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SYNCHRONIZATION OF DIFFERENT DIMENSIONAL CHAOTIC SYSTEMS WITH TIME VARYING PARAMETERS, DISTURBANCES AND INPUT NONLINEARITIES

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Abstract In this paper, the problems of robust exponential generalized and robust exponential Q-S chaos synchronization are investigated between different dimensional chaotic systems. We consider the more practical and realistic cases when unknown time varying parameters with uncertainties, environmental disturbances, and nonlinearity of input control signals are present. The adaptive technique is employed to design the appropriate controllers and the validity of the proposed controllers are proved using Lyapunov stability theorem. Furthermore, numerical simulations are performed to show the efficiency of the presented scheme.

Keywords Chaos synchronization, disturbances, time-varying parameters, input nonlinearity.

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1. Introduction

In various disciplines of physics, biology, chemistry, engineering, and economy, we encounter systems that undergo spatial and temporal evolution. To model, analyze, and understand these phenomena, the study of dynamical systems is a useful tool that helps in achieving these aims.

Chaos is an important, fascinating, and highly complex behavior which exists in some dynamical systems. Chaotic dynamical system is characterized by its sensitive dependence on initial conditions, unpredictability of the long-term future behavior, and by having positive Lyapunov exponents for its attractor.

The applications of dynamical systems and chaos involve mathematical biology Tu [36], economics Tu [36], electronic circuits El-Sayed etc [9, 10], secure communications Stavroulakis [34], cryptography Kocarev & Lian [24], and neuroscience research Izhikevich [18], chaos control and synchronization Hegazi etc [16], Chai etc [3], Chen etc [5] and Boccaletti etc [2].

The research field of chaos synchronization attains increasing interests through the last two decades as it makes chaos usable in many practical applications includ-

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ing image encryption, real time video encryption and secure data communications Zhu [42], Smaoui etc [33] and Gao & Chen [15].

Different types of chaos synchronization are investigated by utilizing several techniques such as adaptive control Chen [4] and Elabbasy etc [7, 8], H-infinity synchronization Wang etc [37], targeting synchronization Padmanaban etc [31], linear feedback synchronization Matouk [29], adaptive sliding mode control Jawaad etc [19], Pourmahmood etc [32], Li & Chang [25] and Yan etc [39], backstepping Matouk & Agiza [30], and nonlinear feedback synchronization Gambino etc [13].

Complete chaos synchronization is achieved when the synchronization errors between the outputs of two chaotic systems are converging to zero. This ensures that the outputs of the two systems evolve on the same chaotic trajectory. On the other hand, in generalized, projective, and function projective synchronization, the output of one system is synchronized to a given function of the output of the second system. When the synchronization error converges to zero with an exponential rate, this is called exponential synchronization. It is suitable for practical applications which need speed and accuracy.

The Q-S type of synchronization is considered a generalization of all the mentioned types of synchronization. In this type, a function Q of the output of the first system is synchronized to a function S of the output of the second system. The usage of Q-S synchronization makes the secure system more complicated as the forms of the target functions are considered secret keys for the encrypted system.

We summarize some achievement related to basic types of chaos synchronization as follows:

Complete chaos synchronization between two identical hyperchaotic systems having unknown time varying parameters is studied in Li & Shi [26]. Also, complete chaos synchronization is presented in Yan etc [39] between different systems -with identical linear parts- when unknown constant parameters and external disturbances exist.

In Zhang etc [41] and Gang etc [14], generalized synchronization between two different dynamical systems having known constant parameters is discussed. The function projective synchronization, generalized synchronization, and generalized functional synchronization of different systems with unknown constant parameters is presented in Sun [35], Long etc [28] and Feng & Pu [11], respectively. The function projective synchronization of different hyperchaotic systems subjected to external disturbances is examined in Fu [12].

The Q-S synchronization scheme between different chaotic systems with known constant parameters is proposed in Hu & Xu [17] whereas Wang & Shi [38] and Yang & Chen [40] studied Q-S synchronization of non-identical chaotic systems with unknown constant parameters. In Li [27], exponential generalized synchronization for different coupled systems with uncertainties in constant system parameters is discussed. The anti-synchronization of hyperchaotic systems with constant uncertainties in linear parts and external disturbances is demonstrated in Jawaad etc [20]. In Aghabab & Heydari [1], complete chaos synchronization between two different uncertain chaotic systems subjected to input nonlinearities is investigated.

Chaos synchronization of some classes of uncertain master and slave systems with mixed types of time delays attracted much attention in recent years, see for example Karimi etc [21–23].

It is observed that the exponential generalized and exponential Q-S synchronization between systems with time varying parameters are not examined in literature.

So, the aim of this paper is to investigate the problems of exponential generalized and exponential Q-S chaos synchronization of different dimensional dynamical systems having different unknown time varying parameters and uncertainties under the effects of environmental disturbances and input nonlinearities of controller. To the best of the authors' knowledge, this is the first work to face this problem which is not investigated in other literature.

This paper is organized as follows: The proposed scheme of synchronization is introduced in section (2). Numerical simulations are performed in section (3). Finally, section (4) contains the conclusion and the general discussions of this work.

2. The proposed scheme of synchronization

In this section, the following master and slave systems are considered:

The master system is given by

$$\dot{x}_i = \sum_{j=1}^{n}(a_{1ij}(t) + d_{1ij}(t))x_j(t) + \sum_{j=1}^{m}(a_{2ij}(t) + d_{2ij}(t))f_{ij}(X) + d_{3i}(t), \tag{2.1}$$
$$i = 1, 2, ..., n,$$

where $X = [x_1\ x_2\ x_3 ... xn]^T \in \mathbb{R}^n$ is the vector of state variables, $a_{1ij}(t)$ and $a_{2ij}(t)$ are the unknown continuous time varying parameters of linear parts and nonlinear parts of the system, respectively, $d_{1ij}(t)$ and $d_{2ij}(t)$ are the disturbances affect linear and nonlinear parts of the system, respectively, $f_{ij}(X) : \mathbb{R}^n \to \mathbb{R}$ are a continuous nonlinear functions, and $d_{3i}(t)$ are external disturbances affect the whole system.

The slave system has the form

$$\dot{y}_i = \sum_{j=1}^{l}(b_{1ij}(t) + d_{4ij}(t))y_j(t) + \sum_{j=1}^{r}(b_{2ij}(t) + d_{5ij}(t))g_{ij}(Y) + d_{6i}(t) + \Psi_i(u_i), \tag{2.2}$$
$$i = 1, 2, ..., l,$$

where the dimension of the master system is n, the dimension of the slave system is l, the parameters of the slave system are defined as the parameters of master system, and $\Psi_i(u_i)$ represent continuous nonlinear functions of control input signals u_i.

Assumption 2.1. The unknown time varying parameters and disturbances of the systems (2.1) and (2.2) are assumed to be bounded such that

$$\vartheta_{ijk}^{\min} < |\vartheta_{ijk}(t)| < \vartheta_{ijk}^{\max}, \quad \xi_{ij}^{\min} < |\xi_{ij}(t)| < \xi_{ij}^{\max}. \tag{2.3}$$

Given the vector $\phi(X) = [\phi_1(X)\ \phi_2(X)\ ...\phi_l(X)]^T$, of smooth continuous target functions, the exponential generalized synchronization is defined as follows.

Definition 2.1. The exponential generalized synchronization is achieved if the norm of the error $e = Y - \phi(X)$ satisfies the inequality $\|e\|_2 \le Ce^{-\xi t}$, $C,\ \xi > 0$ and $t > t_0$.

The error dynamical system for generalized synchronization is defined by

$$e_i = y_i - \phi_i(X), \quad i = 1, 2, \ldots, l, \tag{2.4}$$

hence taking the derivative of e_i with respect to time, we get

$$\begin{aligned}\dot{e}_i =&\dot{y}_i - \sum_{j=1}^{n} \frac{\partial \phi_i(X)}{\partial x_j}\dot{x}_j\\ =&\sum_{j=1}^{l}(b_{1ij}(t)+d_{4ij}(t))y_j(t)+\sum_{j=1}^{r}(b_{2ij}(t)+d_{5ij}(t))g_{ij}(Y)+d_{6i}(t)\\ &+\Psi_i(u_i)-\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial \phi_i(X)}{\partial x_j}(a_{1jk}(t)+d_{1jk}(t))x_k(t)-\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{\partial \phi_i(X)}{\partial x_j}\\ &(a_{2jk}(t)+d_{2jk}(t))f_{jk}(X)-\sum_{j=1}^{n}\frac{\partial \phi_i(X)}{\partial x_j}d_{3j}(t). \end{aligned} \tag{2.5}$$

To design the appropriate controller to achieve the exponential generalized synchronization, the following assumption is necessary.

Assumption 2.2. The continuous nonlinear function $\Psi_i(u_i)$ satisfies the following inequality for positive real numbers ρ_1 and ρ_2

$$\rho_1 u_i^2 \le u_i \Psi_i(u_i) \le \rho_2 u_i^2. \tag{2.6}$$

The controller u_i is designed as follows:

$$\begin{aligned}u_i =&\frac{-1}{\rho_1}[\sum_{j=1}^{l}(b_{1ij}^{es}(t)+d_{4ij}^{es}(t))\left|y_j(t)\right|+\sum_{j=1}^{r}(b_{2ij}^{es}(t)+d_{5ij}^{es}(t))\left|g_{ij}(Y)\right|\\ &+d_{6i}^{es}(t)+\sum_{j=1}^{n}\sum_{k=1}^{n}\left|\frac{\partial \phi_i(X)}{\partial x_j}\right|(a_{1jk}^{es}(t)+d_{1jk}^{es}(t))\left|x_k(t)\right|+\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{\partial \phi_i(X)}{\partial x_j}\right|\\ &(a_{2jk}^{es}(t)+d_{2jk}^{es}(t))\left|f_{jk}(X)\right|+\sum_{j=1}^{n}\left|\frac{\partial \phi_i(X)}{\partial x_j}\right|d_{3j}^{es}(t)+k\left|y_i-\phi_i(X)\right|]\ \mathrm{sign}e_i\\ =&\frac{\chi_i}{\rho_1}\ \mathrm{sign}e_i, \end{aligned} \tag{2.7}$$

where ϑ_{ijk}^{es} and ϑ_{ij}^{es} represent the estimations of parameters $\vartheta_{ijk}^{\max}$ and $\vartheta_{ij}^{\max}$, respectively, and $k>0$ is the control gain.

The implementation of the controller is examined in numerical simulations of examples presented in Section 3. In real implementation of the controller, the tanh function can be used to overcome the problem of chattering phenomenon and enhance the performance of synchronization scheme.

Remark 2.1. From (2.6) and (2.7), the following helpful inequalities hold

$$\begin{aligned}&\rho_1 u_i^2\left|e_i\right| \le u_i\Psi_i(u_i)\left|e_i\right| \le \rho_2 u_i^2\left|e_i\right|,\\ &\rho_1(\frac{\chi_i}{\rho_1})^2\left|e_i\right| \le (\frac{\chi_i}{\rho_1})\Psi_i(u_i)e_i \le \rho_2(\frac{\chi_i}{\rho_1})^2\left|e_i\right|,\\ &\Psi_i(u_i)e_i \le \rho_1(\frac{\chi_i}{\rho_1})\left|e_i\right|,\\ &\Psi_i(u_i)e_i \le \chi_i\left|e_i\right|. \end{aligned}$$

Remark 2.2. The sign function is used in controller (2.7) for two tasks:

(i) To enable the derivation of inequalities in Remark 2.1.

(ii) The controller has negative value when the synchronization error has positive values. Hence, the controller decreases the value of $\dot{y}_i(t)$ when $y_i > \phi_i(X)$. On the other hand, the controller increases the value of $\dot{y}_i(t)$ when $y_i < \phi_i(X)$.

We prove the validation of the proposed controller via utilizing the following Lyapunov function V

$$\begin{aligned} V =&\frac{1}{2}(\sum_{i=1}^{l}(e_i^2 e^{\upsilon t}) + \sum_{i=1}^{n}\sum_{j=1}^{n}(a^2_{df_{1ij}}(t) + d^2_{df_{1ij}}(t)) + \sum_{i=1}^{n}\sum_{j=1}^{m}(a^2_{df_{2ij}}(t) + d^2_{df_{2ij}}(t)) \\ &+ \sum_{i=1}^{n} d^2_{df_{3i}}(t) + \sum_{i=1}^{l}\sum_{j=1}^{l}(b^2_{df_{1ij}}(t) + d^2_{df_{4ij}}(t)) + \sum_{i=1}^{l}\sum_{j=1}^{r}(b^2_{df_{2ij}}(t) + d^2_{df_{5ij}}(t)) \\ &+ \sum_{i=1}^{l} d^2_{df_{6i}}(t)), \end{aligned} \tag{2.8}$$

where $\vartheta_{df_{ijk}} = \vartheta^{\max}_{ijk} - \vartheta^{es}_{ijk}$, $\vartheta_{df_{ij}} = \vartheta^{\max}_{ij} - \vartheta^{es}_{ij}$ and $0 < \upsilon << k$.

Computing the derivative of Lyapunov function defined in (2.8) with respect to time yields

$$\begin{aligned} \dot{V} =&\sum_{i=1}^{l} e_i \dot{e}_i e^{\upsilon t} + \frac{\upsilon}{2}\sum_{i=1}^{l}(e_i^2 e^{\upsilon t}) + \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{df_{1ij}}(t)\dot{a}_{df_{1ij}}(t) + d_{df_{1ij}}(t)\dot{d}_{df_{1ij}}(t)) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m}(a_{df_{2ij}}(t)\dot{a}_{df_{2ij}}(t) + d_{df_{2ij}}(t)\dot{d}_{df_{2ij}}(t)) + \sum_{i=1}^{n} d_{df_{3i}}(t)\dot{d}_{df_{3i}}(t) \\ &+ \sum_{i=1}^{l}\sum_{j=1}^{l}(b_{df_{1ij}}(t)\dot{b}_{df_{1ij}}(t) + d_{df_{4ij}}(t)\dot{d}_{df_{4ij}}(t)) + \sum_{i=1}^{l}\sum_{j=1}^{r}(b_{df_{2ij}}(t)\dot{b}_{df_{2ij}}(t) \\ &+ d_{df_{5ij}}(t)\dot{d}_{df_{5ij}}(t)) + \sum_{i=1}^{l} d_{df_{6i}}(t)\dot{d}_{df_{6i}}(t). \end{aligned} \tag{2.9}$$

After some calculations, we can deduce that

$$\begin{aligned} \dot{V} \leq&\sum_{i=1}^{l}|e_i|\,(\sum_{j=1}^{l}(b^{\max}_{1ij} + d^{\max}_{4ij})\,|y_j(t)| + \sum_{j=1}^{r}(b^{\max}_{2ij} + d^{\max}_{5ij})\,|g_{ij}(Y)| + d^{\max}_{6i} + \chi_i \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n}\left|\frac{\partial\phi_i(X)}{\partial x_j}\right|(a^{\max}_{1jk} + d^{\max}_{1jk})\,|x_k(t)| \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{\partial\phi_i(X)}{\partial x_j}\right|(a^{\max}_{2jk} + d^{\max}_{2jk})\,|f_{jk}(X)| \\ &+ \sum_{j=1}^{n}\left|\frac{\partial\phi_i(X)}{\partial x_j}\right| d^{\max}_{3j}(t))e^{\upsilon t} + \frac{\upsilon}{2}\sum_{i=1}^{l}(e_i^2 e^{\upsilon t}) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{df_{1ij}}(t)\dot{a}_{df_{1ij}}(t) + d_{df_{1ij}}(t)\dot{d}_{df_{1ij}}(t)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\left|\frac{\partial \phi_i(X)}{\partial x_j}\right| d_{3j}^{\max}(t))e^{\upsilon t}+\frac{\upsilon}{2}\sum_{i=1}^{l}(e_i^2 e^{\upsilon t})\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{df_{1ij}}(t)\dot{a}_{df_{1ij}}(t)+d_{df_{1ij}}(t)\dot{d}_{df_{1ij}}(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}(a_{df_{2ij}}(t)\dot{a}_{df_{2ij}}(t)+d_{df_{2ij}}(t)\dot{d}_{df_{2ij}}(t))+\sum_{i=1}^{n} d_{df_{3i}}(t)\dot{d}_{df_{3i}}(t)\\
&+\sum_{i=1}^{l}\sum_{j=1}^{l}(b_{df_{1ij}}(t)\dot{b}_{df_{1ij}}(t)+d_{df_{4ij}}(t)\dot{d}_{df_{4ij}}(t))\\
&+\sum_{i=1}^{l}\sum_{j=1}^{r}(b_{df_{2ij}}(t)\dot{b}_{df_{2ij}}(t)+d_{df_{5ij}}(t)\dot{d}_{df_{5ij}}(t))+\sum_{i=1}^{l} d_{df_{6i}}(t)\dot{d}_{df_{6i}}(t). \qquad (2.10)
\end{aligned}
$$

From (2.10), the updating laws are derived to ensure that $\dot{V}$ has negative values as follows:

$$
\begin{aligned}
\dot{a}_{1ij}^{es} &= \sum_{p=1}^{l}\left|\frac{\partial \phi_p(X)}{\partial x_i}\right| |x_j(t)|\,|e_p|\,e^{\upsilon t},\\
\dot{a}_{2ij}^{es} &= \sum_{p=1}^{l}\left|\frac{\partial \phi_p(X)}{\partial x_i}\right| |f_{ij}(X)|\,|e_p|\,e^{\upsilon t},\\
\dot{b}_{1ij}^{es} &= |y_j(t)|\,|e_i|\,e^{\upsilon t},\\
\dot{b}_{2ij}^{es} &= |g_{ij}(Y)|\,|e_i|\,e^{\upsilon t},\\
\dot{d}_{1ij}^{es} &= \sum_{p=1}^{l}\left|\frac{\partial \phi_p(X)}{\partial x_i}\right| |x_j(t)|\,|e_p|\,e^{\upsilon t}, \qquad (2.11)\\
\dot{d}_{2ij}^{es} &= \sum_{p=1}^{l}\left|\frac{\partial \phi_p(X)}{\partial x_i}\right| |f_{ij}(X)|\,|e_p|\,e^{\upsilon t},\\
\dot{d}_{3i}^{es} &= \sum_{p=1}^{l}\left|\frac{\partial \phi_p(X)}{\partial x_i}\right| |e_p|\,e^{\upsilon t},\\
\dot{d}_{4ij}^{es} &= |y_j(t)|\,|e_i|\,e^{\upsilon t},\\
\dot{d}_{5ij}^{es} &= |g_{ij}(Y)|\,|e_i|\,e^{\upsilon t},\\
\dot{d}_{6i}^{es} &= |e_i|\,e^{\upsilon t},
\end{aligned}
$$

and therefore the derivative of V has the following form

$$
\dot{V} \le -k^*\sum_{i=1}^{l} e_i^2 e^{\upsilon t}, \quad k^* = k-\frac{\upsilon}{2}. \qquad (2.12)
$$

From (2.8) and (2.12) it can be shown that

$$
\frac{1}{2}\sum_{i=1}^{l} e_i^2 e^{\upsilon t} \le V \le V(0), \qquad (2.13)
$$

and therefore

$$\|\mathbf{e}\|_2 \leq \sqrt{2V(0)}e^{-\upsilon t/2}, \quad t \geq 0 \tag{2.14}$$

i.e. exponential generalized synchronization is achieved.

We extend the proposed scheme to the case of exponential Q-S synchronization as follows:

Definition 2.2. The exponential Q-S synchronization is achieved if the norm of the error $\mathbf{e}_{qs} = \mathbf{Q}(\mathbf{Y}) - \mathbf{S}(\mathbf{X})$ satisfies the following inequality

$$\|\mathbf{Q}(\mathbf{Y}) - \mathbf{S}(\mathbf{X})\|_2 \leq Ce^{-\zeta t}, \quad C,\ \zeta > 0 \text{ and } t > t_0, \tag{2.15}$$

where $Q(Y) = [q_1\ q_2 ... q_{l_n}\ ]^T$ and $S(X) = [s_1\ s_2 ... s_{l_n}\ ]^T$ are vectors of continuous smooth target functions $q_i\ (Y)$ and $s_i(X)$, respectively.

The error dynamical system is expressed by

$$\begin{aligned}\dot{\mathbf{e}}_{qs} &= DQ(Y)\dot{Y} - DS(X)\dot{X} \\ &= DQ(Y)(\mathbf{G}(Y, B, \mathbf{d}_s) + \mathbf{\Psi}(\mathbf{u})) - DS(X)\mathbf{F}(X, A, \mathbf{d}_m),\end{aligned} \tag{2.16}$$

where

$$DQ(Y) = \begin{bmatrix} \frac{\partial q_1}{\partial y_1} & \frac{\partial q_1}{\partial y_2} & . & \frac{\partial q_1}{\partial y_l} \\ \frac{\partial q_2}{\partial y_1} & . & . & . \\ . & . & . & . \\ \frac{\partial q_{l_n}}{\partial y_1} & . & . & \frac{\partial q_{l_n}}{\partial y_l} \end{bmatrix}, \quad DS(X) = \begin{bmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} & . & \frac{\partial s_1}{\partial x_n} \\ \frac{\partial s_2}{\partial x_1} & . & . & . \\ . & . & . & . \\ \frac{\partial s_{l_n}}{\partial x_1} & . & . & \frac{\partial s_{l_n}}{\partial x_n} \end{bmatrix},$$

$\mathbf{F}(X, A, \mathbf{d}_m)$, $\mathbf{G}(Y, B, \mathbf{d}_s)$, A, B, $\mathbf{d}_s$ and $\mathbf{d}_m$ represent the right hand side of system (2.1), the right hand side of system (2.2), vector of time varying parameters of master system, vector of time varying parameters of slave system, vector of environmental disturbances of master system, and vector of environmental disturbances of slave system, respectively.

The control input $\mathbf{u}$ is proposed in the following form

$$\begin{aligned}\mathbf{u} = &-\frac{1}{\rho_1}(\mathbf{G}(|Y|, B^{es}, \mathbf{d}_s^{es}) + D_{Abs}^{-1}Q(Y) \\ &(D_{Abs}S(X)\mathbf{F}(|X|, A^{es}, \mathbf{d}_m^{es}) + \mathbf{k}\,|\mathbf{e}_{qs}|))\text{sign } \mathbf{e}_{qs},\end{aligned} \tag{2.17}$$

where $\text{sign}\mathbf{e}_{qs} = [\text{sign}e_{1qs} ... \text{sign}e_{l_n qs}]^T$, $|\mathbf{e}_{qs}| = [|e_{1qs}| ... |e_{l_n qs}|]^T$,

$$e_{iqs} = q_i(Y) - s_i(X),\ D_{Abs}Q(Y) = \begin{bmatrix} \left|\frac{\partial q_1}{\partial y_1}\right| & \left|\frac{\partial q_1}{\partial y_2}\right| & . & \left|\frac{\partial q_1}{\partial y_l}\right| \\ \left|\frac{\partial q_2}{\partial y_1}\right| & . & . & . \\ . & . & . & . \\ \left|\frac{\partial q_{l_n}}{\partial y_1}\right| & . & . & \left|\frac{\partial q_{l_n}}{\partial y_l}\right| \end{bmatrix},$$

$$D_{Abs}S(X) = \begin{bmatrix} \left|\frac{\partial s_1}{\partial x_1}\right| & \left|\frac{\partial s_1}{\partial x_2}\right| & . & \left|\frac{\partial s_1}{\partial x_n}\right| \\ \left|\frac{\partial s_2}{\partial x_1}\right| & . & . & . \\ . & . & . & . \\ \left|\frac{\partial s_{l_n}}{\partial x_1}\right| & . & . & \left|\frac{\partial s_{l_n}}{\partial x_n}\right| \end{bmatrix},$$

$D_{Abs}^{-1}Q(Y)$ is the inverse matrix of $D_{Abs}Q(Y)$, also, $A^{es}, B^{es}, \mathbf{d}_s^{es}$ and $\mathbf{d}_m^{es}$ are the estimations of system parameters and disturbances $A, B, \mathbf{d}_s$ and $\mathbf{d}_m$, respectively. Note that the inverse of nonsingular square matrix can be computed directly, whereas the right inverse matrix $F^{-1} = F^T(FF^T)^{-1}$ is used when the matrix F is not a square matrix [6].

Choosing the following Lyapunov function

$$V(e_{qs}, A, B, D_p) = \frac{1}{2}(\mathbf{e}_{qs}^T\mathbf{e}_{qs})e^{\upsilon t} + \frac{1}{2}(A_{df}^T A_{df} + B_{df}^T B_{df} + d_{m_{df}}^T \mathbf{d}_{m_{df}} + d_{s_{df}}^T \mathbf{d}_{s_{df}}), \tag{2.18}$$

where $A_{df} = A^{\max} - A^{es}$, $B_{df} = B^{\max} - B^{es}$, $\mathbf{d}_{m_{df}} = \mathbf{d}_m^{\max} - \mathbf{d}_m^{es}$, $\mathbf{d}_{s_{df}} = \mathbf{d}_s^{\max} - \mathbf{d}_s^{es}$, and for $\Omega = A, B, \mathbf{d}_s$ or $\mathbf{d}_m$ the symbol $\Omega^{\max}$ represent the vector of maximum values of the elements exist in Ω,

$$\begin{aligned}
A_{df}^T \dot{A}_{df} + d_{m_{df}}^T \dot{d}_{m_{df}} &= -\mathbf{e}_{qs}^T(D_{Abs}S(X)(\mathbf{F}(|X|, A^{\max}, \mathbf{d}_m^{\max}) - \mathbf{F}(|X|, A^{es}, \mathbf{d}_m^{es})))e^{\upsilon t}, \\
B_{df}^T \dot{B}_{df} + d_{s_{df}}^T \dot{d}_{s_{df}} &= -\mathbf{e}_{qs}^T(D_{Abs}Q(Y)(\mathbf{G}(|Y|, B^{\max}, \mathbf{d}_s^{\max}) - \mathbf{G}(|Y|, B^{es}, \mathbf{d}_s^{es})))e^{\upsilon t},
\end{aligned} \tag{2.19}$$

then computing the derivative of V, substituting from (2.19) and performing some calculations to get

$$\dot{V} \le -k^* \sum_{i=1}^{l_n} e_{iqs}^2 e^{\upsilon t}, \tag{2.20}$$

$$\|\mathbf{e}_{qs}\|_2 \le \sqrt{2V(0)}e^{-\upsilon t/2}, \quad t \ge 0 \tag{2.21}$$

and therefore the exponential Q-S synchronization is achieved.

3. Numerical simulations

In this section, illustrative example is presented to verify theoretical findings. The Lorenz system is taken as the master system corresponding to system (2.1) as

$$\begin{aligned}
\dot{x}_1 &= a_1(t)(x_2 - x_1) + d_1, \\
\dot{x}_2 &= a_2(t)x_1 + d_2x_1 - x_2 - a_3(t)x_1x_3, \\
\dot{x}_3 &= a_4(t)x_1x_2 - a_5(t)x_3 + d_3x_3 + d_4,
\end{aligned} \tag{3.1}$$

and the following hyperchaotic system proposed by Chen etc [6], which has two positive Lyapunov exponents larger than most known hyperchaotic systems and therefore more complexity, represents the slave system (2.2)

$$\begin{aligned}
\dot{y}_1 &= b_1(t)(y_2 - y_1) + b_2(t)y_2y_3 + d_5y_2y_3 + d_6 + \psi_1(u_1), \\
\dot{y}_2 &= b_3(t)y_1 + y_2 + y_4 + d_7y_1 - b_4(t)y_1y_3 + d_8y_1y_3 + d_9 + \psi_2(u_2), \\
\dot{y}_3 &= b_5(t)y_1y_2 + d_{10}y_1y_2 - b_6(t)y_3 + d_{11} + \psi_3(u_3), \\
\dot{y}_4 &= -b_7(t)y_2 + d_{12}y_2 + d_{13} + \psi_4(u_4),
\end{aligned} \tag{3.2}$$

where $a_1(t) = 10 + 2\cos 13t$, $a_2(t) = 28 + \cos^2 0.7\pi t$, $a_3(t) = 1 + 0.3\sin 5t$, $a_4(t) = 1 + 0.1\sin 3t\cos 7t$, $a_5(t) = \frac{8}{3}$, $b_1(t) = 35 + 2.3\sin 24t$, $b_2(t) = 35 + 4\cos 15t$, $b_3(t) = 25 + 3\sin^2 5\pi t, b_4(t) = 5 + \sin 11t\cos 27t, b_5(t) = 1 + 0.2\sin 9t, b_6(t) =$

$4.9 + 0.2\cos 19\pi t, b_7(t) = 100 + 14\sin 8\pi t,\ \psi_i(u_i) = (7 + 2\sin t)u_i$, the environmental disturbances d_i are obtained by generating random real numbers from normal distribution with zero mean value and standard deviation value equals 3 as shown in Figure 1.

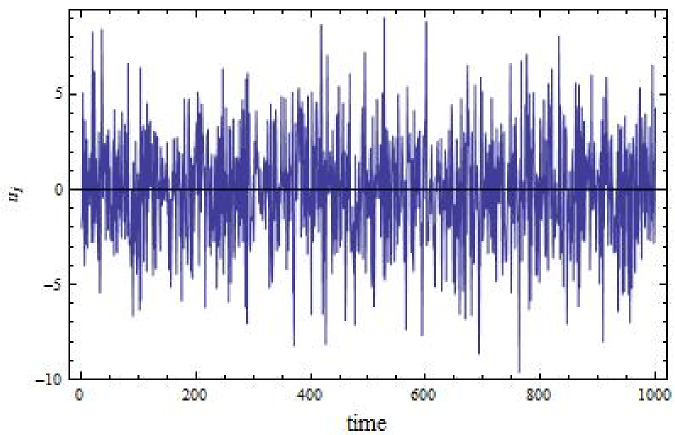

Figure 1. The disturbances d_i are obtained from normal distribution with mean 0 and standard deviation 3 and they are used in master and slave systems

Remark 3.1. The continuous nonlinear function $\psi_i(u_i) = (7 + 2\sin t)u_i$ satisfy Assumption 2.2 with parameters $\rho_1 = 5$ and $\rho_2 = 9$.

We set $\upsilon = 0.01$ and study two cases of exponential chaos synchronization as follows:

Case 1: Exponential generalized synchronization with target functions defined by

$$\phi(X) = \begin{bmatrix} x_1^2 \\ 2x_2 \\ x_2 + 4x_3 \\ -x_2^2 + 0.1x_3 \end{bmatrix}.$$

Case 2: Exponential Q-S synchronization with functions $Q(Y)$ and $S(X)$

$$Q(Y) = \begin{bmatrix} y_1 + y_2 \\ y_2 + y_3 - 3y_4 \\ y_1 + y_2 + y_3 \end{bmatrix},\ S(X) = \begin{bmatrix} x_2 + x_3 \\ \sin(x_1) + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}.$$

Remark 3.2. In case 1 we have $\mathbf{e} = \begin{bmatrix} y_1 - x_1^2 \\ y_2 - 2x_2 \\ y_3 - x_2 - 4x_3 \\ y_4 + x_2^2 - 0.1x_3 \end{bmatrix}$, $\dfrac{\partial\phi_1(X)}{\partial x_1} = 2x_1$, $\dfrac{\partial\phi_1(X)}{\partial x_2}$ $= 0$, …, etc. Therefore, the control signals u_i can be easily obtained from (2.7).

For example,

$$
\begin{aligned}
u_1 =& \frac{-1}{5}[b_1^{es}(t)(|y_1(t)|+|y_2(t)|)+(b_2^{es}(t)+d_5^{es}(t))\,|y_2(t)y_3(t)| \\
&+d_6^{es}(t)+\sum_{j=1}^{3}\left|\frac{\partial\phi_1(X)}{\partial x_j}\right| a_1^{es}(t)(|x_1(t)|+|x_2(t)|) \\
&+\sum_{j=1}^{3}\left|\frac{\partial\phi_1(X)}{\partial x_j}\right| d_1^{es}(t)+k\,|y_1-\phi_1(X)|]\ \mathrm{sign}e_1 \\
=&\frac{\chi_1}{\rho_1}\ \mathrm{sign}e_1,
\end{aligned}
\tag{3.3}
$$

and so on. We choose $k=5$ and replace the sign function by $\tanh 400e_i$ in real implementation of the controller.

Remark 3.3. In case 2 we have $\mathbf{e}=\begin{bmatrix} y_1+y_2-x_2-x_3 \\ y_2+y_3-3y_4-\sin(x_1)-x_2-x_3 \\ y_1+y_2+y_3-x_1-x_2 \end{bmatrix}$, $D_{Abs}Q(Y)=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $D_{Abs}^{-1}Q(Y)=\frac{1}{19}\begin{bmatrix} 9 & -1 & 1 \\ 10 & 1 & -1 \\ -19 & 0 & 19 \\ 3 & 6 & -16 \end{bmatrix}$, $D_{Abs}S(X)=\begin{bmatrix} 0 & 1 & 1 \\ |\cos x_1| & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and we can simply use equations (2.17) to realize the controller.

Results of numerical simulations are shown in Figure 2 to Figure 8 for the following initial values of parameters: $x_1(0)=1,\ x_2(0)=2,\ x_3(0)=3,\ y_1(0)=14,$ $y_2(0)=9,\ y_3(0)=11,\ y_4(0)=15,\ a_1^{es}(0)=5,\ a_2^{es}(0)=15, a_3^{es}(0)=0.3, a_4^{es}(0)=0.5, a_5^{es}(0)=1, b_1^{es}(0)=20, b_2^{es}(0)=25, b_3^{es}(0)=10, b_4^{es}(0)=2, b_5^{es}(0)=0.2, b_6^{es}(0)=2, b_7^{es}(0)=45$, and $d_i^{es}(0)=1, i=1,2,...13$.

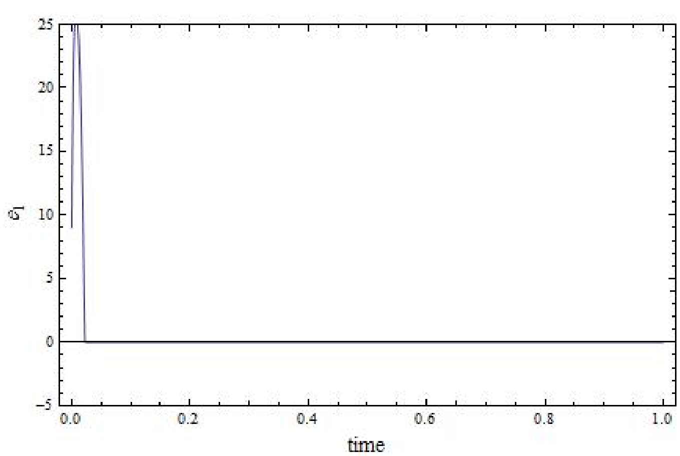

Figure 2. Synchronization error e_1 in the case of generalized synchronization

As an illustration to the computational complexity when performing the process of synchronization from time 0 to 100, the running time of the proposed scheme is 9.2 sec for exponential generalized synchronization and 11.3 sec for exponential Q-S chaos synchronization.

It is observed that the two cases of exponential chaos synchronization are achieved through a small time interval due to exponential rate of decaying of synchronization error which has a norm less than const$*e^{-0.005t}$ for $t > 0$. The value of the constant depends on the form of Lyapunov functions defined in equations (2.8) and (2.18) and depends also on initial values of parameters such that it equals $\sqrt{2V(0)}$.

For example in generalized synchronization case it has an approximate value $\sqrt{2V(0)} \approx 66.7$. So, in theoretical view we have $\sum_{i=1}^{3} e_i^2 \leq 66.7e^{-0.005t}$ which is satisfied by the developed algorithm as shown Figure 2 to Figure 5. Also, Numerical simulations show the good performance of the proposed scheme for Q-S synchronization case, see Figure 6 to Figure 8.

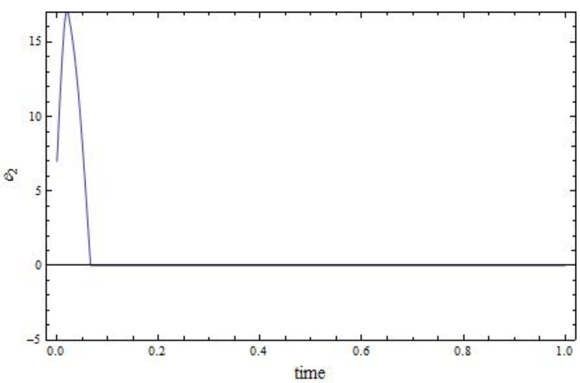

Figure 3. Synchronization error e_2 in the case of generalized synchronization

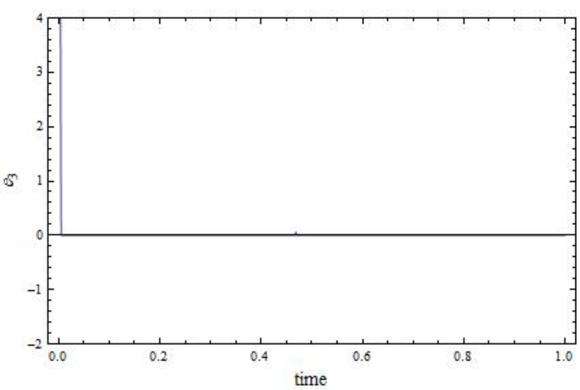

Figure 4. Synchronization error e_3 in the case of generalized synchronization

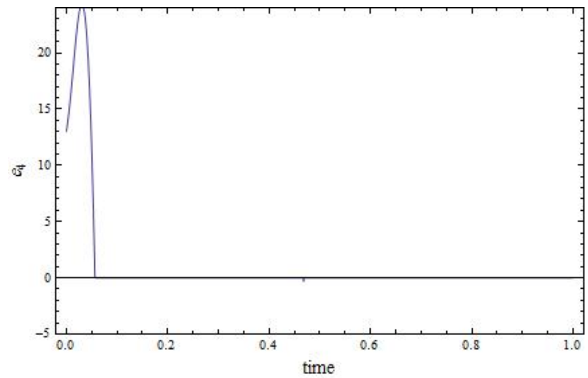

Figure 5. Synchronization error e_4 in the case of generalized synchronization

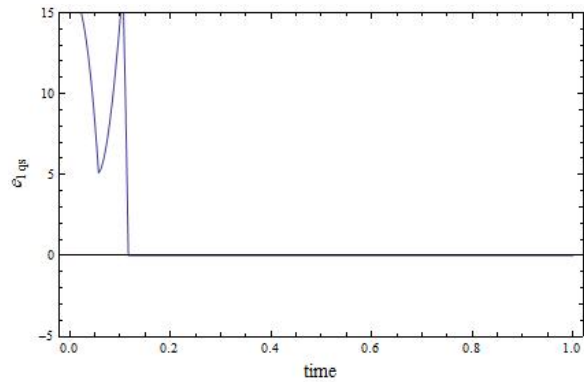

Figure 6. Synchronization error versus time for e_{1qs} in the case of Q-S synchronization

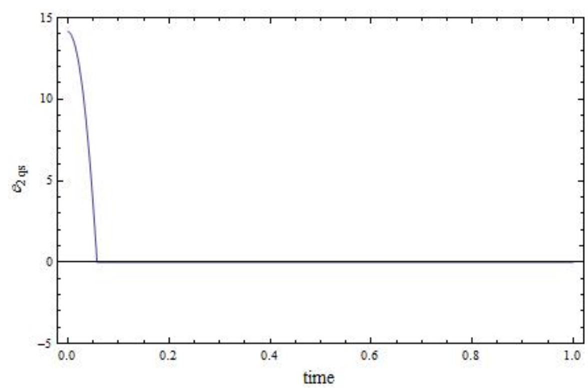

Figure 7. Synchronization error versus time for e_{2qs} in the case of Q-S synchronization

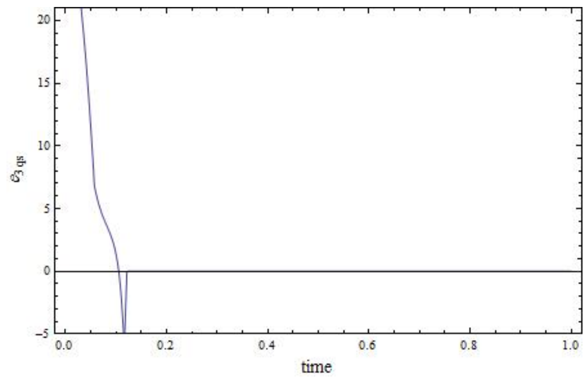

Figure 8. Synchronization error versus time for e_{3qs} in the case of Q-S synchronization

4. Discussion and conclusion

The chaos synchronization in realistic cases involves the presence of unknown time varying parameters, environmental disturbances, and input nonlinearity is achieved. The cases of generalized and Q-S chaos synchronization are studied since they increase and strengthen the security level of communication channel more than other types of chaos synchronization such as complete, projective, and anti synchronization which can be considered as special cases of generalized or Q-S synchronization.

We proposed suitable Lyapunov functions to obtain the updating laws for the two cases of synchronization that are examined and theoretically proved the validity of the presented new controllers. Results obtained illustrate the good performance of the proposed controllers that succeed in achieving the intended two cases of exponential chaos synchronization through a small time interval due to exponential rate of decaying of synchronization error.

Future work can include circuit implementation of the proposed scheme of synchronization in practical secure telecommunications systems.

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