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STABILITY AND TRAVELING WAVES OF AN EPIDEMIC MODEL WITH RELAPSE AND SPATIAL DIFFUSION*

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Abstract An epidemic model with relapse and spatial diffusion is studied. Such a model is appropriate for tuberculosis, including bovine tuberculosis in cattle and wildlife, and for herpes. By using the linearized method, the local stability of each of feasible steady states to this model is investigated. It is proven that if the basic reproduction number is less than unity, the disease-free steady state is locally asymptotically stable; and if the basic reproduction number is greater than unity, the endemic steady state is locally asymptotically stable. By the cross-iteration scheme companied with a pair of upper and lower solutions and Schauder's fixed point theorem, the existence of a traveling wave solution which connects the two steady states is established. Furthermore, numerical simulations are carried out to complement the main results.

Keywords Traveling waves, relapse, spatial diffusion, upper-lower solutions, time delay.

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1. Introduction

In most epidemic models, individuals are often divided into several classes such as susceptible, infective and recovered classes. For some diseases, such as human tuberculosis and Herpes simplex virus type 2 (herpes), recovered individuals may relapse with reactivation of latent infection and revert back to the infective class. For human tuberculosis, incomplete treatment can lead to relapse, but relapse can also occur in patients who took a full course of treatment and were declared cured (see, e.g., Chin Martin [7] and van den Driessche [12]). Important features of herpes are that an individual once infected remains infected for life, and the virus reactivates regularly with reactivation producing a relapse period of infectiousness (see, e.g., Blower etc. [1], Chin [3], van den Driessche [14], VanLandingham [15] and the references therein). Many relapse phenomenon of disease observed in clinical study is an important feature of some animal and human disease (see, e.g., Bowong & Aziz-Alaoui [2], Noble [8] and Tudor [11]).

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van den Driessche and Zou [14] formulated and studied a more realistic model by considering a more general relapse distribution and investigating the consequences of different assumptions about the relapse period. The model is given by

$$\begin{aligned}
\dot{S}(t) &= d - dS(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta S(t)I(t) - (d+\gamma)I(t) - \int_0^t \gamma I(\xi)e^{-d(t-\xi)}P(t-\xi)d\xi, \\
\dot{R}(t) &= \gamma I(t) - dR(t) + \int_0^t \gamma I(\xi)e^{-d(t-\xi)}P(t-\xi)d\xi,
\end{aligned} \tag{1.1}$$

where $S(t), I(t)$ and $R(t)$ are the proportions of susceptible, infectious and recovered individuals at time t, respectively. $d > 0$ is the birth rate and death rate constant, per capita natural death rate of the population (Note that for simplicity, the authors only consider a closed community in which the birth rate and death rate constants are equal). $\beta > 0$ is the average number of effective contacts of an infectious individual per unit time. $\gamma > 0$ is the recovery rate constant assuming that the infective period is exponentially distributed. The term $e^{-d(t-\xi)}$ in the integral accounts for the death of infective individuals. $P(t)$ is the fraction of recovered individuals remaining in the recovered class t time units after recovery, which satisfies the following properties:

(H1) $P : [0,\infty) \to [0,1]$ is differentiable except at possibly finitely many points where it may have jump discontinuities, non-increasing and satisfies $P(0) = 1$, $\lim_{t\to\infty} P(t) = 0$ and $\int_0^\infty P(u)du$ is positive and finite.

In [14], by utilizing the theory of asymptotically autonomous system, van den Driessche and Zou studied the dynamic behavior of solutions of (1.1). Three particular forms for $P(t)$, such as negative exponential relapse distribution, compact support, and step function, are investigated.

If all individuals remain in the recovered class τ time units before relapsing, $P(t)$ is the step function given by

$$P(t) = \begin{cases} 1, & t \in [0,\tau], \\ 0, & t > \tau, \end{cases}$$

and system (1.1) becomes

$$\begin{aligned}
\dot{S}(t) &= d - dS(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta S(t)I(t) - (d+\gamma)I(t) + \gamma e^{-d\tau}I(t-\tau), \\
\dot{R}(t) &= \gamma I(t) - \gamma e^{-d\tau}I(t-\tau) - dR(t).
\end{aligned} \tag{1.2}$$

For this case, the endemic equilibrium of (1.1) is proved to be locally asymptotically stable if the basic reproductive number is greater than unity, and globally asymptotically stable if, in addition, the relapse time is short.

Note that it is implicitly assumed that the population are well mixed, and the spatial mobility of individuals has been ignored in model (1.2) as well as most other epidemic models. In reality, the environment in which an individual lives is often heterogeneous making it necessary to distinguish the locations. For this reason, the spatial effects cannot be neglected in studying the spread of epidemics. In fact, many investigators have introduced population movements into related equations for epidemiological modeling and simulations in efforts to understand the most basic features of spatially distributed interactions (see, e.g., Gan etc. [4], Kuperman & Wio [5], Maidan & Yang [6], Peng & Liu [9], Ruan & Xiao [10], Wang & Wang [16], Wang & Zhao [17], Xu & Ma [18], Yu etc. [19], Zhang & Xu [20]).

Motivated by the works of van den Driessche & Zou [14], Kuperman [5], Yu etc. [19], and Gan etc. [4], in this paper, we consider the effect of spatial diffusion and disease relapse on the dynamics of infectious disease. To this end, we are concerned with the following delayed reaction diffusion system:

$$\begin{aligned}\frac{\partial S(x,t)}{\partial t} &= d_1\frac{\partial^2 S(x,t)}{\partial x^2} + d - dS(x,t) - \beta S(x,t)I(x,t),\\ \frac{\partial I(x,t)}{\partial t} &= d_2\frac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)I(x,t) - (d+\gamma)I(x,t) + \gamma e^{-d\tau}I(x,t-\tau),\\ \frac{\partial R(x,t)}{\partial t} &= d_3\frac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t) - \gamma e^{-d\tau}I(x,t-\tau) - dR(x,t),\end{aligned} \tag{1.3}$$

where $S(x,t), I(x,t)$ and $R(x,t)$ represent the proportions of the susceptible, infectious and recovered individuals at time t and location x, respectively. The parameters d_1, d_2 and d_3 are the corresponding diffusion rates for the three populations, respectively.

Accompanied with (1.3), we take the initial condition

$$\begin{aligned}&S(x,t) = \rho_1(x,t) \geqslant 0, I(x,t) = \rho_2(x,t) \geqslant 0,\\ &R(x,t) = \rho_3(x,t) \geqslant 0, \rho_i(x,0) > 0,\end{aligned} \tag{1.4}$$

where $t \in [-\tau, 0], x \in \mathbb{R}, i = 1, 2, 3$. It is easy to see that the solution of the initial value problem (1.3) and (1.4) exists globally and remains nonnegative.

The organization of this paper is as follows. In the next section, we investigate the local stability of each of feasible steady states of system (1.3) by using the linearized method. In Section 3, by constructing a pair of upper-lower solutions, we use the cross iteration method and Schauder's fixed point theorem to prove the existence of traveling wave solutions to system (1.3). In Section 4, numerical simulations are presented that complement the theoretical results. We briefly summarize our results in Section 5.

2. Local stability of steady states

In this section, by analyzing the corresponding characteristic equations, we investigate the local stability of steady states to system (1.3) with the initial conditions (1.4). Note that in system (1.3), the last equation is decoupled from the first two equations and thus it is sufficient to consider the following subsystem

$$\begin{aligned}\frac{\partial S(x,t)}{\partial t} &= d_1\frac{\partial^2 S(x,t)}{\partial x^2} + d - dS(x,t) - \beta S(x,t)I(x,t),\\ \frac{\partial I(x,t)}{\partial t} &= d_2\frac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)I(x,t) - (d+\gamma)I(x,t) + \gamma e^{-d\tau}I(x,t-\tau).\end{aligned} \tag{2.1}$$

By means of the method of the next generation matrix (see, for example, van den Driessche & Watmough [13]), one obtains the basic reproduction number of system (2.1) as follows:

$$\mathscr{R}_0 = \frac{\beta}{d+\gamma-\gamma e^{-d\tau}},$$

which describes the average number of newly infected individuals at the beginning of the infectious process. It is easy to show that system (2.1) always has a disease-free uniform steady state $E_0(1,0)$. Furthermore, if $\mathscr{R}_0 > 1$, system (2.1) has a unique endemic steady state $E^*(S^*, I^*)$, where

$$S^* = \frac{d+\gamma-\gamma e^{-d\tau}}{\beta},\ I^* = \frac{d-dS^*}{\beta S^*}.$$

For any feasible uniform steady state (S^0, I^0), the linearized system of (2.1) at (S^0, I^0) is

$$\begin{aligned}\frac{\partial S(x,t)}{\partial t} =& d_1\frac{\partial^2 S(x,t)}{\partial x^2} - \beta I^0 S(x,t) - dS(x,t) - \beta S^0 I(x,t),\\ \frac{\partial I(x,t)}{\partial t} =& d_2\frac{\partial^2 I(x,t)}{\partial x^2} + \beta I^0 S(x,t) + \beta S^0 I(x,t) - (d+\gamma)I(x,t)\\ &+\gamma e^{-d\tau}I(x,t-\tau),\end{aligned} \tag{2.2}$$

which admits non-trivial solutions with the form

$$\begin{pmatrix} S(x,t)\\ I(x,t)\end{pmatrix} = \begin{pmatrix} c_1\\ c_2\end{pmatrix} e^{\lambda t + ikx}$$

if and only if

$$\begin{vmatrix} \lambda + d_1k^2 + d + \beta I^0 & \beta S^0\\ -\beta I^0 & \lambda + d_2k^2 - \beta S^0 + d + \gamma - \gamma e^{-d\tau}e^{-\lambda\tau}\end{vmatrix} = 0,$$

where λ is a complex and k is a real number.

Theorem 2.1. *If $\mathscr{R}_0 < 1$, E_0 is locally asymptotically stable. If $\mathscr{R}_0 > 1$, E_0 is unstable, and E^* is locally asymptotically stable.*

Proof. Letting $(S^0, I^0) = (1,0)$ in (2.2), it follows that

$$(\lambda + d_1k^2 + d)(\lambda + d_2k^2 + d + \gamma - \beta - \gamma e^{-d\tau}e^{-\lambda\tau}) = 0. \tag{2.3}$$

From the first factor of (2.3), we obtain that $\lambda = -d_1k^2 - d < 0$, and from the second factor of (2.3), we get

$$\lambda = -d_2k^2 - d - \gamma + \beta + \gamma e^{-d\tau}e^{-\lambda\tau}. \tag{2.4}$$

If $\mathscr{R}_0 < 1$, we claim that all roots of (2.4) satisfy $\mathrm{Re}\lambda < 0$. Otherwise, there exists a root λ_0 of (2.4) with $\mathrm{Re}\lambda_0 \geqslant 0$. Hence, from (2.4), we could deduce that

$$\mathrm{Re}\lambda_0 = -d_2k^2 - d - \gamma + \beta + \gamma e^{-d\tau}e^{-\tau\mathrm{Re}\lambda_0}\cos(\tau\mathrm{Im}\lambda_0) \leqslant (d+\gamma-\gamma e^{-d\tau})(\mathscr{R}_0 - 1) < 0,$$

a contradiction. Therefore, E_0 is linearly asymptotically stable if $\mathscr{R}_0 < 1$.

If $\mathscr{R}_0 > 1$, we claim that (2.4) has at least one positive real root. Let

$$g_1(\lambda, k) := \lambda + d_2k^2 + d + \gamma - \beta - \gamma e^{-d\tau}e^{-\lambda\tau}),$$

then for k small, we have

$$g_1(0,k) = d_2k^2 + (d+\gamma-\gamma e^{-d\tau})(1-R_0) < 0, \quad g_1(\infty, k) = \infty,$$

which indicates that $g_1(\lambda, k)$ has at least one positive real root, and accordingly, E_0 is linearly unstable.

Letting $(S^0, I^0) = (S^*, I^*)$ in (2.2), it follows that

$$\lambda = -\frac{\beta^2 S^* I^*}{\lambda + d_2 k^2 + \gamma e^{-d\tau}(1 - e^{-\lambda\tau})} - (d_1 k^2 + d + \beta I^*). \tag{2.5}$$

We claim that all roots of (2.5) have negative real parts. Otherwise, suppose that there exists a $(a_0 + i\omega_0, k_0)$ satisfying with $a_0 \geqslant 0$. Then from (2.5), we could deduce that

$$\begin{aligned} a_0 = &-\frac{\beta^2 S^* I^* [a_0 + d_2 k_0^2 + \gamma e^{-d\tau}(1 - e^{-a_0\tau}\cos(\omega_0\tau))]}{[a_0 + d_2 k_0^2 + \gamma e^{-d\tau}(1 - e^{-a_0\tau}\cos(\omega_0\tau))]^2 + [\gamma e^{-d\tau} e^{-a_0\tau}\sin(\omega_0\tau)) + \omega_0]^2} \\ &- (d_1 k^2 + d + \beta I^*) \\ <&0, \end{aligned}$$

a contradiction. Therefore, E^* is linearly asymptotically stable for $\mathscr{R}_0 > 1$. The proof is complete. $\square$

3. Existence of traveling waves

In this section, we investigate the existence of traveling wave solutions to system (1.3). The technique of the proofs is to use the Schauder's fixed point theorem, the method of upper-lower solutions and its associated cross iteration scheme. For simplicity, in this section, we assume that $d_1 = d_2 = D$.

Denoting $N = S + I$, then system (2.1) is equivalent to the following system

$$\begin{aligned} \frac{\partial N}{\partial t} =& D\frac{\partial^2 N}{\partial x^2} + d - dN(x,t) - \gamma I(x,t) + \gamma e^{-d\tau} I(x, t-\tau), \\ \frac{\partial I}{\partial t} =& D\frac{\partial^2 I}{\partial x^2} + \beta(N(x,t) - I(x,t))I(x,t) - (d+\gamma)I(x,t) \\ &+ \gamma e^{-d\tau} I(x, t-\tau). \end{aligned} \tag{3.1}$$

Let $\hat{N} = 1 - N$, then system (3.1) is transformed into (omitting the hat on N for simplicity)

$$\begin{aligned} \frac{\partial N}{\partial t} =& D\frac{\partial^2 N}{\partial x^2} + \gamma I(x,t) - dN(x,t) - \gamma e^{-d\tau} I(x, t-\tau), \\ \frac{\partial I}{\partial t} =& D\frac{\partial^2 I}{\partial x^2} + \beta(1 - N(x,t) - I(x,t))I(x,t) - (d+\gamma)I(x,t) \\ &+ \gamma e^{-d\tau} I(x, t-\tau). \end{aligned} \tag{3.2}$$

A traveling wave solution of (3.2) is a solution $(S(x,t), I(x,t))$ of the form $S(x,t) = \phi(x+ct), I(x,t) = \varphi(x+ct)$, where $\phi, \varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $c > 0$ is a constant corresponding to the wave speed. On substituting $S(x,t) = \phi(x+ct), I(x,t) = \varphi(x+ct)$ and denoting the traveling wave coordinate $x + ct$ still by t, we derive from (3.2) that

$$\begin{aligned} D\phi''(t) - c\phi'(t) + f_1(\phi, \varphi)(t) = 0, \\ D\varphi''(t) - c\varphi'(t) + f_2(\phi, \varphi)(t) = 0, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} f_1(\phi,\varphi)(t) &= f_1(\phi,\varphi(0),\varphi(-\tau))(t) := \gamma\varphi(t) - d\phi(t) - \gamma e^{-d\tau}\varphi(t-c\tau), \\ f_2(\phi,\varphi)(t) &= \beta(1-\phi(t)-\varphi(t))\varphi(t) - (d+\gamma)\varphi(t) + \gamma e^{-d\tau}\varphi(t-c\tau). \end{aligned} \tag{3.4}$$

Eq. (3.3) will be solved subject to the following boundary value conditions:

$$\lim_{t\to-\infty}(\phi(t),\varphi(t)) = (0,0), \lim_{t\to+\infty}(\phi(t),\varphi(t)) = (k_1,k_2) := (1-S^*-I^*, I^*). \tag{3.5}$$

We define the upper and lower solutions of system (3.3) as follows.

Definition 3.1. A pair of continuous functions $\overline{\Phi} = (\overline{\phi},\overline{\varphi})$ and $\underline{\Phi} = (\underline{\phi},\underline{\varphi})$ are called a pair of upper-lower solutions of (3.3), if there exists a set $\mathcal{S} = \{T_i \in \mathbb{R}, i = 1,2,\cdots,n\}$ with finite points such that, $\overline{\Phi}'$ and $\underline{\Phi}'$ are twice continuously differentiable on $\mathbb{R}\setminus\mathcal{S}$ and satisfy

$$\begin{aligned} &D\overline{\phi}''(t) - c\overline{\phi}'(t) + f_1(\overline{\phi},\overline{\varphi}(0),\underline{\varphi}(-\tau))(t) \leqslant 0, \\ &D\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_2(\underline{\phi},\overline{\varphi})(t) \leqslant 0, \end{aligned}$$

and

$$\begin{aligned} &D\underline{\phi}''(t) - c\underline{\phi}'(t) + f_1(\underline{\phi},\underline{\varphi}(0),\overline{\varphi}(-\tau))(t) \geqslant 0, \\ &D\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_2(\overline{\phi},\underline{\varphi})(t) \geqslant 0 \end{aligned}$$

for $t \in \mathbb{R}\setminus\mathcal{S}$.

Lemma 3.1. *There exist constants $\beta_1,\beta_2 > 0$ such that*

$$\begin{aligned} &f_1(\phi_1,\varphi_1(0),\varphi_2(-\tau)) - f_1(\phi_2,\varphi_2(0),\varphi_1(-\tau)) + \beta_1[\phi_1(0)-\phi_2(0)] \geqslant 0, \\ &f_2(\phi_1,\varphi_1) - f_2(\phi_1,\varphi_2) + \beta_2[\varphi_1(0)-\varphi_2(0)] \geqslant 0, \end{aligned}$$

where $\phi_i,\varphi_i \in C([-\tau,0],R), i = 1,2$ with $(0,0) \leqslant (\phi_2,\varphi_2) \leqslant (\phi_1,\varphi_1) \leqslant (M_1,M_2)$, $M_j > k_j (j = 1,2)$ are positive constants.

Proof. It is not difficult to verify that

$$\begin{aligned} f_1(\phi_1,\varphi_1(0),\varphi_2(-\tau)) - f_1(\phi_2,\varphi_2(0),\varphi_1(-\tau)) =& \gamma\varphi_1(0) - d\phi_1(0) + d\phi_2(0) - \gamma\varphi_2(0) \\ &+ \gamma e^{-d\tau}(\varphi_2(-\tau)-\varphi_1(-\tau)) \\ \geqslant& -d(\phi_1(0)-\phi_2(0)), \\ f_2(\phi_1,\varphi_1) - f_2(\phi_1,\varphi_2) =& \beta(1-\phi_2(0)-\varphi_1(0))\varphi_1(-\tau) + \gamma e^{-d\tau}\varphi_1(-\tau) \\ &- \beta(1-\phi_1(0)-\varphi_2(0))\varphi_2(0) - \gamma e^{-d\tau}\varphi_2(-\tau) \\ &- (d+\gamma)(\varphi_1(0)-\varphi_2(0)) \\ \geqslant& -(\beta\varphi_1(0)+d+\gamma)(\varphi_1(0)-\varphi_2(0)). \end{aligned}$$

Let $\beta_1 = d$ and $\beta_2 = \beta M_2 + d + \gamma$, then the proof is complete. □

Define a set of functions

$$\mathcal{C}_{[0,M]}(\mathbb{R},\mathbb{R}^2) := \{(\phi,\varphi) \in \mathcal{C}(\mathbb{R},\mathbb{R}^2) | 0 \leqslant \phi(s) \leqslant M_2, 0 \leqslant \varphi(s) \leqslant M_3 \text{ for } s \in \mathbb{R}\},$$

and two operators $H = (H_1,H_2)$ and $F = (F_1,F_2)$ from $\mathcal{C}_{[0,M]}(\mathbb{R},\mathbb{R}^2)$ to $(\mathbb{R},\mathbb{R}^2)$ by

$$H_1(\phi,\varphi)(t) = f_1(\phi,\varphi)(t) + \beta_1\phi(t), \quad H_2(\phi,\varphi)(t) = f_2(\phi,\varphi)(t) + \beta_2\varphi(t),$$

$$F_i(\phi,\varphi)(t)$$
$$=\frac{1}{D(\lambda_{i2}-\lambda_{i1})}\left[\int_{-\infty}^{t} e^{\lambda_{i1}(t-s)}H_i(\phi,\varphi)(s)\mathrm{d}s+\int_{t}^{\infty} e^{\lambda_{i2}(t-s)}H_i(\phi,\varphi)(s)\mathrm{d}s\right],$$

where

$$\lambda_{i1}=\frac{c-\sqrt{c^2+4\beta_i D}}{2D},\quad \lambda_{i2}=\frac{c+\sqrt{c^2+4\beta_i D}}{2D},\quad i=1,2.$$

For $\mu\in(0,\min\{-\lambda_{11},\lambda_{12},-\lambda_{21},\lambda_{22}\})$, define $B_\mu(\mathbb{R},\mathbb{R}^2)=\{\Phi\in\mathcal{C}(\mathbb{R},\mathbb{R}^2):|\Phi|_\mu<\infty\}$, where $|\Phi|_\mu=\sup_{t\in\mathbb{R}}e^{-\mu|t|}|\Phi(t)|_{\mathbb{R}^2}$. Then $(B_\mu(\mathbb{R},\mathbb{R}^2),|\cdot|_\mu)$ is a Banach space.

By Lemma 3.1 and the definitions of H and F, the following properties of H and F are obvious.

Lemma 3.2. *For $\beta_1\geqslant d$ and $\beta_2\geqslant\beta M_2+d+\gamma$, we have*

$$\begin{aligned}&H_1(\phi_2,\varphi_2(0),\varphi_1(-\tau))(t)\leqslant H_1(\phi_1,\varphi_1(0),\varphi_2(-\tau))(t),\\&F_1(\phi_2,\varphi_2(0),\varphi_1(-\tau))(t)\leqslant F_1(\phi_1,\varphi_1(0),\varphi_2(-\tau))(t),\\&H_2(\phi_1,\varphi_2)(t)\leqslant H_2(\phi_2,\varphi_1)(t),\quad F_2(\phi_1,\varphi_2)(t)\leqslant F_2(\phi_2,\varphi_1)(t)\end{aligned}$$

for $\phi_i,\varphi_i\in C([-\tau,0],R), i=1,2$ with $(0,0)\leqslant(\phi_2,\varphi_2)(t)\leqslant(\phi_1,\varphi_1)(t)\leqslant(M_1,M_2)$.

Lemma 3.3. *F is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$.*

Proof. Letting $\Phi=(\phi_1,\varphi_1),\Psi=(\phi_2,\varphi_2)$, we have

$$\begin{aligned}&|H_1[\Phi](t)-H_1[\Psi](t)|e^{-\mu|t|}\\\leqslant&|f_1(\phi_1,\varphi_1)(t)-f_1(\phi_2,\varphi_2)(t)|e^{-\mu|t|}+\beta_1|\phi_1(t)-\phi_2(t)|e^{-\mu|t|}\\\leqslant&(\gamma|\varphi_1(t)-\varphi_2(t)|+d|\phi_1(t)-\phi_2(t)|+\gamma e^{-d\tau}|\varphi_1(t-c\tau)-\varphi_2(t-c\tau)|)e^{-\mu|t|}\\&+\beta_1|\Phi-\Psi|_\mu\\\leqslant&B_1|\Phi-\Psi|_\mu,\end{aligned}$$

where $B_1:=\gamma+d+\gamma e^{-d\tau}e^{c\mu\tau}+\beta_1$. Then for $t>0$, we obtain

$$\begin{aligned}&|F_1[\Phi](t)-F_1[\Psi](t)|e^{-\mu|t|}\\\leqslant&\frac{e^{-\mu t}}{D(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right]|H_1(\phi_1,\varphi_1)(s)-H_1(\phi_2,\varphi_2)(s)|ds\\\leqslant&\frac{B_1}{D(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}e^{(\lambda_{11}-\mu)t}+\frac{\lambda_{12}-\lambda_{11}}{(\mu-\lambda_{11})(\lambda_{12}-\mu)}\right]|\Phi-\Psi|_\mu\\\leqslant&\frac{B_1}{D(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}+\frac{\lambda_{12}-\lambda_{11}}{(\mu-\lambda_{11})(\lambda_{12}-\mu)}\right]|\Phi-\Psi|_\mu.\end{aligned}$$

Similarly, for $t<0$, we have

$$\begin{aligned}&|F_1[\Phi](t)-F_1[\Psi](t)|e^{-\mu|t|}\\\leqslant&\frac{B_1}{D(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}-\frac{\lambda_{12}-\lambda_{11}}{(\mu+\lambda_{11})(\lambda_{12}+\mu)}\right]|\Phi-\Psi|_\mu,\end{aligned}$$

which implies that F_1 is continuous. By a similar argument as above, we can also prove that F_2 is continuous. Then the proof is complete. □

Define a profile set Γ as follows:

$$\Gamma((\underline{\phi},\underline{\varphi}),(\overline{\phi},\overline{\varphi}))=\{(\phi,\varphi)\in C(\mathbb{R},\mathbb{R}^2)|(\underline{\phi},\underline{\varphi})(t)\leqslant(\phi,\varphi)(t)\leqslant(\overline{\phi},\overline{\varphi})(t)\ \text{ for } t\in\mathbb{R}\}.$$

Then we have the following results.

Lemma 3.4. $F(\Gamma)\subset\Gamma$, *if the following assumption holds:*

(C1) $\overline{\Phi}'(t+)\leqslant\overline{\Phi}'(t-),\ \underline{\Phi}'(t+)\geqslant\underline{\Phi}'(t-),\ t\in\mathbb{R}.$

Proof. From Lemma 3.2, we have that

$$\begin{aligned}&F_1(\underline{\phi},\underline{\varphi}(0),\overline{\varphi}(-\tau))(t)\leqslant F_1(\phi,\varphi(0),\varphi(-\tau))(t)\leqslant F_1(\overline{\phi},\overline{\varphi}(0),\underline{\varphi}(-\tau))(t),\\&F_2(\overline{\phi},\underline{\varphi})(t)\leqslant F_2(\phi,\varphi)(t)\leqslant F_2(\underline{\phi},\overline{\varphi})(t)\end{aligned}$$

for any $(\phi,\varphi)\in\Gamma$. Then we only need to prove that

$$\begin{aligned}&F_1(\underline{\phi},\underline{\varphi}(0),\overline{\varphi}(-\tau))(t)\geqslant\underline{\phi}(t),\quad F_1(\overline{\phi},\overline{\varphi}(0),\underline{\varphi}(-\tau))(t)\leqslant\overline{\phi}(t),\\&F_2(\overline{\phi},\underline{\varphi})(t)\geqslant\underline{\varphi}(t),\qquad\qquad\quad\ \ F_2(\underline{\phi},\overline{\varphi})(t)\leqslant\overline{\varphi}(t).\end{aligned}\tag{3.6}$$

Noting that for any $t\in\mathbb{R}\setminus\mathcal{S}$, there exists a $T_i\in\mathcal{S}$ such that $t\in(T_{i-1},T_i)$, then by the definitions of F and upper-lower solutions, we have

$$\begin{aligned}&F_1(\overline{\phi},\overline{\varphi}(0),\underline{\varphi}(-\tau))(t)\\=&\frac{1}{D(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right]H_1(\overline{\phi},\overline{\varphi}(0),\underline{\varphi}(-\tau))(s)ds\\\leqslant&\frac{1}{D(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right](\beta_1\overline{\phi}+c\overline{\phi}'-D\overline{\phi}'')(s)ds\\=&\overline{\phi}(t)+\frac{1}{\lambda_{12}-\lambda_{11}}\left[\sum_{j=1}^{i-1}\overline{\phi}'(s)|_{T_j-}^{T_j+}+\sum_{j=i}^{k}\overline{\phi}'(s)|_{T_j-}^{T_j+}\right]\\\leqslant&\overline{\phi}(t),\end{aligned}$$

if (C1) holds, where $T_{i-1}<T_i, i=1,2,\cdots,k+1$ and $T_0=-\infty, T_{k+1}=\infty$. In view of the continuity of F_1 and $\overline{\phi}(t)$, we have that $F_1(\underline{\phi},\underline{\varphi}(0),\overline{\varphi}(-\tau))(t)\geqslant\underline{\phi}(t)$ for all $t\in\mathbb{R}$.

By a similar argument, we can obtain that (3.6) holds for $t\in\mathbb{R}$. The proof is complete. □

Lemma 3.5. $F:\Gamma\to\Gamma$, *is compact.*

The proof of Lemma 3.5 is similar to that in [4], we therefore omit it here.

From Lemmas 3.1 and 3.5, we see that the existence of traveling wave solutions for system (2.1) follows from the existence of a pair of upper and lower solutions $(\overline{\phi},\overline{\varphi})$ and $(\underline{\phi},\underline{\varphi})$ of (3.3) satisfying (C1) and the following conditions:

(C2) $(0,0)\leqslant(\underline{\phi}(t),\underline{\varphi}(t))\leqslant(\overline{\phi}(t),\overline{\varphi}(t))\leqslant(M_1,M_2),\ t\in\mathbb{R}.$

(C3) $\lim_{t\to-\infty}(\underline{\phi}(t),\underline{\varphi}(t))=(0,0),\lim_{t\to+\infty}(\overline{\phi}(t),\overline{\varphi}(t))=(k_1,k_2).$

Now, to construct the upper-lower solutions, consider the following two functions:

$$\begin{aligned}\Delta_1(\lambda, c) :=& D\lambda^2 - c\lambda + \frac{\gamma l_2 k_2}{l_1 k_1} - d,\\ \Delta_2(\lambda, c) :=& D\lambda^2 - c\lambda + \beta - d - \gamma + \gamma e^{-d\tau} e^{-\lambda c\tau},\end{aligned}$$

where $l_1, l_2 > 0$ satisfy $\Delta_1(0, c) \geqslant \Delta_2(0, c)$.

Lemma 3.6. *There exists $c_i > 0$ $(i = 1, 2)$ such that the following conclusions hold.*

(i) For any given $c > c_i, \Delta_i(\lambda, c) = 0$ has two distinct positive roots $\lambda_{2i-1}(c) < \lambda_{2i}(c)$, $(i = 1, 2)$, and

$$\Delta_i(\lambda, c)\begin{cases} > 0, & 0 < \lambda < \lambda_{2i-1}(c),\\ < 0, & \lambda_{2i-1}(c) < \lambda < \lambda_{2i}(c),\\ > 0, & \lambda > \lambda_{2i}(c).\end{cases}$$

(ii) If $c < c_i$, then $\Delta_i(\lambda, c) = 0$ has no real roots $(i = 1, 2)$.

Proof. By direct calculations, we have

$$\begin{aligned}
&\Delta_2(0, c) = \beta - d - \gamma + \gamma e^{-d\tau} > 0, && \text{for all } c > 0,\\
&\Delta_2(\lambda, 0) = D\lambda^2 + \beta - d - \gamma + \gamma e^{-d\tau} > 0, && \text{for all } \lambda > 0,\\
&\frac{\partial \Delta_2(\lambda, c)}{\partial c} = -\lambda - \lambda\tau\gamma e^{-d\tau} e^{-\lambda c\tau} < 0, && \text{for all } \lambda > 0,\\
&\frac{\partial^2 \Delta_2(\lambda, c)}{\partial \lambda^2} = 2D + c^2\tau^2\gamma e^{-d\tau} e^{-\lambda c\tau} > 0, && \text{for all } \lambda > 0,\\
&\Delta_2(\infty, c) = \infty, && \text{for all } c > 0.
\end{aligned}$$

In view of the above observations on function $\Delta_2(\lambda, c)$, the conclusion is true for $i = 2$. Similarly, one can show the other conclusions. The proof is complete. $\square$

Denoting $c^* = \max\{c_1, c_2\}$, fixing $c > c^*$, and noting that $\lambda_3 < \lambda_1 < \lambda_2 < \lambda_4$ from $\Delta_1(0, c) \geqslant \Delta_2(0, c)$, we can fix η_1, η_2 such that

$$\eta_1 \in (1, \min\{\frac{\lambda_2}{\lambda_1}, \frac{2\lambda_3}{\lambda_1}\}), \quad \eta_2 \in (\frac{\eta_1\lambda_1}{\lambda_3}, \min\{\frac{\lambda_4}{\lambda_3}, 2\}). \tag{3.7}$$

Furthermore, we have a result as follows.

Lemma 3.7. *If $\mathscr{R}_0 > 1$, $d > \gamma + \gamma e^{-d\tau}$ hold, there exist $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$ such that*

$$\begin{aligned}&\gamma\varepsilon_2 - d\varepsilon_1 + \gamma e^{-d\tau}\varepsilon_4 < 0,\ \varepsilon_3 - \varepsilon_2 < 0,\\ &\gamma\varepsilon_4 - d\varepsilon_3 + \gamma e^{-d\tau}\varepsilon_2 < 0,\ \varepsilon_1 - \varepsilon_4 < 0.\end{aligned} \tag{3.8}$$

Proof. For $\varepsilon_2 > 0$, we can find $\varepsilon_3, \varepsilon_4, \varepsilon_1 > 0$ and $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\varepsilon_3 = \varepsilon_2 - \alpha_1,\ \gamma\varepsilon_4 = d\varepsilon_3 - \gamma e^{-d\tau}\varepsilon_2 - \alpha_2,\ d\varepsilon_1 = \gamma\varepsilon_2 + \gamma e^{-d\tau}\varepsilon_4 + \alpha_3. \tag{3.9}$$

Then by a direct calculation, the following equation could be obtained from (3.9),

$$\varepsilon_1 - \varepsilon_4 = \frac{\gamma^2 - (d - \gamma e^{-d\tau})^2}{\gamma d}\varepsilon_2 + \frac{(d - \gamma e^{-d\tau})(d\alpha_1 + \alpha_2) + \gamma\alpha_3}{\gamma d}. \tag{3.10}$$

Since $d > \gamma + \gamma e^{-d\tau}$ is equivalent to $d - \gamma e^{-d\tau} > 0$ and $\gamma^2 - (d - \gamma e^{-d\tau})^2 < 0$, then by choosing α_i $(i = 1, 2, 3)$ sufficiently small, we can obtain $\varepsilon_1 - \varepsilon_4 < 0$. The proof is complete. □

Define continuous functions $\overline{\Phi}(t) = (\phi_1(t), \varphi_1(t))$ and $\underline{\Phi}(t) = (\phi_2(t), \varphi_2(t))$ as follows:

$$\phi_1(t) = \begin{cases} l_1 k_1 e^{\lambda_1 t}, & t \leqslant t_1, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1, \end{cases} \qquad \varphi_1(t) = \begin{cases} l_2 k_2 e^{\lambda_4 t}, & t \leqslant t_2, \\ k_2 + \varepsilon_2 e^{-\lambda t}, & t > t_2, \end{cases}$$

$$\phi_2(t) = \begin{cases} l_1 k_2 (e^{\lambda_1 t} - M e^{\eta_1 \lambda_1 t}), & t \leqslant t_3, \\ k_1 - \varepsilon_3 e^{-\lambda t}, & t > t_3, \end{cases} \qquad \varphi_2(t) = \begin{cases} l_2 k_2 (e^{\lambda_3 t} - M e^{\eta_2 \lambda_3 t}), & t \leqslant t_4, \\ k_2 - \varepsilon_4 e^{-\lambda t}, & t > t_4, \end{cases}$$

where η_i $(i = 1, 2)$ is defined as in (3.7), ε_i $(i = 1, 2, 3, 4)$ is defined as in (3.8), $M > 1$ is a large enough constant and $\lambda > 0$ is a small constant. We can see that $\overline{\Phi}(t)$ and $\underline{\Phi}(t)$ satisfy the conditions (C1), (C2) and (C3).

Lemma 3.8. *Assume that $\mathscr{R}_0 > 1$, $d > \gamma + \gamma e^{-d\tau}$, and M is large enough. Then $(\phi_1(t), \varphi_1(t))$ and $(\phi_2(t), \varphi_2(t))$ are a pair of upper-lower solutions of* (3.3).

Proof. Denote

$$\begin{aligned} p_1(t) =& D\phi_1''(t) - c\phi_1'(t) + \gamma\varphi_1(t) - d\phi_1(t) - \gamma e^{-d\tau}\varphi_2(t - c\tau), \\ p_2(t) =& D\varphi_1''(t) - c\varphi_1'(t) + \beta(1 - \phi_2(t) - \varphi_1(t))\varphi_1(t) - (d + \gamma)\varphi_1(t) \\ &+ \gamma e^{-d\tau}\varphi_1(t - c\tau), \\ q_1(t) =& D\phi_2''(t) - c\phi_2'(t) + \gamma\varphi_2(t) - d\phi_2(t) - \gamma e^{-d\tau}\varphi_1(t - c\tau), \\ q_2(t) =& D\varphi_2''(t) - c\varphi_2'(t) + \beta(1 - \phi_1(t) - \varphi_2(t))\varphi_2(t) - (d + \gamma)\varphi_2(t) \\ &+ \gamma e^{-d\tau}\varphi_2(t - c\tau). \end{aligned}$$

If $t \leqslant t_1$, $\phi_1(t) = l_1 k_1 (e^{\lambda_1 t}, \varphi_1(t) \leqslant l_2 k_2 e^{\lambda_4 t}$, then

$$\begin{aligned} p_1(t) &\leqslant l_1 k_1 e^{\lambda_1 t}[D\lambda_1^2 - c\lambda_1 + \gamma l_2 k_2 e^{(\lambda_4 - \lambda_1)t}/(l_1 k_1) - d] \\ &< k_1 e^{\lambda_1 t}\Delta_1(\lambda_1, c) = 0. \end{aligned}$$

If $t > t_1$, $\phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}, \varphi_1(t) \leqslant k_2 + \varepsilon_2 e^{-\lambda t}$. It follows that

$$p_1(t) \leqslant e^{-\lambda t}(D\varepsilon_1\lambda^2 + c\varepsilon_1\lambda + \gamma\varepsilon_2 - d\varepsilon_1 + \gamma\varepsilon_4 e^{-d\tau}e^{\lambda c\tau}) =: P_1(\lambda).$$

Noting that $P_1(0) = \gamma\varepsilon_2 - d\varepsilon_1 + \gamma\varepsilon_4 e^{-d\tau} < 0$, there exists a $\lambda_1^* > 0$ such that $p_1(t) \leqslant P_1(\lambda) < 0$ for $\lambda \in (0, \lambda_1^*)$.

If $t \leqslant t_2$, $\varphi_1(t) = l_2 k_2 e^{\lambda_4 t}, \varphi_1(t - c\tau) = l_2 k_2 e^{\lambda_4 (t - c\tau)}$, then

$$\begin{aligned} p_2(t) &= l_2 k_2 e^{\lambda_4 t}[D\lambda_4^2 - c\lambda_4 + \beta(1 - \phi_2(t) - k_2 e^{\lambda_4 t}) - d - \gamma + \gamma e^{-d\tau}e^{-\lambda_4 c\tau}] \\ &< l_2 k_2 e^{\lambda_4 t}\Delta_2(\lambda_4, c) = 0. \end{aligned}$$

If $t > t_2$, $\phi_2(t) \geqslant k_1 - \varepsilon_3 e^{-\lambda t}, \varphi_1(t) = k_2 + \varepsilon_2 e^{-\lambda t}, \varphi_1(t - c\tau) = k_2 + \varepsilon_2 e^{-\lambda(t - c\tau)}$, we obtain that

$$p_2(t) \leqslant \varepsilon_2 e^{-\lambda t}[D\lambda^2 + c\lambda + \beta k_2(\varepsilon_3 - \varepsilon_2)/\varepsilon_2 + \gamma e^{-d\tau}(e^{\lambda c\tau} - 1)] := P_2(\lambda).$$

Since $P_2(0) = \beta k_2(\varepsilon_3 - \varepsilon_2) < 0$, there exists a $\lambda_2^* > 0$ such that $p_2(t) < 0$ for all $\lambda \in (0, \lambda_2^*)$.

If $t \leqslant t_3$, $\phi_2(t) = l_1k_1(e^{\lambda_1 t} - Me^{\eta_1\lambda_1 t}), \varphi_2(t) \geqslant l_2k_2(e^{\lambda_3 t} - Me^{\eta_2\lambda_3 t}), \varphi_1(t) \leqslant l_2k_2e^{\lambda_4(t-c\tau)}$, then from (3.7) we get

$$\begin{aligned} q_1(t) \geqslant & l_1k_1e^{\lambda_1 t}[D\lambda_1^2 - c\lambda_1 + \gamma l_2k_2e^{(\lambda_3-\lambda_1)t}/(l_1k_1) - d] - l_2k_2\gamma e^{-d\tau}e^{\lambda_4(t-c\tau)} \\ & - l_1Mk_1e^{\eta_1\lambda_1 t}[D\eta_1^2\lambda_1^2 - c\eta_1\lambda_1 + \gamma l_2k_2e^{(\eta_2\lambda_3-\eta_1\lambda_1)t}/(l_1k_1) - d] \\ \geqslant & - Ml_1k_1e^{\eta_1\lambda_1 t}\Delta_1(\eta_1\lambda_1, c) - l_2k_2\gamma e^{-d\tau} > 0, \end{aligned}$$

for M large enough.

If $t > t_3$, $\phi_2(t) = k_1 - \varepsilon_3e^{-\lambda t}, \varphi_2(t) \geqslant k_2 - \varepsilon_4e^{-\lambda t}, \varphi_1(t-c\tau) \leqslant k_2 + \varepsilon_2e^{-\lambda t}$, then

$$q_1(t) \geqslant -\varepsilon_3e^{-\lambda t}(D\lambda^2 + c\lambda + \gamma\varepsilon_4/\varepsilon_3 - d + \gamma e^{-d\tau}\varepsilon_2/\varepsilon_3) := P_3(\lambda).$$

Since $P_3(0) = d\varepsilon_3 - \gamma\varepsilon_4 - \gamma e^{-d\tau}\varepsilon_2 > 0$, there exists a $\lambda_3^* > 0$ such that $q_1(t) > 0$ for all $\lambda \in (0, \lambda_3^*)$.

If $t \leqslant t_4$, $\varphi_2(t) = l_2k_2(e^{\lambda_3 t} - Me^{\eta_2\lambda_3 t}), \varphi_2(t-c\tau) = l_2k_2(e^{\lambda_3(t-c\tau)} - Me^{\eta_2\lambda_3(t-c\tau)})$, $\phi_1(t) \leqslant l_1k_1e^{\lambda_1 t}$, then from (3.7) we get

$$\begin{aligned} q_2(t) \geqslant & l_2k_2e^{\lambda_3 t}\Delta_2(\lambda_3, c) - l_2k_2\beta e^{\lambda_3 t}(l_1k_1e^{\lambda_1 t} + l_2k_2e^{\lambda_3 t}) - l_2Mk_2e^{\eta_2\lambda_3 t}\Delta_2(\eta_2\lambda_3, c) \\ > & 0 \end{aligned}$$

for M large enough.

If $t > t_4$, $\phi_1(t) \leqslant k_1 + \varepsilon_1e^{-\lambda t}, \varphi_2(t) = k_2 - \varepsilon_4e^{-\lambda t}, \varphi_2(t-c\tau) \geqslant k_2 - \varepsilon_4e^{-\lambda(t-c\tau)}$. It then follows that

$$q_2(t) \geqslant - \varepsilon_4e^{-\lambda t}[D\lambda^2 + c\lambda + \beta(1 - k_2/\varepsilon_4)(\varepsilon_4 - \varepsilon_1) + \gamma e^{-d\tau}(e^{\lambda c\tau} - 1)] := P_4(\lambda).$$

Since $P_4(0) = \beta(k_2 - \varepsilon_4)(\varepsilon_4 - \varepsilon_1) < 0$, there exists a $\lambda_4^* > 0$ such that $q_2(t) > 0$ for all $\lambda \in (0, \lambda_4^*)$.

Taking $\lambda \in (0, \min\{\lambda_i^*, i = 1, 2, 3, 4\})$, we see that $(\phi_1(t), \varphi_1(t))$ and $(\phi_2(t), \varphi_2(t))$ are a pair of upper-lower solutions of (3.3). □

Combining Lemmas 3.1-3.8 with Schauder's fixed point theorem, we know that there exists a fixed point $(\phi^*(t), \varphi^*(t))$ of F in $\Gamma((\underline{\phi}, \underline{\varphi}), (\overline{\phi}, \overline{\varphi}))$, which gives a solution of (3.3). Furthermore, from (C3), we obtain that

$$\lim_{t\to-\infty}(\phi^*(t), \varphi^*(t)) = (0, 0), \quad \lim_{t\to+\infty}(\phi^*(t), \varphi^*(t)) = (k_1, k_2),$$

which indicates that the fixed point satisfies the asymptotic boundary conditions (3.5). Therefore, there exists a traveling wave solution for system (3.2) connecting the steady state $(0, 0)$ and (k_1, k_2). Accordingly, we have the following result.

Theorem 3.1. *Suppose $d_1 = d_2, \mathscr{R}_0 > 1$, and $d > \gamma + \gamma e^{-d\tau}$, then for every $c > c^*$, system* (1.3) *has a traveling wave solution with speed c connecting the disease-free steady state E_0 and the endemic steady state E^*.*

4. Numerical simulations

In this section, focusing on the traveling wave solutions of system (2.1), we perform some numerical simulations.

In system (1.3), we fix $d = 0.1,\ \beta = 0.6,\ \gamma = 0.5,\ d_1 = d_2 = d_3 = 0.01$ and $\tau = 0.75$, then system (1.3) with above coefficients has a disease-free steady state

$E_0(1,0,0)$ and an endemic steady state $E^*(0.2269, 0.5679, 0.2052)$. For convenience, we truncate the spatial domain $\mathbb{R}$ by $[-10, 10]$. By calculation, we obtain the basic reproductive number $\mathscr{R}_0 = 4.4076 > 1$. To illustrate the existence of traveling wave solutions, we choose initial conditions:

$$(S(x,t), I(x,t)) = \begin{cases} E_0, & -10 \leqslant x < 0, \quad -\tau \leqslant t \leqslant 0, \\ E^*, & 0 \leqslant x \leqslant 10, \quad -\tau \leqslant t \leqslant 0. \end{cases} \tag{4.1}$$

The numerical simulations shown in Figure 1 indicate that system (1.3) has a traveling wave solution connecting E_0 and E^*.

Now we address the effects of parameters on the dynamics of system (1.3). We first investigate the impact of the diffusion rates. To this end, we fix $d = 0.1$, $\beta = 0.6$, $\gamma = 0.5$, $\tau = 0.75$, and let the diffusivity vary. By contrasting Fig. 1 and Fig. 2, we can see that the traveling wave in Fig. 2(b) is much fast than that in Fig. 1, but the wave speeds in Fig. 2(a) and (c) are just the same as that in Fig. 1, which indicates that the diffusion rate of infective can advance the time to arrive at the infection steady state, but the diffusion rates of susceptible individuals and recovered individuals can not. Even so, there is an obvious difference between the wave profiles in Fig. 2(a) and the others: the former has a hump in wave profile for I, i.e., the diffusion rate of susceptible individuals could lead to non-monotone traveling waves. In biological meaning, large diffusion of susceptible individuals may enhance the accumulative effect of infection.

Changing the parameter from $\tau = 0.75$ to 2, we find monotone traveling front profiles, lower infection load and slower wave speed for system (1.3) in Fig. 3(a), by contrasting with Fig. 1. In biological meaning, long average relapse period could lead to low and slow infection, but it can enhance the accumulative effect of infection.

To investigate the impact of transmission coefficient, we increase change the parameter from $\beta = 0.6$ to 0.8. By contrasting Fig. 1 with Fig. 3(b), we see that large transmission coefficient could lead to fast infection.

5. Conclusions

In this paper, we formulated an epidemic model with relapse, time delay and spatial diffusion. The dynamics of problem (1.3)-(1.4) was addressed. It was shown that the basic reproductive number of system (1.3) is given by $\mathscr{R}_0 = \beta/(d + \gamma - \gamma e^{-d\tau})$, which describes the average number of newly infected individuals at the beginning of the infectious process. We have shown when the average number of newly infected individuals is less than unity, i.e., $\mathscr{R}_0 < 1$, system (1.3) has a unique disease-free steady state $E_0(1,0,0)$, which is asymptotically stable; when the average number of newly infected individuals is greater than unity, i.e., $\mathscr{R}_0 > 1$, system (1.3) admits two steady states, E_0 and $E^*(S^*, I^*, R^*)$. In this case, the endemic steady state E^* is always stable, while the disease-free steady state E_0 is unstable. Clearly, the spatial diffusion cannot destabilize the spatially homogenous steady state. Then by using the technique of upper and lower solutions and Schauder's fixed point theorem, we derived the existence of a traveling wave solution connecting the disease-free steady state and the endemic steady state. Furthermore, we gave some numeric simulations to illustrate the main results, and combining with numeric simulations, we discussed the effects of some parameters on the dynamics of system

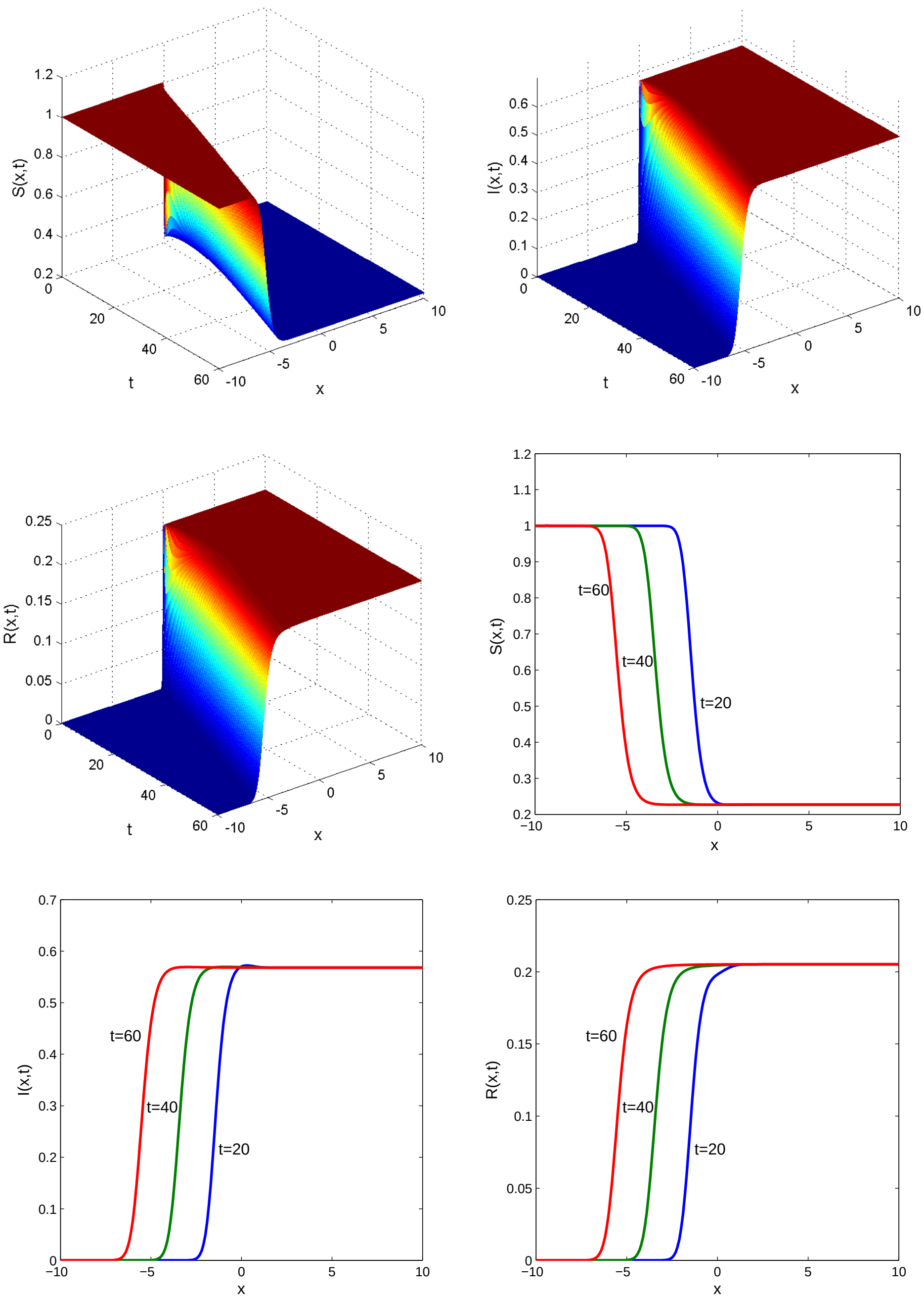

Figure 1. The traveling wave observed in system (1.3) with parameters:$d = 0.1, \beta = 0.6, \gamma = 0.5, d_1 = d_2 = d_3 = 0.01, \tau = 0.75$, and initial conditions (4.1).

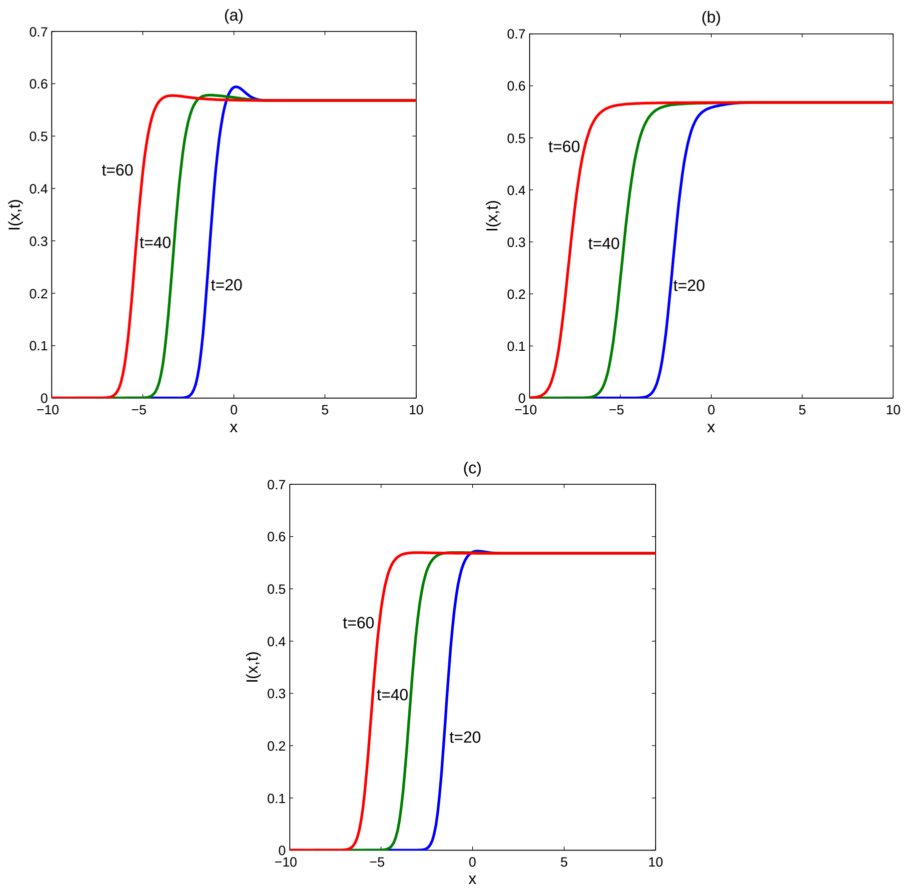

Figure 2. The traveling wave observed in system (1.3) with parameters: $d = 0.1, \beta = 0.6, \gamma = 0.5, \tau = 0.75, d_1 = 0.02, d_2 = d_3 = 0.01$ in (a); $d_1 = d_3 = 0.01, d_2 = 0.02$ in (b) and $d_1 = d_2 = 0.01, d_3 = 0.02$ in (c).

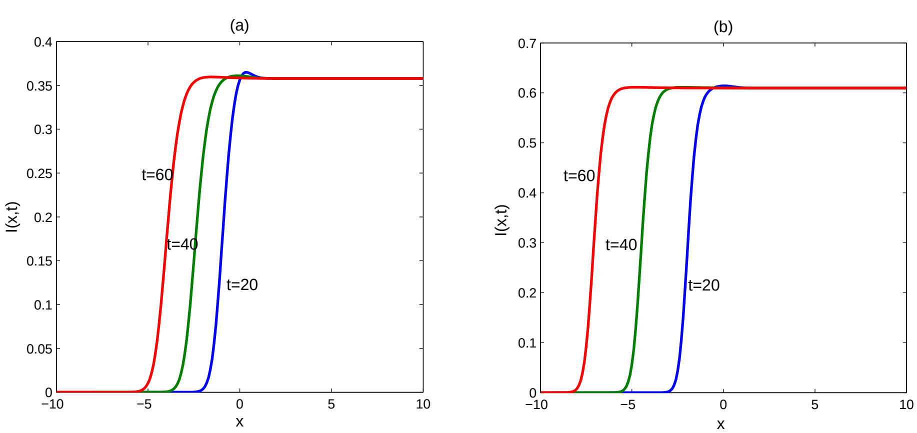

Figure 3. The traveling wave observed in system (1.3) with parameters: $d = 0.1, \gamma = 0.5, d_1 = d_2 = d_3 = 0.01, \beta = 0.6, \tau = 2$ in (a); $\beta = 0.8, \tau = 0.75$ in (b).

(1.3). From the discussion in Section 4, we could conclude that the traveling wave speed is increasing in the diffusion rates of infective individuals and the transmission coefficient, but decreasing in the average relapse period of the disease. In addition, large diffusion rate of susceptible individuals or long average relapse period may lead to non-monotone traveling waves.

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