rightarrow &f_i'(x; \eta(x, x_0)) + f_i''(x, \eta(x, x_0))p + \rho_0\|\theta(x, x_0)\|^2 \leq 0 \end{aligned}$$

and

$$\begin{aligned}&-b_1(x,x_0)\psi_1\{h_i)(x_0)-\frac{1}{2}q^T h_i''(x;\eta(x,x_0)q\}\leq 0\\ \Rightarrow & h_i'(x;\eta(x,x_0))+h_i''(x,\eta(x,x_0))q+\rho_1\|\theta(x,x_0)\|^2\leq 0.\end{aligned}$$

If the second (implied) inequalities in f_i is strict (whenever $x \neq u$), then (f_i, h_i) is said to be second order semi-strictly quasi (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$.

Definition 2.7. (f_i, h_i) is said to be second order pseudo (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned}& f_i'(x;\eta(x,x_0))+f_i''(x,\eta(x,x_0))p+\rho_0\|\theta(x,x_0)\|^2\geq 0\\ \Rightarrow & b_0(x,x_0)\psi_0\{f_i(x)-f_i(x_0)-\frac{1}{2}p^T f_i''(x;\eta(x,x_0)p\}\geq 0\end{aligned}$$

and

$$\begin{aligned}& h_i'(x;\eta(x,x_0))+h_i''(x,\eta(x,x_0))q+\rho_1\|\theta(x,x_0)\|^2\geq 0\\ \Rightarrow & -b_1(x,x_0)\psi_1\{h_i(x_0)\}\geq 0.\end{aligned}$$

Definition 2.8. (f_i, h_i) is said to be second order pseudo quasi (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned}& f_i'(x;\eta(x,x_0))+f_i''(x,\eta(x,x_0))p+\rho_0\|\theta(x,x_0)\|^2\geq 0\\ \Rightarrow & b_0(x,x_0)\psi_0\{f_i(x)-f_i(x_0)-\frac{1}{2}p^T f_i''(x;\eta(x,x_0)p\}\geq 0\end{aligned}$$

and

$$\begin{aligned}&-b_1(x,x_0)\psi_1\{h_i)(x_0)-\frac{1}{2}q^T h_i''(x;\eta(x,x_0)q\}\leq 0\\ \Rightarrow & h_i'(x;\eta(x,x_0))+h_i''(x,\eta(x,x_0))q+\rho_1\|\theta(x,x_0)\|^2\leq 0.\end{aligned}$$

Definition 2.9. (f_i, h_i) is said to be second order strictly pseudo quasi (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned}& b_0(x,x_0)\psi_0\{f_i(x)-f_i(x_0)-\frac{1}{2}p^T f_i''(x;\eta(x,x_0)p\}\leqq 0\\ \Rightarrow & f_i'(x;\eta(x,x_0))+f_i''(x,\eta(x,x_0))p+\rho_0\|\theta(x,x_0)\|^2< 0\end{aligned}$$

and

$$\begin{aligned}&-b_1(x,x_0)\psi_1\{h_i)(x_0)-\frac{1}{2}q^T h_i''(x;\eta(x,x_0)q\}\leqq 0\\ \Rightarrow & h_i'(x;\eta(x,x_0))+h_i''(x,\eta(x,x_0))q+\rho_1\|\theta(x,x_0)\|^2\leqq 0.\end{aligned}$$

Definition 2.10. (f_i, h_i) is said to be second order quasi pseudo (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned} & b_0(x, x_0)\psi_0\{f_i(x) - f_i(x_0) - \frac{1}{2}p^T f_i''(x; \eta(x, x_0)p\} \leq 0 \\ \Rightarrow & f_i'(x; \eta(x, x_0)) + f_i''(x, \eta(x, x_0))p + \rho_0\|\theta(x, x_0)\|^2 \leq 0 \end{aligned}$$

and

$$\begin{aligned} & h_i'(x; \eta(x, x_0)) + h_i''(x, \eta(x, x_0))q + \rho_1\|\theta(x, x_0)\|^2 \geq 0 \\ \Rightarrow & - b_1(x, x_0)\psi_1\{h_i(x_0)\} \geq 0. \end{aligned}$$

Definition 2.11. (f_i, h_i) is said to be second order quasi strictly pseudo (d, ρ, η, θ)-type 1 univex at $x_0 \in D$ for all $x \in D$,if there exist $\psi : R \to R$, $b : D \times D \to R$ and $\theta : D \times D \to R^n$ and $\rho : R \to R$ such that

$$\begin{aligned} & b_0(x, x_0)\psi_0\{f_i(x) - f_i(x_0) - \frac{1}{2}p^T f_i''(x; \eta(x, x_0)p\} \leqq 0 \\ \Rightarrow & f_i'(x; \eta(x, x_0)) + f_i''(x, \eta(x, x_0))p + \rho_0\|\theta(x, x_0)\|^2 \leq 0 \end{aligned}$$

and

$$\begin{aligned} & h_i'(x; \eta(x, x_0)) + h_i''(x, \eta(x, x_0))q + \rho_1\|\theta(x, x_0)\|^2 \geqq 0 \\ \Rightarrow & - b_1(x, x_0)\psi_1\{h_i(x_0)\} > 0. \end{aligned}$$

In this paper, we consider the following multiple objective nonlinear fractional programming problems:

- **(MFP)**

$$\begin{aligned} & \text{Maximize } \frac{f(x)}{g(x)} = (\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}), \\ & \text{Subject to } h(x) \leqq 0, \end{aligned} \tag{2.1}$$

where $f_i : D \times D \to R$, $g_i : D \times D \to R$,for $i = 1, 2, ..., p$; and $h : D \times D \to R^m$ are second order directionally differentiable at $x_0 \in D$. In the sequel, we assume that $f_i \geq 0$ and $g_i > 0$ on R for $i = 1, 2, .., k$.

Let $P = \{x \in D \subseteq R^n : h_j(x) \leq 0, j = 1, 2, .., m\}$ be the set of feasible solution for the problem (MFP) and denote $I = \{1, 2, .., k\}, M = \{1, 2, .., m\}$,
$J_1 = \{j \in M : h_j(x) = 0\}$ and $J_2 = \{j \in M : h_j(x) < 0\}$. It is obvious that $J_1 \cup J_2 = M$.

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of efficient or weak efficient solution in following sense.

Definition 2.12. A point $x_0 \in P$ is said to be efficient solution of (MFP), if there exists no $x \in P$ such that $\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x_0)}{g_i(x_0)}$ for all $i = 1, 2, .., k$ and $\frac{f_r(x)}{g_r(x)} < \frac{f_r(x_0)}{g_r(x_0)}$ for some $r \in \{1, 2, ...k\}$.

Definition 2.13. The problem (MFP) is said to satisfy the generalized Slater's constraint qualification at $x_0 \in P$ if h is second order (d, ρ, η, θ)-type 1 univex functions at $x_0 \in P$ and there exist $x_0 \in P$ such that $h_j(x_0) < 0, j \in J_2$.

Theorem 2.1. *(Karush-Khun-Tucker type necessary condition) Assume that x_0 is an efficient solution for (MFP) at which the generalized Slater's constraint qualification is satisfied. Then there exist multipliers $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, .., \overline{\mu}_k) \in R^k$ and $\overline{\lambda} \in R^m$, such that*

$$\sum_{i=1}^{k} \mu_i [f_i'(u; \eta(x,u) - v_i g_i'(u; \eta(x,u)] + \lambda^T h'(u; \eta(x,u)) = 0, \tag{2.2}$$

$$f_i(u) - v_i g_i(u) = 0, i = 1, 2, .., k, \tag{2.3}$$

$$\lambda^T h(u) = 0, \tag{2.4}$$

$$u \geq, v \geq 0, \tag{2.5}$$

3. Second order fractional duality

- **(SFD)** Minimize $v = \{v_1, v_2, ..., v_k\}$
  Subject to

$$\sum_{i=1}^{k} \mu_i [(f_i'(u; \eta(x,u) + f_i''(u; \eta(x,u))p) - v_i(g_i'(u; \eta(x,u) + g_i''(u; \eta(x,u))p)]$$
$$+ \lambda^T h'(u; \eta(x,u)) + \lambda^T h''(u; \eta(x,u))q = 0, \tag{3.1}$$

$$[f_i(u) - \frac{1}{2} p^T f_i''(u; \eta(x,u))p] - v_i[g_i(u) - \frac{1}{2} p^T g_i''(u; \eta(x,u))p] \geq 0, i = 1, 2, ..k, \tag{3.2}$$

$$\lambda^T h(u) - \frac{1}{2} q^T h''(u; \eta(x,u))q \geq 0, \tag{3.3}$$

$$u \geq, v \geq 0, \mu = (\mu_1, \mu_2, ..., \mu_k) \geq 0, \ \lambda(\geq 0) \in R^m. \tag{3.4}$$

Theorem 3.1. *(Weak Duality) Let x and (u, μ, λ, p, q) be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i [f_i - v_i g_i], \lambda^T h)$ *is second order strictly pseudo quasi (d, ρ, η, θ) type 1 univex function at $u \in D$,*

(ii) for any $a \in R, \psi_0(a) > 0 \Rightarrow a > 0$ and $c \in R, c \geq 0 \Rightarrow \psi_1(c) \geq 0, b_0 \geq 0$ and $b_1 \geq 0$.

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleqslant (v_1, v_2, ..., v_k)$.

Proof. Suppose to the contrary that

$$(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \leqslant (v_1, v_2, ..., v_k).$$

Then there exist an index $r \in \{1, 2, .., k\}$ such that $\frac{f_r(x)}{g_r(x)} < v_r$ and $\frac{f_i(x)}{g_i(x)} \leq v_i$, for all $i \neq r$.

From which we have $f_r(x) - v_r g_r(x) < 0$, for some $r \in \{1, 2, .., k\}$ and $f_i(x) - v_i g_i(x) \leq 0$, for all $i \neq r$.

Since $\mu \geq 0$, we obtained

$$\sum_{i=1}^{k} \mu_i [f_i(x) - v_i g_i(x)] \leq 0. \tag{3.5}$$

From (3.3), we have $\lambda^T h(u) - \frac{1}{2} q^T h''(u;\eta(x,u))q \geq 0$.

So by hypothesis (ii), the above inequality becomes

$$b_1(x,u)\psi_1[\lambda^T h(u) - \frac{1}{2} q^T h''(u;\eta(x,u))q] \geq 0$$
$$\Rightarrow - b_1(x,u)\psi_1[\lambda^T h(u) - \frac{1}{2} q^T h''(u;\eta(x,u))q] \leq 0. \tag{3.6}$$

So by hypothesis (i), we get

$$\lambda^T h'(u;\eta(x,u)) + \lambda^T h''(u;\eta(x,u))p + \rho_1 ||\theta(x,u)||^2 \leq 0. \tag{3.7}$$

Now from (3.1) and (3.7), we obtained

$$\sum_{i=1}^{k} \mu_i \{[f_i'(u;\eta(x,u)) + f_i''(u;\eta(x,u))p] - v_i[g_i'(u;\eta(x,u)) + g_i''(u;\eta(x,u))p]\}$$
$$- \rho_1 ||\theta(x,u)||^2 \geq 0. \tag{3.8}$$

Now from hypothesis (iii), we get $\rho_0 \geq -\rho_1$

$$\Rightarrow \rho_0 ||\theta(x,u)||^2 \geq -\rho_1 ||\theta(x,u)||^2. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\sum_{i=1}^{k} \mu_i \{[f_i'(u;\eta(x,u)) + f_i''(u;\eta(x,u))p] - v_i[g_i'(u;\eta(x,u)) + g_i''(u;\eta(x,u))p]\}$$
$$+ \rho_0 ||\theta(x,u)||^2 \geq 0. \tag{3.10}$$

So from hypothesis (i) we obtained

$$b_0(x,u)\psi_0 \left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i [f_i(x) - v_i g_i(x))] - \sum_{i=1}^{k} \mu_i (f_i(u) - v_i g_i(u)) \\ + \frac{1}{2} p^T \sum_{i=1}^{k} \mu_i [f_i''(u;\eta(x,u)) - v_i g_i''(u,\eta(x,u))]p \end{array} \right\} > 0. \tag{3.11}$$

Now by hypothesis (ii) and dual constraint (3.2), (3.11) becomes

$$\sum_{i=1}^{k} \mu_i [f_i(x) - v_i g_i(x))] > 0$$

which contradicts (3.5). Hence

$$(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k).$$

□

Corollary 3.1. *Let x and $(u, \mu, \lambda, 0, 0)$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i[f_i - v_i g_i], \lambda^T h)$ *is strictly pseudo quasi (d, ρ, η, θ) type 1 univex function at $u \in D$,*

(ii) for any $a \in R, \psi_0(a) > 0 \Rightarrow a > 0$ and $c \in R, c \geq 0 \Rightarrow \psi_1(c) \geq 0, b_0 \geq 0$ and $b_1 \geq 0$.

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k)$.

Theorem 3.2. *(Weak Duality)Let x and (u, λ, η, p, q) be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is second order semi-strictly quasi (d, ρ, η, θ) type 1 univex function at $u \in D$,*

(ii) for any $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ and $c \in R, c \leq 0 \Rightarrow \psi_1(c) \leq 0$, $b_0 \geq 0$ and $b_1 \geq 0$.

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k)$.

Proof. Suppose to the contrary that

$$(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \leqslant (v_1, v_2, ..., v_k).$$

Then there exist an index $r \in \{1, 2, .., k\}$ such that $\frac{f_r(x)}{g_r(x)} < v_r$ and $\frac{f_i(x)}{g_i(x)} \leq v_i$, for all $i \neq r$.

From which we have $f_r(x) - v_r g_r(x) < 0$, for some $r \in \{1, 2, .., k\}$ and $f_i(x) - v_i g_i(x) \leq 0$,for all $i \neq r$.

Since $\mu \geq 0$, we obtained

$$\sum_{i=1}^{k} \mu_i[f_i(x) - v_i g_i(x)] \leq 0. \tag{3.12}$$

Now from (3.2) from (3.12), we get

$$\left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i[f_i(x) - v_i g_i(x))] - \sum_{i=1}^{k} \mu_i(f_i(u) - v_i g_i(u)) \\ +\frac{1}{2} p^T \sum_{i=1}^{k} \mu_i[f_i''(u; \eta(x, u)) - v_i g_i''(u, \eta(x, u))]p \end{array} \right\} \leq 0. \tag{3.13}$$

So by hypothesis (ii), (3.13) implies

$$b_0(x, u)\psi_0 \left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i[f_i(x) - v_i g_i(x))] - \sum_{i=1}^{k} \mu_i(f_i(u) - v_i g_i(u)) \\ +\frac{1}{2} p^T \sum_{i=1}^{k} \mu_i[f_i''(u; \eta(x, u)) - v_i g_i''(u, \eta(x, u))]p \end{array} \right\} \leq 0 \tag{3.14}$$

Again From (3.3), we have $\lambda^T h(u) - \frac{1}{2} q^T h''(u; \eta(x, u))q \geq 0$.

So by hypothesis (ii), we get

$$\begin{aligned} & b_1(x,u)\psi_1[\lambda^T h(u) - \frac{1}{2} q^T h''(u;\eta(x,u))q] \geq 0 \\ \Rightarrow & - b_1(x,u)\psi_1[\lambda^T h(u) - \frac{1}{2} q^T h''(u;\eta(x,u))q] \leq 0. \end{aligned} \tag{3.15}$$

So from (3.14), (3.15) and hypothesis (i), we get

$$\begin{aligned} & \sum_{i=1}^{k} \mu_i[(f_i'(u;\eta(x,u)) + f_i''(u;\eta(x,u))p) - v_i(g_i'(u;\eta(x,u)) + g_i''(u;\eta(x,u))p)] \\ & + \rho_0||\theta(x,u)||^2 < 0 \end{aligned} \tag{3.16}$$

and

$$\lambda^T h'(u;\eta(x,u)) + \lambda^T h''(u;\eta(x,u))p + \rho_1||\theta(x,u)||^2 \leq 0. \tag{3.17}$$

Adding (3.16) and (3.17) we get

$$\begin{aligned} & \sum_{i=1}^{k} \mu_i[(f_i'(u;\eta(x,u) + f_i''(u;\eta(x,u))p) - v_i(g_i'(u;\eta(x,u) + g_i''(u;\eta(x,u))p)] \\ & + \lambda^T h'(u;\eta(x,u)) + \lambda^T h''(u;\eta(x,u))q + \rho_0||\theta(x,u)||^2 + \rho_1||\theta(x,u)||^2 < 0 \\ \Rightarrow & \sum_{i=1}^{k} \mu_i[(f_i'(u;\eta(x,u) + f_i''(u;\eta(x,u))p) - v_i(g_i'(u;\eta(x,u) + g_i''(u;\eta(x,u))p)] \\ & + \lambda^T h'(u;\eta(x,u)) + \lambda^T h''(u;\eta(x,u))q < -(\rho_0||\theta(x,u)||^2 + \rho_1||\theta(x,u)||^2) < 0. \end{aligned}$$

This contradicts (3.1).

Hence $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k)$. □

Corollary 3.2. *Let x and $(u,\lambda,\eta,0,0)$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is semi-strictly quasi (d,ρ,η,θ) type 1 univex function at $u \in D$,*

(ii) for any $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ and $c \in R, c \leq 0 \Rightarrow \psi_1(c) \leq 0$, $b_0 \geq 0$ and $b_1 \geq 0$,

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k)$.

Theorem 3.3. *(Weak Duality) Let x and (u,λ,η,p,q) be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is second order quasi strictly pseudo (d,ρ,η,θ) type 1 univex function at $u \in D$,*

(ii) for any $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ and $c \in R, \psi_1(c) \geq 0 \Rightarrow c \geq 0, b_0 \geq 0$ and $b_1 \geq 0$,

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleqslant (v_1, v_2, ..., v_k)$.

Proof. Suppose to the contrary that

$$(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \leqslant (v_1, v_2, ..., v_k).$$

$$\Rightarrow \frac{f_i(x)}{g_i(x)} \leq v_i, \forall i = 1, 2, .., k \text{ and } \frac{f_r(x)}{g_r(x)} < v_r, \text{ for some } r \in \{1, 2, .., k\}.$$

From which we have $f_i(x) - v_i g_i(x) \leq 0, \forall i = 1, 2, .., k$ and $f_r(x) - v_r g_r(x) < 0$, for some $r \in \{1, 2, .., k\}$.

So

$$f_i(x) - v_i g_i(x) \leq 0, \forall i = 1, 2, .., k. \tag{3.18}$$

So for $\mu \geq 0$, (3.18) can be written as

$$\sum_{i=1}^{k} \mu_i(f_i(x) - v_i g_i(x)) \leq 0, \forall i = 1, 2, .., k. \tag{3.19}$$

Now subtracting (3.2) from (3.19), we get

$$\left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i(f_i(x) - v_i g_i(x)) - \sum_{i=1}^{k} \mu_i(f_i(u) - v_i g_i(u)) \\ +\frac{1}{2} p^T \sum_{i=1}^{k} \mu_i(f_i''(u; \eta(x, u)) - v_i g_i''(u, \eta(x, u)))p \end{array} \right\} \leq 0. \tag{3.20}$$

Now by hypothesis (ii), (3.20) becomes

$$b_0(x, u)\psi_0 \left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i(f_i(x) - v_i g_i(x)) - \sum_{i=1}^{k} \mu_i(f_i(u) - v_i g_i(u)) \\ +\frac{1}{2} p^T \sum_{i=1}^{k} \mu_i(f_i''(u; \eta(x, u)) - v_i g_i''(u, \eta(x, u)))p \end{array} \right\} \leq 0. \tag{3.21}$$

So from (3.21) and hypothesis (i), we get

$$\begin{aligned} &\sum_{i=1}^{k} \mu_i([f'(u; \eta(x, u)) + f''(u; \eta(x, u))p] - v_i[g'(u; \eta(x, u)) + g''(u; \eta(x, u))p]) \\ &+ \rho_0||\theta(x, u)||^2 < 0. \end{aligned} \tag{3.22}$$

From (3.22) and (3.1), we obtained

$$\lambda^T h'(u; \eta(x, u)) + \lambda^T h''(u; \eta(x, u))p - \rho_0||\theta(x, u)||^2 \geq 0$$

and by hypothesis (iii) the above inequality becomes

$$\Rightarrow \lambda^T h'(u; \eta(x, u)) + \lambda^T h''(u; \eta(x, u))p + \rho_1||\theta(x, u)||^2 \geq 0.$$

So by hypothesis (i), we get

$$\Rightarrow -b_1(x, u)\psi_1[\lambda^T h(u) - \frac{1}{2} q^T h''(u; \eta(x, u))q] > 0,$$

which by hypothesis (ii) implies

$$\Rightarrow \lambda^T h(u) - \frac{1}{2} q^T h''(u; \eta(x, u))q] < 0.$$

This is a contradiction to (3.3).

So $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleqslant (v_1, v_2, ..., v_k)$. □

Corollary 3.3. *Let x and $(u, \lambda, \eta, 0, 0)$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is quasi strictly pseudo* (d, ρ, η, θ) *type 1 univex function at* $u \in D$,

(ii) for any $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ *and* $c \in R, \psi_1(c) \geq 0 \Rightarrow c \geq 0, b_0 \geq 0$ *and* $b_1 \geq 0$.

(iii) $\rho_0 + \rho_1 \geq 0$.

Then $(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_k(x)}{g_k(x)}) \nleq (v_1, v_2, ..., v_k)$.

Theorem 3.4. *(Strong Duality) Let* $\overline{x}$ *be an efficient solution for (MFP) at which Slater's constraint qualification is satisfied. Then there exist* $\overline{\mu} \in R^k$ *and* $\overline{\lambda} \in R^m$ *such that* $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ *is feasible for (SFD) and two objective values are equal.Furthermore if* $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is second order strictly pseudo quasi* (d, ρ, η, θ) *type 1 univex function at* $\overline{u} \in D$ *with* $\rho_0 + \rho_1 \geq 0$, *then* $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ *is efficient solution for (SFD).*

Proof. Since $\overline{x}$ is an efficient solution for (MFP) and Slater's constraint qualification is satisfied at $\overline{x}$, then from Theorem 2.1, there exist $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, .., \overline{\mu}_k) \in R^k$ and $\overline{\lambda} \in R^m$, such that

$$
\begin{aligned}
&\sum_{i=1}^{k} \overline{\mu}_i [f_i'(\overline{u}; \eta(\overline{x}, \overline{u}) - v_i g_i'(\overline{u}; \eta(\overline{x}, \overline{u})] + \lambda^T h'(\overline{u}; \eta(\overline{x}, \overline{u})) = 0, \\
&f_i(\overline{u}) - v_i g_i(\overline{u}) = 0, i = 1, 2, .., k, \\
&\overline{\lambda}^T h(\overline{u}) = 0, \\
&\lambda \geq 0, v_i \geq 0, i = 1, 2, ..., k,
\end{aligned}
$$

which gives that $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ is feasible for (SFD) and the two objectives are equal i.e. $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$.

Now, if $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ is not an efficient solution for (SFD),then there exist a feasible solution for (SFD) such that $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})} < v$, which contradicts the weak duality Theorem 3.1. Hence $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ is an efficient solution for (SFD). □

Theorem 3.5. *(Strong Duality) Let* $\overline{x}$ *be an efficient solution for (MFP) at which Slaters constraint qualification is satisfied. Then there exist* $\overline{\mu} \in R^k$ *and* $\overline{\lambda} \in R^m$ *such that* $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ *is feasible for (SFD) and two objective values are equal.Furthermore,if* $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is second order semi-strictly quasi* (d, ρ, η, θ) *type 1 univex function at* $u \in D$ *with* $\rho_0 + \rho_1 \geq 0$, $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ *is efficient solution for (SFD).*

Proof. The proof of this theorem is similar to that of Theorem 3.5, except that here we invoke the weak duality Theorem 3.2. □

Theorem 3.6. *(Strong Duality) Let* $\overline{x}$ *be an efficient solution for (MFP) at which Slaters constraint qualification is satisfied. Then there exist* $\overline{\mu} \in R^k$ *and* $\overline{\lambda} \in R^m$ *such that* $(\overline{x}, \overline{\mu}, \overline{\lambda}, \overline{p} = 0, \overline{q} = 0)$ *is feasible for (SFD) and two objective values are equal.Furthermore,if* $(\sum_{i=1}^{k} \mu_i(f_i - v_i g_i), \lambda^T h)$ *is second order quasi strictly pseudo*

(d,ρ,η,θ) type 1 univex function at $\overline{u}\in D$ with $\rho_0+\rho_1\geq 0$, $(\overline{x},\overline{\mu},\overline{\lambda},\overline{p}=0,\overline{q}=0)$ is efficient solution for (SFD).

Proof. The proof of this theorem is similar to that of Theorem 3.5, except that here we invoke the weak duality Theorem 3.3. □

Theorem 3.7. *(Strict Converse Duality) Let $\overline{x}$ and $(\overline{u},\overline{\mu},\overline{\lambda},\overline{p},\overline{q})$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $\frac{f_i(\overline{x})}{g_i(\overline{x}}\leq v_i, i=1,2,..,k,$

(ii) $(\sum_{i=1}^k \mu_i[f_i-v_ig_i],\lambda^Th)$ *is second order strictly pseudo quasi (d,ρ,η,θ) type 1 univex function at $u\in D$,*

(iii) for any $a\in R,\psi_0(a)>0\Rightarrow a>0$ and $c\in R, c\geq 0\Rightarrow \psi_1(c)\geq 0, b_0\geq 0$ and $b_1\geq 0$,

(iv) $\rho_0+\rho_1\geq 0.$

Then $\overline{x}=\overline{u}$.

Proof. We assume that $\overline{x}\neq\overline{u}$ and exhibit a contradiction.

Since $\mu\geq 0$ and hypothesis (i) holds, therefore,

$$\sum_{i=1}^k \mu_i[f_i(\overline{x})-v_ig_i(\overline{x})]\leq 0. \tag{3.23}$$

As $(\overline{u},\overline{\mu},\overline{\lambda},\overline{p},\overline{q})$ is the feasible solutions of (SFD)from (3.3) we have

$$\lambda^Th(\overline{u})-\frac{1}{2}\overline{q}^Th''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}\geq 0,$$

and by hypothesis (iii), we get

$$-b_1(\overline{x},\overline{u})\psi_1\{\lambda^Th(\overline{u})-\frac{1}{2}\overline{q}^Th''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}\}\leq 0,$$

which by hypothesis (ii) implies

$$\lambda^Th'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^Th''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}+\rho_1||\theta(\overline{x},\overline{u})||^2\leq 0. \tag{3.24}$$

Now from (3.1), (3.24) and hypothesis (iv), we obtained

$$\begin{aligned}&\sum_{i=1}^k \overline{\mu}_i[(f_i'(\overline{u};\eta(\overline{x},\overline{u}))+f_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p})-\overline{v_i}(g_i'(u;\eta(x,u))+g_i''(u;\eta(x,u))\overline{p})]\\&+\rho_0||\theta(x,u)||^2\geq 0.\end{aligned} \tag{3.25}$$

So from hypothesis (ii) we obtained

$$b_0(\overline{x},\overline{u})\psi_0\left\{\begin{array}{l}\sum_{i=1}^k \mu_i(f_i(\overline{x})-\overline{v_i}g_i(\overline{x}))-\sum_{i=1}^k \mu_i(f_i(\overline{u})-\overline{v_i}g_i(\overline{u}))\\+\frac{1}{2}\overline{p}^T\sum_{i=1}^k \mu_i(f_i''(\overline{u};\eta(\overline{x},\overline{u}))-\overline{v_i}g_i''(\overline{u},\eta(\overline{x},\overline{u})))\overline{p}\end{array}\right\}>0. \tag{3.26}$$

Since $\psi_0(a) > 0 \Rightarrow a > 0$ and $b_0(\overline{x}, \overline{u}) > 0$,(3.26) implies

$$\left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i(f_i(\overline{x}) - \overline{v_i} g_i(\overline{x})) - \sum_{i=1}^{k} \mu_i(f_i(\overline{u}) - \overline{v_i} g_i(\overline{u})) \\ + \frac{1}{2}\overline{p}^T \sum_{i=1}^{k} \mu_i(f_i''(\overline{u}; \eta(\overline{x}, \overline{u})) - \overline{v_i} g_i''(\overline{u}, \eta(\overline{x}, \overline{u})))\overline{p} \end{array} \right\} > 0. \tag{3.27}$$

Since feasibility condition (3.2) holds, therefore (3.27) implies

$$\sum_{i=1}^{k} \mu_i(f_i(\overline{x}) - \overline{v_i} g_i(\overline{x})) > 0,$$

which is a contradiction to (3.23). Hence $\overline{x} = \overline{u}$. □

Theorem 3.8. *(Strict Converse Duality) Let $\overline{x}$ and $(\overline{u}, \overline{\mu}, \overline{\lambda}, \overline{p}, \overline{q})$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $\frac{f_i(\overline{x})}{g_i(\overline{x}} \leq v_i, i = 1, 2, .., k,$

(ii) $(\sum_{i=1}^{k}(f_i - v_i g_i), \lambda^T h)$ *is second order semi-strictly quasi* (d, ρ, η, θ) *type I univex function at* $u \in D$,

(iii) for any $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ *and* $c \in R, c \geq 0 \Rightarrow \psi_1(c) \geq 0, b_0 \geq 0$ *and* $b_1 \geq 0$,

(iv) $\rho_0 + \rho_1 \geq 0$.

Then $\overline{x} = \overline{u}$.

Proof. We assume that $\overline{x} \neq \overline{u}$ and exhibit a contradiction.

Since hypothesis (i) holds, we get

$$f_i(\overline{x}) - v_i g_i(\overline{x}) \leq 0, \forall i = 1, 2, .., k. \tag{3.28}$$

So for $\mu \geq 0$, (3.28) can be written as

$$\sum_{i=1}^{k} \mu_i(f_i(\overline{x}) - v_i g_i(\overline{x})) \leq 0, \forall i = 1, 2, .., k. \tag{3.29}$$

Now subtracting (3.2) from (3.29), we get

$$\left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i(f_i(\overline{x}) - v_i g_i(\overline{x})) - \sum_{i=1}^{k} \mu_i(f_i(\overline{u}) - v_i g_i(\overline{u})) \\ + \frac{1}{2}\overline{p}^T \sum_{i=1}^{k} \mu_i(f_i''(\overline{u}; \eta(\overline{x}, \overline{u})) - v_i g_i''(\overline{u}, \eta(\overline{x}, \overline{u})))\overline{p} \end{array} \right\} \leq 0. \tag{3.30}$$

Since $a \in R, a \leq 0 \Rightarrow \psi_0(a) \leq 0$ and $b_0(\overline{x}, \overline{u}) > 0$, (3.30) implies

$$b_0(\overline{x}, \overline{u})\psi_0 \left\{ \begin{array}{l} \sum_{i=1}^{k} \mu_i(f_i(\overline{x}) - v_i g_i(\overline{x})) - \sum_{i=1}^{k} \mu_i(f_i(\overline{u}) - v_i g_i(\overline{u})) \\ + \frac{1}{2}\overline{p}^T \sum_{i=1}^{k} \mu_i(f_i''(\overline{u}; \eta(\overline{x}, \overline{u})) - v_i g_i''(\overline{u}, \eta(\overline{x}, \overline{u})))\overline{p} \end{array} \right\} \leq 0. \tag{3.31}$$

Again From (3.3), we have $\lambda^T h(\overline{u}) - \frac{1}{2}\overline{q}^T h''(\overline{u}; \eta(\overline{x}, \overline{u}))\overline{q} \geq 0$.

So for $b_1 \geq 0$ and $c \in R, c \geq 0 \Rightarrow \psi_1(c) \geq 0,\ b_0 \geq 0$ we get

$$\begin{aligned} & b_1(\overline{x}, \overline{u})\psi_1[\lambda^T h(\overline{u}) - \frac{1}{2}\overline{q}^T h''(\overline{u}; \eta(\overline{x}, \overline{u}))\overline{q}] \geq 0 \\ \Rightarrow & - b_1(\overline{x}, \overline{u})\psi_1[\lambda^T h(\overline{u}) - \frac{1}{2}\overline{q}^T h''(\overline{u}; \eta(\overline{x}, \overline{u}))\overline{q}] \leq 0. \end{aligned} \tag{3.32}$$

So from (3.31), (3.32) and hypothesis (i), we get

$$\sum_{i=1}^{k}\{[f_i'(\overline{u};\eta(\overline{x},\overline{u}))+f_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]-v_i[g_i'(\overline{u};\eta(\overline{x},\overline{u}))+g_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]\}$$
$$+\rho_0||\theta(\overline{x},\overline{u})||^2<0. \tag{3.33}$$

and

$$\lambda^T h'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}+\rho_1||\theta(\overline{x},\overline{u})||^2\leq 0. \tag{3.34}$$

Adding (3.33) and (3.34) we get

$$\sum_{i=1}^{k}\{[f_i'(\overline{u};\eta(\overline{x},\overline{u}))+f_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]-v_i[g_i'(\overline{u};\eta(\overline{x},\overline{u}))+g_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]\}$$
$$+\lambda^T h'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}+\rho_0||\theta(\overline{x},\overline{u})||^2+\rho_1||\theta(\overline{x},\overline{u})||^2<0,$$
$$\Rightarrow\sum_{i=1}^{k}\{[f_i'(\overline{u};\eta(\overline{x},\overline{u}))+f_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]-v_i[g_i'(\overline{u};\eta(\overline{x},\overline{u}))+g_i''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]\}$$
$$+\lambda^T h'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}<-(\rho_0||\theta(\overline{x},\overline{u})||^2+\rho_1||\theta(\overline{x},\overline{u})||^2)<0,$$

This contradicts (3.1).

So $\overline{x}=\overline{u}$. □

Theorem 3.9. *(Strict Converse Duality) Let $\overline{x}$ and $(\overline{u},\overline{\mu},\overline{\lambda},\overline{p},\overline{q})$ be the feasible solutions of (MFP) and (SFD) respectively. If*

(i) $\frac{f_i(\overline{x})}{g_i(\overline{x}}\leq v_i, i=1,2,..,k,$

(ii) $(\sum_{i=1}^{k}\mu_i[f_i-v_ig_i],\lambda^T h)$ *is second order quasi strictly pseudo* (d,ρ,η,θ) *type 1 univex function at* $\overline{u}\in D$*,with* $\rho_0+\rho_1\geq 0.$

Then $\overline{x}=\overline{u}$.

Proof. We assume that $\overline{x}\neq\overline{u}$ and exhibit a contradiction.

Since hypothesis (i) holds, we get

$$f_i(\overline{x})-v_ig_i(\overline{x})\leq 0,\forall i=1,2,..,k. \tag{3.35}$$

So for $\mu\geq 0$, (3.35) can be written as

$$\sum_{i=1}^{k}\mu_i(f_i(\overline{x})-v_ig_i(\overline{x}))\leq 0,\forall i=1,2,..,k. \tag{3.36}$$

Now subtracting (3.2) from (3.36),we get

$$\left\{\begin{array}{l}\sum_{i=1}^{k}\mu_i(f_i(\overline{x})-v_ig_i(\overline{x}))-\sum_{i=1}^{k}\mu_i(f_i(\overline{u})-v_ig_i(\overline{u}))\\ +\frac{1}{2}\overline{p}^T\sum_{i=1}^{k}\mu_i(f_i''(\overline{u};\eta(\overline{x},\overline{u}))-v_ig_i''(\overline{u},\eta(\overline{x},\overline{u})))\overline{p}\end{array}\right\}\leq 0. \tag{3.37}$$

Now by hypothesis (ii), (3.37) becomes

$$b_0(x,u)\psi_0\left\{\begin{array}{l}\sum_{i=1}^{k}\mu_i(f_i(\overline{x})-v_ig_i(\overline{x}))-\sum_{i=1}^{k}\mu_i(f_i(\overline{u})-v_ig_i(\overline{u}))\\ +\frac{1}{2}p^T\sum_{i=1}^{k}\mu_i(f_i''(\overline{u};\eta(\overline{x},\overline{u}))-v_ig_i''(\overline{u},\eta(\overline{x},\overline{u})))p\end{array}\right\}\leq 0. \tag{3.38}$$

So from (3.38) and hypothesis (i), we get

$$\sum_{i=1}^{k}\mu_i([f'(\overline{u};\eta(\overline{x},\overline{u}))+f''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}]-v_i[g'(\overline{u};\eta(\overline{x},\overline{u}))+g''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}])$$
$$+\rho_0||\theta(\overline{x},\overline{u})||^2<0. \tag{3.39}$$

From (3.39) and (3.1), we obtained

$$\lambda^T h'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}-\rho_0||\theta(\overline{x},\overline{u})||^2\geq 0$$

and by hypothesis (iii) the above inequality becomes

$$\Rightarrow \lambda^T h'(\overline{u};\eta(\overline{x},\overline{u}))+\lambda^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{p}+\rho_1||\theta(\overline{x},\overline{u})||^2\geq 0.$$

So by hypothesis (i), we get

$$\Rightarrow -b_1(\overline{x},\overline{u})\psi_1[\lambda^T h(\overline{u})-\frac{1}{2}q^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}]>0,$$

which by hypothesis (ii) implies $\Rightarrow \lambda^T h(\overline{u})-\frac{1}{2}q^T h''(\overline{u};\eta(\overline{x},\overline{u}))\overline{q}]<0$. This is a contradiction to (3.3). So $\overline{x}=\overline{u}$. □

4. Conclusion

In this paper, a new class of second order(d,ρ,η,θ)-type 1 univex function is introduced. The Wolfe type second order dual problem (SFD) of the nondifferentiable multiobjective fractional programming problem (MFP) is considered, where the objective and constraint functions involved are directionally differentiable. Also the duality results under second order(d,ρ,η,θ)-type 1 univex functions are established. This results can be further extended to higher order fractional dual problem. Also the sufficient optimality conditions for (MFP) under second order (d,ρ,η,θ)-type 1 univex functions can be established.

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