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REMARKS ON THE REGULARITY CRITERIA OF THE SOLUTIONS OF THE 3D MICROPOLAR FLUID EQUATIONS*

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Abstract We provide two regularity criteria for the weak solutions of the 3D micropolar fluid equations, the first one in terms of one directional derivative of the velocity, i.e., $\partial_3 u$, while the second one is is in terms of the behavior of the direction of the velocity $\frac{u}{|u|}$. More precisely, we prove that if

$$\partial_3 u \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)) \quad \text{with } \frac{2}{\beta}+\frac{3}{\alpha}\le 1+\frac{1}{\alpha}, 2<\alpha\le\infty, 2\le\beta<\infty;$$

or

$$\operatorname{div}\left(\frac{u}{|u|}\right) \in L^{\frac{4}{1-2r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0\le r<\frac{1}{2},$$

then the weak solution $(u(x,t),\omega(x,t))$ is regular on $\mathbb{R}^3\times[0,T]$. Here $\dot{X}_r(\mathbb{R}^3)$ is the multiplier space.

Keywords Micropolar fluid equations, regularity criteria, Navier-Stokes equations, weak solution.

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1. Introduction

In this paper, we study the following initial value problem of the incompressible micropolar fluid equations:

$$\partial_t u-\Delta u-\nabla\times\omega+\nabla P+(u\cdot\nabla)u=0 \qquad \text{in } \mathbb{R}^3\times(0,T), \tag{1.1}$$

$$\partial_t \omega-\Delta\omega-\nabla\operatorname{div}\omega+2\omega+(u\cdot\nabla)\omega-\nabla\times u=0 \qquad \text{in } \mathbb{R}^3\times(0,T), \tag{1.2}$$

$$\operatorname{div} u=0 \qquad \text{in } \mathbb{R}^3\times(0,T), \tag{1.3}$$

$$(u,\omega)|_{t=0}=(u_0,\omega_0) \qquad \text{in } \mathbb{R}^3. \tag{1.4}$$

Here $u=(u_1(x,t),u_2(x,t),u_3(x,t))$, $P=P(x,t)$ and $\omega=(\omega_1(x,t),\omega_2(x,t),\omega_3(x,t))$ are respectively the unknown velocity field of the flow, the pressure of the flow, and the micro-rotational velocity, (u_0,d_0) is a given initial data with $\operatorname{div} u_0=0$ in the sense of distributions.

The micropolar fluid equations was introduced by Eringen [7] in 1960s. It is a special model of microfluids which exhibits the microrotational effects and microrotational inertia and can be viewed as a non-Newtonian fluid. In a physical

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sense, micropolar fluid may represent fluids that consists of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. It describe many phenomena such as animal blood and certain anisotropic fluids, e.g., liquid crystals which cannot be characterized appropriately by the Navier-Stokes equations. For more detail background, we refer the readers to see Lukaszewicz [14], Rojas-Medar [19] and the references therein. Besides their physical applications, the micropolar fluid equations are also mathematically significant.When $\omega = 0$, system (1.1)-(1.4) reduces to be the well-known Navier-Stokes equations (see [12]). Fundamental mathematical issues such as the existences of weak and strong solutions for micropolar fluid equations were treated by Galdi and Rionero [9] and Yamaguchi [21], respectively. However, the problem of global regularity of the weak solutions of the 3D micropolar fluids with any initial value still remains unsolve since system (1.1)-(1.4) includes the 3D Navier–Stokes equations. Therefore, it is an interesting thing that the regularity of a given weak solution of the 3D micropolar fluids or the 3D Navier–Stokes equations can be shown under some additional conditions, and over the years different criteria for regularity of the weak solutions have been proposed. For the Navier–Stokes equations, the well-known Prodi-Serrin conditions (see [15, 17, 18]) shows that any weak solution $u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} \leq 1$, $3 < q \leq \infty$ and $2 \leq p < \infty$ is regular on $\mathbb{R}^3 \times [0, T]$. Beirãa da Veiga [1] established a Serrin's type regularity criterion on the gradient of the velocity field, i.e., $\nabla u \in L^\beta(0, T; L^\alpha(\mathbb{R}^3))$ with $\frac{3}{2} \leq \alpha \leq \infty$, $1 \leq \beta \leq \infty$. Constantin and Fefferman [4] introduced a criterion involving the direction of the vorticity, they showed that under a Lipshit-like assumption on the direction of the vorticity $\frac{\mathrm{rot}\ u}{|\mathrm{rot}\ u|}$, the solution is smooth, here $\mathrm{rot}\ u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$ is the vorticity. In [20], Vasseur established that the condition

$$\mathrm{div}(\frac{u}{|u|}) \in L^p(0, T; L^q) \text{ with } \frac{2}{p} + \frac{3}{q} \leq \frac{1}{2} \text{ and } 6 \leq q \leq \infty, \tag{1.5}$$

still ensures the regular of solutions to the 3D Navier-Stokes equations. On the other hand, it is desirable to show that show that there are many regularity results of the weak solutions on the 3D Navier–Stokes equations if some partial derivatives of the velocity satisfy certain growth conditions, i.e., Cao and Titi [2, 3], Zhou and Pokorný [22] refined that one of the following conditions

$$\partial_3 u \in L^p(0, T; L^q) \text{ with } \frac{2}{p} + \frac{3}{q} \leq 2, p > \frac{27}{16}, q > 1;$$
$$\nabla u_3 \in L^p(0, T; L^q) \text{ with } \frac{2}{p} + \frac{3}{q} \leq \frac{19}{12} + \frac{1}{2q}, \text{ if } q \in (\frac{30}{19}, 3] \text{ or } \frac{2}{p} + \frac{3}{q} \leq \frac{3}{2} + \frac{3}{4q}, q > 3;$$
$$\partial_3 u_3 \in L^p(0, T; L^q) \text{ with } \frac{2}{p} + \frac{3}{q} \leq \frac{3}{4} + \frac{3}{2q}, q > 2,$$

implies the regularity of weak solutions to the 3D Navier-Stokes equations.

As the case of the 3D micropolar fluids, there still also many interesting results have been obtained, Dong and Chen [5] proved the regularity of weak solutions under the velocity condition

$$\nabla u \in L^q(0, T; \dot{B}^0_{p,r}(\mathbb{R}^3)) \text{ with } \frac{2}{q} + \frac{3}{p} \leq 2, \frac{3}{2} < p \leq \infty, 1 \leq r \leq \frac{2p}{3}.$$

Recently, Jia et al [10] obtained the following regularity criterion only in terms of one derivative of the velocity

$$\partial_3 u \in L^p(0,T;L^q) \ \text{ with } \ \frac{2}{q}+\frac{3}{p}=1,\ 3<p<\infty.$$

For other regularity criteria results of the 3D micropolar fluids, we refer the reader to see [6, 8, 10] and their references therein.

Motivated by the regularity criteria results for the Navier-Stokes equations and for the micropolar fluids equations cited above, the purpose of the present paper is focused on the regularity criteria of the weak solutions of the 3D micropolar fluids. To introduce the main results, let us first recall the definition of weak solutions the 3D micropolar fluids (1.1)-(1.4) (see [9, 14]).

Definition 1.1. (Weak solutions). Let $(u_0,\omega_0)\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ and $T>0$. A measurable function (u,ω) on $\mathbb{R}^3\times(0,T)$ is called a weak solution of system (1.1)–(1.4) on $(0,T)$ if (u,ω) satisfies the following properties

(i) $(u,\omega)\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;

(ii) (u,ω) verifies (1.1)–(1.4) in the sense of distribution, i.e.,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}(\partial_t\phi+(u\cdot\nabla)\phi)u\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3}\nabla\times\omega\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^3}u_0\phi(x,0)\mathrm{d}x\\
=&\int_0^T\int_{\mathbb{R}^3}\nabla u:\nabla\phi\mathrm{d}x\mathrm{d}t;\\
&\int_0^T\int_{\mathbb{R}^3}(\partial_t\varphi+(u\cdot\nabla)\varphi)\omega\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3}\nabla\times u\varphi\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^3}\omega_0\varphi(x,0)\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{R}^3}(\nabla\omega:\nabla\varphi+\operatorname{div}\omega\operatorname{div}\varphi)\mathrm{d}x\mathrm{d}t+2\int_0^T\int_{\mathbb{R}^3}\omega\varphi\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for all $\phi,\varphi\in C_0^\infty(\mathbb{R}^3\times[0,T))$ with $\operatorname{div}\phi=0$. $\operatorname{div}u=0$ in distribution sense, i.e.,

$$\int_0^T\int_{\mathbb{R}^3}u\cdot\nabla\phi\mathrm{d}x\mathrm{d}t=0,$$

for all $\phi\in C_0^\infty(\mathbb{R}^3\times[0,T))$.

(iii) (u,ω) satisfies the energy inequality, i.e.,

$$\begin{aligned}
&\|u(T)\|_{L^2}^2+\|\omega(T)\|_{L^2}^2+2\int_0^T\|\nabla u\|_{L^2}^2\mathrm{d}t+2\int_0^T\|\nabla\omega\|_{L^2}^2\mathrm{d}t\\
&+2\int_0^T\|\operatorname{div}\omega\|_{L^2}^2\mathrm{d}t+2\int_0^T\|\omega\|_{L^2}^2\mathrm{d}t\le\|u(0)\|_{L^2}^2+\|\omega(0)\|_{L^2}^2
\end{aligned}\tag{1.6}$$

for $0<t\le T\le\infty$.

Our main results read as follows:

Theorem 1.1. *Let* $(u_0,\omega_0)\in H^1(\mathbb{R}^3)$ *with* $\operatorname{div}u_0=0$. *Assume* (u,ω) *is the weak solution of system* (1.1)-(1.4) *on* $\mathbb{R}^3\times(0,T)$. *Suppose that the one derivative of the* *velocity, i.e.,* $\partial_3 u$ *satisfies*

$$\partial_3 u \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)) \quad \text{with } \frac{2}{\beta}+\frac{3}{\alpha}\le 1+\frac{1}{\alpha},\ 2<\alpha\le\infty.$$

Then the solution (u,ω) *is regular on* $\mathbb{R}^3\times[0,T]$.

Theorem 1.2. *Let* $(u_0,\omega_0)\in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$ *with* $\operatorname{div} u_0=0$. *Assume* (u,ω) *is the weak solution of system* (1.1)-(1.4) *on* $\mathbb{R}^3\times(0,T)$. *Suppose that* $\operatorname{div}\left(\frac{u}{|u|}\right)$ *satisfies*

$$\operatorname{div}\left(\frac{u}{|u|}\right)\in L^{\frac{4}{1-2r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0\le r<\frac{1}{2}, \tag{1.7}$$

or $\left\|\operatorname{div}\left(\frac{u}{|u|}\right)\right\|_{L^\infty(0,T;\dot{X}_{\frac{1}{2}})}$ *is small. Then the solution* (u,ω) *is regular on* $\mathbb{R}^3\times[0,T]$. *Here* $\dot{X}_r(\mathbb{R}^3)$ *is the multiplier space.*

Remark 1.1.

1. It is easy to see that our result of Theorem 1.1 is an improvement of the recent work by Jia et al [11].
2. Since the multiplier space $\dot{X}_r(\mathbb{R}^3)$ (see Definition 3.1 below) with $0\le r\le 1$ is wider that the Lebesgue space $L^{\frac{3}{r}}(\mathbb{R}^3)$, hence our result (1.7) gives that the condition (1.5) still implies the weak solution (u,d) is regular on $\mathbb{R}^3\times(0,T)$. We also notice that the result of Theorem 1.2 is still valid for the direction regularity problem of the three dimensional Navier-Stokes equations. So it is an improvement of the recent result obtained by Vasseur [20].

The remainder of the paper is organized as follows. In Section 2, we shall present the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Throughout the paper, we use the letter C to denote the constants which may change from line to line; and use $\|\cdot\|_X$ to denote the norm of space $X(\mathbb{R}^3)$.

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. In order to prove Theorem 1.1, we need to quote the following lemma from [2,3], which will play an important role in our discussion.

Lemma 2.1. *For* $\phi,f,g\in C_c^\infty(\mathbb{R}^3)$, *we have*

$$\left|\int_{\mathbb{R}^3}\phi f g\,dx_1\,dx_2\,dx_3\right|\le C\|\phi\|_{L^2}^{\frac{r-1}{r}}\|\partial_3 u\|_{L^{\frac{2}{3-r}}}^{\frac{1}{r}}\|f\|_{L^2}^{\frac{r-2}{r}}\|\partial_1 f\|_{L^2}^{\frac{1}{r}}\|\partial_2 f\|_{L^2}^{\frac{1}{r}}\|g\|_{L^2},$$

where $2<r<3$, $C>0$ *is a positive bounded constant.*

We now give the proof of Theorem 1.1. By differentiating (1.1) and (1.2) with respect to space variable, then multiply the resulting equations with ∇u and $\nabla\omega$, respectively, one obtains that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\omega(\cdot,t)\|_{L^2}^2)+\|\nabla^2 u(\cdot,t)\|_{L^2}^2\\
&\quad+\|\nabla^2\omega(\cdot,t)\|_{L^2}^2+\|\nabla\operatorname{div}\omega(\cdot,t)\|_{L^2}^2\\
\le&\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla(u\cdot\nabla\omega)\cdot\nabla\omega\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\nabla(\nabla\times\omega)\cdot\nabla\omega+\nabla(\nabla\times u)\cdot\nabla\omega\mathrm{d}x+2\|\nabla\omega\|_{L^2}^2\\
:=&I_1+I_2+I_3+2\|\nabla\omega\|_{L^2}^2.
\end{aligned}
\tag{2.1}
$$

By using the Hölder inequality and integration by parts, it follows that

$$
I_3+2\|\nabla\omega\|_{L^2}^2\le\frac{1}{8}(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\omega\|_{L^2}^2)+C(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2).
$$

By using Lemma 2.1 (with $r=\frac{3\alpha-2}{\alpha}$, $2<\alpha\le\infty$) and the Young inequality, we have

$$
\begin{aligned}
I_1&=-\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u\mathrm{d}x=-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}u_i\partial_i u_j\Delta u_j\mathrm{d}x\\
&\le C\|u\|_{L^2}^{\frac{2\alpha-2}{3\alpha-2}}\|\partial_3 u\|_{L^\alpha}^{\frac{\alpha}{3\alpha-2}}\|\nabla u\|_{L^2}^{\frac{\alpha-2}{3\alpha-2}}\|\nabla^2 u\|_{L^2}^{\frac{2\alpha}{3\alpha-2}}\|\Delta u\|_{L^2}\\
&\le C\|\partial_3 u\|_{L^\alpha}^{\frac{\alpha}{3\alpha-2}}\|\nabla u\|_{L^2}^{\frac{\alpha-2}{3\alpha-2}}\|\nabla^2 u\|_{L^2}^{\frac{5\alpha-2}{3\alpha-2}}\\
&\le\frac{1}{8}\|\nabla^2 u\|_{L^2}^2+C\|\partial_3 u\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where we have used the fact that $\|\Delta u\|_{L^2}\le\|\nabla^2 u\|_{L^2}$. Similarly,

$$
\begin{aligned}
I_2&=-\int_{\mathbb{R}^3}(u\cdot\nabla\omega)\cdot\Delta\omega\mathrm{d}x=-\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}u_i\partial_i\omega_j\Delta\omega_j\mathrm{d}x\\
&\le C\|u\|_{L^2}^{\frac{2\alpha-2}{3\alpha-2}}\|\partial_3 u\|_{L^\alpha}^{\frac{\alpha}{3\alpha-2}}\|\nabla\omega\|_{L^2}^{\frac{\alpha-2}{3\alpha-2}}\|\nabla^2\omega\|_{L^2}^{\frac{2\alpha}{3\alpha-2}}\|\Delta\omega\|_{L^2}\\
&\le C\|\partial_3 u\|_{L^\alpha}^{\frac{\alpha}{3\alpha-2}}\|\nabla\omega\|_{L^2}^{\frac{\alpha-2}{3\alpha-2}}\|\nabla^2\omega\|_{L^2}^{\frac{5\alpha-2}{3\alpha-2}}\\
&\le\frac{1}{8}\|\nabla^2\omega\|_{L^2}^2+C\|\partial_3 u\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla\omega\|_{L^2}^2.
\end{aligned}
$$

Inserting the estimates of I_1, I_2 and I_3 into the inequality (2.1), it follows that

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\omega(\cdot,t)\|_{L^2}^2)+\|\nabla^2 u(\cdot,t)\|_{L^2}^2\\
&\quad+\|\nabla^2\omega(\cdot,t)\|_{L^2}^2+2\|\nabla\operatorname{div}\omega(\cdot,t)\|_{L^2}^2\\
\le&C\|\partial_3 u\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2),
\end{aligned}
$$

which together with the Gronwall inequality lead to the a priori estimate

$$
\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla\omega(\cdot,t)\|_{L^2}^2)\\
\le&(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2)\exp\left\{C\int_0^T\|\partial_3 u(s)\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}}\mathrm{d}s\right\}<\infty.
\end{aligned}
\tag{2.2}
$$

Combining the a priori estimate (2.2) with the energy inequality (1.6), and by standard arguments of continuation of local solutions, we conclude that the solution $(u(x,t),\omega(x,t))$ can be extended beyond $t=T$ provided $\partial_3 u\in L^\beta(0,T;L^\alpha)$ for $\frac{2}{\beta}+\frac{3}{\alpha}\le 1+\frac{1}{\alpha}$ with $2<\alpha\le\infty$. This completes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

In this section, we shall give the proof of Theorem 1.2. Before going to do it, let us first recall the definition of the multiplier space.

Definition 3.1. For $0\le r<\frac{3}{2}$, the space $\dot X_r(\mathbb{R}^3)$ is defined as the space of $f(x)\in L^2_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot X_r}=\sup_{\|g\|_{\dot H^r}\le 1}\|fg\|_{L^2}<+\infty,$$

where we denote the completion of the space $C_0^\infty(\mathbb{R}^3)$ by $\dot H^r(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot H^r}:=\|(-\Delta)^{\frac{r}{2}}u\|_{L^2}$.

We have the homogeneity properties: $\forall x_0\in\mathbb{R}^3$ and $\lambda>0$,

$$f(\cdot+x_0)_{\dot X_r}=\|f(\cdot)\|_{\dot X_r};\quad \|f(\lambda\cdot)\|_{\dot X_r}=\frac{1}{\lambda^r}\|f(\cdot)\|_{\dot X_r}.$$

The following imbedding holds:

$$L^{\frac{3}{r}}(\mathbb{R}^3)\hookrightarrow L^{\frac{3}{r},\infty}(\mathbb{R}^3)\hookrightarrow \dot X_r(\mathbb{R}^3)\quad\text{ for } 0\le r<\frac{3}{2}.$$

Moreover, the vector function $v:=(\frac{x_2}{|x|^{1+r}},-\frac{x_1}{|x|^{1+r}},0)\in\dot X_r(\mathbb{R}^3)$ for $0\le r<\frac{3}{2}$ and $\operatorname{div} v=0$, but $v\notin L^{\frac{3}{2}}(\mathbb{R}^3)$. For more detailed properties of the space $\dot X_r(\mathbb{R}^3)$, we refer to [12,13].

We now present the proof of Theorem 1.2. Firstly, multiplying both sides of the equation (1.1) by $|u|^2u$, and integrating over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\|u(\cdot,t)\|_{L^4}^4+\||u||\nabla u|(\cdot,t)\|_{L^2}^2+\frac{1}{2}\|\nabla|u|^2(\cdot,t)\|_{L^2}^2\\&=-\int_{\mathbb{R}^3}\nabla P\cdot|u|^2u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\times\omega\cdot|u|^2u\mathrm{d}x,\end{aligned}\tag{3.1}$$

where we used the following identities due to the divergence free condition:

$$\begin{aligned}&\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot|u|^2u\mathrm{d}x=\frac{1}{4}\int_{\mathbb{R}^3}u\cdot\nabla|u|^4\mathrm{d}x=0;\\&\int_{\mathbb{R}^3}(\Delta u)\cdot|u|^2u\mathrm{d}x=-\int_{\mathbb{R}^3}|\nabla u|^2|u|^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla|u|^2|\mathrm{d}x.\end{aligned}$$

Multiplying both sides of the equation (1.2) by $|\omega|^2\omega$, and integrating over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^4}^4+\||\omega||\nabla\omega|(\cdot,t)\|_{L^2}^2+\|\nabla|\omega|^2(\cdot,t)\|_{L^2}^2+\frac{1}{2}\||\nabla\times\omega||\omega|(\cdot,t)\|_{L^2}^2\\&+2\int_{\mathbb{R}^3}|\omega|^4\mathrm{d}x=\int_{\mathbb{R}^3}\nabla\times u\cdot|\omega|^2\omega\mathrm{d}x,\end{aligned}\tag{3.2}$$

where we have used the fact that $\nabla \operatorname{div} \omega = \nabla \times (\nabla \times \omega) + \Delta\omega$ implies

$$
\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla \operatorname{div} \omega \cdot |\omega|^2 \omega \mathrm{d}x \\
=&-\int_{\mathbb{R}^3} (\nabla \times (\nabla \times \omega) + \Delta\omega) \cdot |\omega|^2 \omega \mathrm{d}x \\
=&\int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \nabla \times \omega \cdot \nabla |\omega|^2 \times \omega \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |\omega|^2|^2 \mathrm{d}x \\
\geq& \int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla \times \omega|^2 |\omega|^2 + |\nabla |\omega|^2|^2) \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 \mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |\omega|^2|^2 \mathrm{d}x \\
=&\frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times \omega|^2 |\omega|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \omega|^2 |\omega|^2 \mathrm{d}x.
\end{aligned}
$$

Combining (3.1) and (3.2) together, it follows that

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|u(\cdot,t)\|_{L^4}^4 + \|\omega(\cdot,t)\|_{L^4}^4) + \||u||\nabla u|(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|\nabla |u|^2(\cdot,t)\|_{L^2}^2 \\
&+ \||\omega||\nabla \omega|(\cdot,t)\|_{L^2}^2 + \|\nabla |\omega|^2(\cdot,t)\|_{L^2}^2 + 2\|\omega(\cdot,t)\|_{L^4}^4 \\
\leq& \int_{\mathbb{R}^3} |Pu||\nabla |u|^2| \mathrm{d}x + \int_{\mathbb{R}^3} |\omega||u|^2|\nabla u| \mathrm{d}x + \int_{\mathbb{R}^3} |u||\omega|^2 |\nabla \omega| \mathrm{d}x \\
\equiv& I + I_2 + I_3,
\end{aligned}
\tag{3.3}
$$

where we use the following identities

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla P \cdot |u|^2 u \mathrm{d}x = -\int_{\mathbb{R}^3} Pu \cdot \nabla |u|^2 \mathrm{d}x; \\
&\int_{\mathbb{R}^3} \nabla \times \omega \cdot |u|^2 u \mathrm{d}x = -\int_{\mathbb{R}^3} |u|^2 \omega \cdot \nabla \times u \mathrm{d}x - \int_{\mathbb{R}^3} \omega \cdot \nabla |u|^2 \times u \mathrm{d}x; \\
&\int_{\mathbb{R}^3} \nabla \times u \cdot |\omega|^2 \omega \mathrm{d}x = -\int_{\mathbb{R}^3} |\omega|^2 u \cdot \nabla \omega \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla |\omega|^2 \times \omega \mathrm{d}x
\end{aligned}
$$

and the facts that $|\nabla \times u| \leq |\nabla u|$, $|\nabla |u|| \leq |\nabla u|$ and $|\nabla |u|^2| \leq 2|u||\nabla u|$.

Then we shall estimate the above terms I_1, I_2 and I_3 one by one. For the term I_2, by using the Hölder inequality and the Young inequality

$$
\begin{aligned}
I_2 &\leq \||\omega||u|\|_{L^2} \||u||\nabla u|\|_{L^2} \\
&\leq \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + C\||\omega||u|\|_{L^2}^2 \\
&\leq \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + C\|\omega\|_{L^4}^2 \|u\|_{L^4}^2 \\
&\leq \frac{1}{2}\||u||\nabla u|\|_{L^2}^2 + C(\|\omega\|_{L^4}^4 + \|u\|_{L^4}^4).
\end{aligned}
\tag{3.4}
$$

Similarly, we can estimate I_3 as

$$
I_3 \leq \frac{1}{2}\||\omega||\nabla \omega|\|_{L^2}^2 + C(\|\omega\|_{L^4}^4 + \|u\|_{L^4}^4).
\tag{3.5}
$$

Now, we estimate I_1 under the assumption Theorem (1.1). Using the hölder inequality, the interpolation inequality, and the fact that

$$|u|\,\mathrm{div}(\frac{u}{|u|}) = -\frac{u}{|u|}\cdot\nabla|u|,$$

we can do estimate I_1 for $p \geq 6$ as

$$\begin{aligned} I_1 &\leq C\int_{\mathbb{R}^3}|Pu||u||\nabla|u||\mathrm{d}x \leq C\int_{\mathbb{R}^3}|Pu||u|\left|\frac{u}{|u|}\cdot\nabla|u|\right|\mathrm{d}x \\ &= C\int_{\mathbb{R}^3}|Pu||u|\left||u|\,\mathrm{div}\left(\frac{u}{|u|}\right)\right|\mathrm{d}x \leq C\|Pu\|_{L^2}\left\||u|^2\,\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{L^2} \\ &\leq C\||P||u|\|_{L^2}\||u|^2\|_{\dot{H}^r}\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r} \\ &\leq C\||P||u|\|_{L^2}\||u|^2\|_{L^2}^{1-r}\|\nabla|u|^2\|_{L^2}^{r}\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r} \\ &\leq C\||P||u|\|_{L^2}\|u\|_{L^4}^{2-2r}\|\nabla|u|^2\|_{L^2}^{r}\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r}, \end{aligned} \tag{3.6}$$

where we have used the fact that the condition $\mathrm{div}\,u = 0$ implies that (see, e.g., [20])

$$\frac{u}{|u|}\cdot\nabla|u| = |u|\,\mathrm{div}\left(\frac{u}{|u|}\right).$$

For estimate $\||P||u|\|_{L^2}^2$. Taking the gradient on (1.1) and using the facts $\nabla\cdot u = 0$ and $\mathrm{div}(\nabla\times\omega) = 0$ yield

$$-\Delta P = \sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j).$$

By using the Calderon-Zygmund inequality, it is easy to obtain that there exists a absolute positive constant C such that

$$\|P\|_{L^r} \leq C\|u\|_{L^{2r}}^2 \text{ for } 1 < r < \infty.$$

Hence, it is easy to find that

$$\begin{aligned} \|Pu\|_{L^2}^2 &\leq \||P|^2|u|^2\|_{L^1} \leq \|P\|_{L^2}\|Pu^2\|_{L^2} \\ &\leq \|u\|_{L^4}^2\|P\|_{L^3}\||u|^2\|_{L^6} \leq \|u\|_{L^4}^2\|u\|_{L^6}^2\|\nabla|u|^2\|_{L^2} \\ &\leq \|u\|_{L^4}^2\|\nabla u\|_{L^2}^2\|\nabla|u|^2\|_{L^2}. \end{aligned}$$

Hence, by inserting above estimate into (3.6), it follows that for $0 \leq r < \frac{1}{2}$

$$I_1 \leq C\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r}\|u\|_{L^4}^{3-2r}\|\nabla|u|^2\|_{L^2}^{r+\frac{1}{2}}\|\nabla u\|_{L^2}.$$

By using the Young inequality two times, we have

$$\begin{aligned} I_1 &\leq \frac{3}{2}\|\nabla|u|^2\|_{L^2}^2 + C\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r}^{\frac{4}{3-2r}}\|\nabla u\|_{L^2}^{\frac{4}{3-2r}}\|u\|_{L^4}^4 \\ &\leq \frac{3}{2}\|\nabla|u|^2\|_{L^2}^2 + C\left(\left\|\mathrm{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r}^{\frac{4}{1-2r}} + \|\nabla u\|_{L^2}^2\right)\|u\|_{L^4}^4. \end{aligned} \tag{3.7}$$

Inserting the estimates (3.4), (3.5) and (3.7) into (3.3) gives that

$$\begin{aligned}
&\frac{d}{dt}(\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4)+\||u||\nabla u|(\cdot,t)\|_{L^2}^2+\|\nabla|u|^2(\cdot,t)\|_{L^2}^2\\
&+\||\omega||\nabla\omega|(\cdot,t)\|_{L^2}^2+\|\nabla|\omega|^2(\cdot,t)\|_{L^2}^2\\
\le& C\left(\left\|\operatorname{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_r}^{\frac{4}{1-2r}}+\|\nabla u\|_{L^2}^2\right)(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4).
\end{aligned}\tag{3.8}$$

Applying the Growall's inequality to the above estimate, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\{\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4\}+\int_0^T\||u||\nabla u|(\cdot,t)\|_{L^2}^2+\|\nabla|u|^2(\cdot,t)\|_{L^2}^2\mathrm{d}t\\
&+\int_0^T\||\omega||\nabla\omega|(\cdot,t)\|_{L^2}^2+\|\nabla|\omega|^2(\cdot,t)\|_{L^2}^2\mathrm{d}t\\
\le& e^{C\int_0^T\left(\|\operatorname{div}(\frac{u}{|u|})\|_{\dot{X}_r}^{\frac{4}{1-2r}}+\|\nabla u\|_{L^2}^2\right)\mathrm{d}t}(\|u_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4)\\
\le& e^{C(\int_0^T\|\operatorname{div}(\frac{u}{|u|})\|_{\dot{X}_r}^{\frac{4}{1-2r}}\mathrm{d}t+\|(u_0,\omega_0)\|_{L^2}^2)}(\|u_0\|_{L^4}^4+\|\omega_0\|_{L^4}^4)<\infty,
\end{aligned}\tag{3.9}$$

where we have used the energy inequality (1.5) in the last inequality.

When $r=\frac{1}{2}$, the estimate (3.7) reduces to

$$\begin{aligned}
&\frac{d}{dt}(\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4)+\||u||\nabla u|(\cdot,t)\|_{L^2}^2+\|\nabla|u|^2(\cdot,t)\|_{L^2}^2\\
&+\||\omega||\nabla\omega|(\cdot,t)\|_{L^2}^2+\|\nabla|\omega|^2(\cdot,t)\|_{L^2}^2\\
\le& C\left\|\operatorname{div}\left(\frac{u}{|u|}\right)\right\|_{\dot{X}_{\frac{1}{2}}}^2\|\nabla u\|_{L^2}^2(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4).
\end{aligned}$$

Hence, the condition $\left\|\operatorname{div}\left(\frac{u}{|u|}\right)\right\|_{L^\infty(0,T;\dot{X}_{\frac{1}{2}})}$ is small enough together with the Growall's inequality give that

$$\sup_{0\le t\le T}\{\|u(\cdot,t)\|_{L^4}^4+\|\omega(\cdot,t)\|_{L^4}^4\}+\int_0^T\||u||\nabla u|(\cdot,t)\|_{L^2}^2+\|\nabla|u|^2(\cdot,t)\|_{L^2}^2\mathrm{d}t<\infty.\tag{3.10}$$

From (3.9) and (3.10), we get by using the assumption (1.6) that

$$\|u\|_{L^4(0,T;L^{12})}+\|\omega\|_{L^4(0,T;L^{12})}<\infty.$$

Combining the above inequality and the standard Serrin regularity criterion (see e.g., [19]), we have (u,ω) smooth on $\mathbb{R}^3\times(0,T)$. Then, by using the standard arguments of the continuation of local solutions, we conclude that the solution $(u(x,t),\omega(x,t))$ can be extended to T. This completes the proof of Theorem 1.2. □

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