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ELLIPTIC RECONSTRUCTION AND A POSTERIORI ERROR ESTIMATES FOR PARABOLIC OPTIMAL CONTROL PROBLEMS*

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Abstract In this article, a semidiscrete finite element method for parabolic optimal control problems is investigate. By using elliptic reconstruction, a posteriori error estimates for finite element discretizations of optimal control problem governed by parabolic equations with integral constraints are derived.

Keywords A posteriori error estimates, elliptic reconstruction, finite element method, optimal control problems, parabolic equation.

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1. Introduction

There has been extensive research on the a posteriori error estimates of finite element methods for PDEs and optimal control problems, mostly focused on the elliptic case. A systematic introduction of a posteriori error estimates of finite element method for partial differential equations can be found in [1,3,4]. A posteriori error estimates of linear elliptic optimal control problems were established in [11,15], and for mixed finite element approximation of Stokes optimal control problems in [14]. Some results on a posteriori error estimates of mixed finite element methods applied to elliptic equations or optimal control problems have also been obtained in [5,7–9,20].

Parabolic optimal control problems are frequently met in the mathematical model for describing petroleum reservoir simulation, environmental modeling, groundwater contaminant transport, and many other applications. A priori and a posteriori error estimates of finite element methods for optimal control problems were established in [13] and [16,22,23], respectively. A priori estimates of space-time finite element discretization for parabolic control problems have obtained in [18,19], and a characteristic finite element approximation for optimal control problems governed by transient advection-diffusion equations were also investigated in [10]. Recently, an optimal control system governed by hyperbolic equations with strong nonlinearity is considered in [21]. To the best of our knowledge there has been little work done on the a posteriori estimates of finite element methods for parabolic control

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problems. The purpose of this work is to investigate a posteriori error estimates of semidiscrete finite element method for parabolic equations by using elliptic reconstruction.

We are interested in the following parabolic optimal control problems:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\|y-y_d\|^2+\|u\|^2\right) dt, & \\ y_t-\operatorname{div}(A\nabla y)=f+u, & x\in\Omega,\ t\in J, \\ y|_{\partial\Omega}=0, & t\in J, \\ y(0)=y_0, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$, $J=[0,T]\,(T>0)$. The coefficient $A=(a_{ij}(x))_{2\times 2}\in(W^{1,\infty}(\bar{\Omega}))^{2\times 2}$ is an uniformly symmetric positive definite matrix, i.e., there exists a constant $c>0$ such that

$$(A\xi)\cdot\xi\ge c\mid\xi\mid^2,\qquad \forall\,\xi\in\mathbb{R}^2.$$

Moreover, we assume that $f,y_d\in C(J;L^2(\Omega))$, $y_0\in H_0^1(\Omega)$ and K is a nonempty closed convex subset in $L^2(J;L^2(\Omega))$, defined by

$$K=\{\ v|\,v\in L^2(J;L^2(\Omega))\,\text{and}\int_0^T\int_\Omega v\,dx\,dt\ge 0\ \}.$$

Here we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $H_0^1(\Omega)\equiv\left\{v\in H^1(\Omega):v|_{\partial\Omega}=0\right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. We denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the space $H^l(J;W^{m,q}(\Omega))$ and $C^k(J;W^{m,q}(\Omega))$ (cf. Ref. [13]). In addition, c or C denotes a generic positive constant.

The plan of this paper is as follows. In Section 2, we shall construct a semidiscrete finite element approximation for the model problem (1.1). In Section 3, we give some useful error estimates for intermediate variables. We derive a posteriori error estimates for finite element approximation of parabolic optimal control problems in Section 4. We give a conclusion and introduce our future works in the last Section.

2. A semidiscrete finite element discretization

A semidiscrete finite element approximation for the model problem (1.1) is now considered. For ease of exposition, we denote $L^p(J;W^{m,q}(\Omega))$ by $L^p(W^{m,q})$. Let $W=H_0^1(\Omega)$ and $U=L^2(\Omega)$. Moreover, we denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively. Let

$$a(v,w)=\int_\Omega(A\nabla v)\cdot\nabla w,\qquad\qquad \forall\,v,\,w\in W,$$

$$(f_1,f_2)=\int_\Omega f_1\cdot f_2,\qquad\qquad \forall\,f_1,\,f_2\in U.$$

It follows from the assumptions on A that

$$a(v,v) \geq c\|v\|_1^2, \quad |a(v,w)| \leq C\|v\|_1\|w\|_1, \qquad \forall\, v,w \in W.$$

Thus a possible weak formula for the model problem (1.1) reads:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \big(\|y-y_d\|^2 + \|u\|^2\big)\,dt, \\ (y_t,w) + a(y,w) = (f+u,w), & \forall\, w \in W,\ t \in J, \\ y(0) = y_0, & \forall\, x \in \Omega. \end{cases} \tag{2.1}$$

It is well known (e.g. see [12]) that the problem (2.1) has a unique solution (y,u), and the pair $(y,u) \in \big(H^1(L^2)\cap L^2(H^1)\big)\times K$ is the solution of the formulation (2.1) if and only if there is an adjoint state $p \in H^1(L^2)\cap L^2(H^1)$ such that the triplet (y,p,u) satisfies the following optimality conditions:

$$\begin{aligned} &(y_t,w) + a(y,w) = (f+u,w), && \forall\, w \in W,\ t \in J, \\ &y(0) = y_0, && \forall\, x \in \Omega, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &-(p_t,q) + a(q,p) = (y-y_d,q), && \forall\, q \in W,\ t \in J, \\ &p(T) = 0, && \forall\, x \in \Omega, \end{aligned} \tag{2.3}$$

$$\int_0^T (u+p, v-u)dt \geq 0, \qquad \forall\, v \in K. \tag{2.4}$$

Lemma 2.1. *Let (y,p,u) be the solution of (2.2)-(2.4). Then $u = \max(0,\overline{p}) - p$, where*

$$\overline{p} = \frac{\int_0^T \int_\Omega p\,dx\,dt}{\int_0^T \int_\Omega 1\,dx\,dt} \tag{2.5}$$

denotes the integral average on $\Omega \times J$ of the function p.

Proof. For any function $p \in H^1(L^2)$, we show that

$$u = \max(0,\overline{p}) - p$$

satisfies the variational inequality (2.4).

If $\overline{p} > 0$, then $u = \overline{p} - p$ and

$$\begin{aligned} \int_0^T (u+p, v-u)dt &= \int_0^T \int_\Omega (\overline{p} - p + p)(v - \overline{p} + p)dxdt \\ &= \overline{p}\int_0^T \int_\Omega v dxdt \geq 0, \qquad \forall\, v \in K. \end{aligned} \tag{2.6}$$

If $\overline{p} \leq 0$, then $u = -p$ and

$$\int_0^T (u+p, v-u)dt = 0, \qquad \forall\, v \in K. \tag{2.7}$$

Note that the solution of (2.3) is unique. Thus we complete the proof of the lemma. □

Now let $\mathcal{T}^h$ be regular triangulations of Ω such that $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$ and $h = \max_{\tau \in \mathcal{T}^h} \{h_\tau\}$, where h_τ denotes the diameter of the element τ. Moreover, we set

$$\begin{aligned} W_h =& \left\{ \ v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathbb{P}_l, \ \forall \tau \in \mathcal{T}^h, \ w_h|_{\partial\Omega} = 0 \ \right\}, \\ K_h =& L^2(W_h) \cap K, \end{aligned}$$

where $\mathbb{P}_l$ is the space of polynomials up to order l.

A semidiscrete finite element approximation of the weak formulation (2.1) is

$$\begin{cases} \min\limits_{u_h \in K_h} \dfrac{1}{2} \displaystyle\int_0^T \left(\|y_h - y_d\|^2 + \|u_h\|^2 \right) dt, \\ (y_{h,t}, w_h) + a\left(y_h, w_h\right) = \left(f + u_h, w_h\right), \quad \forall \ w_h \in W_h, \, t \in J, \\ y_h(0) = y_0^h, \qquad \forall \, x \in \Omega, \end{cases} \tag{2.8}$$

where y_0^h is an appropriate approximation of y_0.

It follows (e.g. see [16]) that the control problem (2.8) has a unique solution (y_h, u_h), and $(y_h, u_h) \in H^1(W_h) \times K_h$ is the solution of (2.8) if and only if there is a adjoint state $p_h \in H^1(W_h)$ such that the triplet (y_h, p_h, u_h) satisfies the optimality conditions

$$\begin{aligned} &(y_{h,t}, w_h) + a(y_h, w_h) = (f + u_h, w_h), && \forall \, w_h \in W_h, \, t \in J, \\ &y_h(0) = y_0^h, && \forall \, x \in \Omega, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &-(p_{h,t}, q_h) + a(q_h, p_h) = (y_h - y_d, q_h), && \forall \, q_h \in W_h, \, t \in J, \\ &p_h(T) = 0, && \forall \, x \in \Omega, \end{aligned} \tag{2.10}$$

$$\int_0^T (u_h + p_h, v - u_h) dt \geq 0, \qquad \forall \, v \in K_h. \tag{2.11}$$

Similar to Lemma 2.1, we can derive the following relationship between u_h and p_h.

Lemma 2.2. *Let (y_h, p_h, u_h) be the solution of (2.9)-(2.11). Then we have $u_h = \max(0, \overline{p_h}) - p_h$, where*

$$\overline{p_h} = \frac{\int_0^T \int_\Omega p_h \, dx \, dt}{\int_0^T \int_\Omega 1 \, dx \, dt} \tag{2.12}$$

denotes the integral average on $\Omega \times J$ of the function p_h.

Proof. For any function $p_h \in H^1(W_h)$, we show that

$$u_h = \max(0, \overline{p_h}) - p_h$$

satisfies the variational inequality (2.11).

If $\overline{p_h} > 0$, then $u_h = \overline{p_h} - p_h$ and

$$\begin{aligned} \int_0^T (u_h + p_h, v - u_h) dt &= \int_0^T \int_\Omega (\overline{p_h} - p_h + p_h)(v - \overline{p_h} + p_h) dx dt \\ &= \overline{p_h} \int_0^T \int_\Omega v dx dt \geq 0, \qquad \forall \, v \in K_h. \end{aligned} \tag{2.13}$$

If $\overline{p_h} \le 0$, then $u_h = -p_h$ and

$$\int_0^T (u_h + p_h, v - u_h) dt = 0, \qquad \forall\, v \in K_h. \tag{2.14}$$

Note that the solution of (2.10) is unique. Thus we complete the proof of the lemma. □

3. Error estimates of intermediate variables

We now give some error estimates of intermediate variables. For any control function $u_h \in K_h$, let $(y(u_h), p(u_h)) \in H^1(H_0^1) \times H^1(H_0^1)$ be the solution of the following equations:

$$\begin{aligned} &(y_t(u_h), w) + a\,(y(u_h), w) = (f + u_h, w)\,, && \forall\, w \in W,\ t \in J,\\ &y(u_h)(0) = y_0, && \forall\, x \in \Omega, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &-(p_t(u_h), q) + a\,(q, p(u_h)) = (y(u_h) - y_d, q)\,, && \forall\, q \in W,\ t \in J,\\ &p(u_h)(T) = 0, && \forall\, x \in \Omega. \end{aligned} \tag{3.2}$$

We define the errors as follows:

$$e_y = y(u_h) - y_h,$$

and

$$e_p = p(u_h) - p_h.$$

Then, from (2.9)-(2.10) and (3.1)-(3.2), the above errors satisfy the following equations

$$(e_{y,t}, w) + a(e_y, w) = -r_1(w), \qquad \forall\, w \in W, \tag{3.3}$$

$$-(e_{p,t}, q) + a(q, e_p) = (e_y, q) - r_2(q), \qquad \forall\, q \in W, \tag{3.4}$$

where

$$\begin{aligned} r_1(w) &= (y_{h,t}, w) + a(y_h, w) - (f + u_h, w),\\ r_2(q) &= -(p_{h,t}, q) + a(q, p_h) - (y_h - y_d, q). \end{aligned}$$

We now introduce elliptic reconstructions $\tilde{y}(t), \tilde{p}(t) \in H_0^1(\Omega)$ of y_h, p_h for $t \in J$, respectively. For given y_h, p_h, let $\tilde{y}(t), \tilde{p}(t) \in H_0^1(\Omega)$ satisfy

$$a(\tilde{y} - y_h, w) = -r_1(w), \qquad \forall\, w \in W, \tag{3.5}$$

$$a(q, \tilde{p} - p_h) = (\tilde{y} - y_h, q) - r_2(q), \qquad \forall\, q \in W. \tag{3.6}$$

Since for any $w_h, q_h \in W_h$, $r_1(w_h) = 0$ and $r_2(q_h) = 0$, let us note that y_h and p_h are elliptic projection of $\tilde{y}$ and $\tilde{p}$, respectively. By using elliptic reconstructions, we rewrite:

$$e_y = (\tilde{y} - y_h) - (\tilde{y} - y(u_h)) := \eta_y - \xi_y,$$

and

$$e_p = (\tilde{p} - p_h) - (\tilde{p} - p(u_h)) := \eta_p - \xi_p.$$

Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (2.2)-(2.4) and (2.9)-(2.11), respectively. We decompose the errors as follows:

$$y-y_h=(y-y(u_h))+(y(u_h)-y_h):=r_y-e_y,$$

and

$$p-p_h=(p-p(u_h))+(p(u_h)-p_h):=r_p-e_p.$$

From (2.2)-(2.3) and (3.1)-(3.2), we derive

$$(r_{y,t},w)+a(r_y,w)=(u-u_h,w), \qquad \forall\, w\in W,\ t\in J, \tag{3.7}$$

$$-(r_{p,t},q)+a(q,r_p)=(r_y,q), \qquad \forall\, q\in W,\ t\in J. \tag{3.8}$$

Lemma 3.1. *Let r_y,r_p satisfy (3.7)-(3.8). Then we have*

$$\|r_y\|_{L^\infty(L^2)}\le C\|u-u_h\|_{L^2(L^2)}, \tag{3.9}$$

$$\|r_p\|_{L^\infty(L^2)}\le C\|u-u_h\|_{L^2(L^2)}. \tag{3.10}$$

Proof. By selecting $w=r_y$ in (3.7), we obtain

$$(r_{y,t},r_y)+a(r_y,r_y)=(u-u_h,r_y). \tag{3.11}$$

From Hölder's inequality and Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}(\|r_y\|^2)+c\|r_y\|_1^2\le C(\delta)\|u-u_h\|^2+\delta\|r_y\|^2. \tag{3.12}$$

Let us note that $r_y(0)=0$, on integrating (3.12) with respect to time from 0 to t and using Gronwall's lemma, we have

$$\|r_y\|^2_{L^\infty(L^2)}\le C(\delta)\|u-u_h\|^2_{L^2(L^2)}. \tag{3.13}$$

By choosing $q=r_p$ in (3.8), we obtain

$$-(r_{p,t},r_p)+a(r_p,r_p)=(r_y,r_p). \tag{3.14}$$

From Hölder's inequality and Young's inequality, we derive

$$-\frac{1}{2}\frac{d}{dt}(\|r_p\|^2)+c\|r_p\|_1^2\le C(\delta)\|r_y\|^2+\delta\|r_p\|^2. \tag{3.15}$$

Note that $r_p(T)=0$, on integrating (3.15) with respect to time from t to T and using Gronwall's lemma, we have

$$\|r_p\|^2_{L^\infty(L^2)}\le C(\delta)\|r_y\|^2_{L^2(L^2)}. \tag{3.16}$$

According to embedding theorem, so inequality (3.10) follows from (3.13) and (3.16). □

Lemma 3.2. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (2.2)-(2.4) and (2.9)-(2.11), respectively. Assume that $u_h+p_h|_\tau\in H^1(\tau)$ and that exists $w\in K_h$ such that*

$$\left|\int_0^T(u_h+p_h,w-u)dt\right|\le C\int_0^T\sum_\tau h_\tau|u_h+p_h|_{H^1(\tau)}\|u-u_h\|_{L^2(\tau)}dt. \tag{3.17}$$

Then

$$\|u-u_h\|_{L^2(L^2)} \leq C\left(\eta_1 + \|p_h - p(u_h)\|_{L^2(L^2)}\right), \tag{3.18}$$

where

$$\eta_1 = \left(\int_0^T \sum_\tau h_\tau^2 |u_h + p_h|_{H^1(\tau)}^2 dt\right)^{\frac{1}{2}}. \tag{3.19}$$

Proof. It follows from (2.4) and (2.11) that

$$\begin{aligned}
\|u-u_h\|_{L^2(L^2)}^2 &= \int_0^T (u-u_h, u-u_h)dt \\
&= \int_0^T (u+p, u-u_h)dt + \int_0^T (u_h+p_h, u_h-u)dt \\
&\quad + \int_0^T (p_h - p(u_h), u-u_h)dt + \int_0^T (p(u_h)-p, u-u_h)dt \\
&\leq \int_0^T (u_h+p_h, w-u)dt + \int_0^T (p_h-p(u_h), u-u_h)dt \\
&\quad + \int_0^T (p(u_h)-p, u-u_h)dt \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.20}$$

According to the assumption (3.17) and Young's inequality, for the first term we have

$$I_1 = \int_0^T (u_h+p_h, w-u)dt \leq C(\delta)\eta_1^2 + \delta\|u-u_h\|_{L^2(L^2)}^2. \tag{3.21}$$

From Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
I_2 &= \int_0^T (p_h - p(u_h), u-u_h)dt \\
&\leq C(\delta)\|p_h - p(u_h)\|_{L^2(L^2)}^2 + \delta\|u-u_h\|_{L^2(L^2)}^2.
\end{aligned} \tag{3.22}$$

Let us note that $y(0)-y(u_h)(0)=0$ and $p(T)-p(u_h)(T)=0$, from (2.2) minus (3.1) and select $w=p(u_h)-p$, then integral from 0 to T in the equation two sides and use integral by parts, we have

$$\begin{aligned}
&-\int_0^T (y-y(u_h), p_t(u_h)-p_t)dt + \int_0^T a(y-y(u_h), p(u_h)-p)dt \\
&= \int_0^T (u-u_h, p(u_h)-p)dt.
\end{aligned} \tag{3.23}$$

Similarly, from (2.3) minus (3.2) and select $q=y(u_h)-y$, then integral from 0 to T in the equation two sides, we have

$$\begin{aligned}
&-\int_0^T (p_t - p_t(u_h), y(u_h)-y)dt + \int_0^T a(y(u_h)-y, p-p(u_h))dt \\
&= \int_0^T (y-y(u_h), y(u_h)-y)dt.
\end{aligned} \tag{3.24}$$

By using (3.23) and (3.24), we derive

$$I_3 = \int_0^T (p(u_h) - p, u - u_h)dt = -\int_0^T (y - y(u_h), y - y(u_h))dt \leq 0. \tag{3.25}$$

Then (3.18) follows from (3.20)-(3.25). □

From (3.1)-(3.2) and (3.5)-(3.6), we have the following error equations:

$$(\xi_{y,t}, w) + a(\xi_y, w) = (\eta_{y,t}, w), \qquad \forall\, w \in W,\ t \in J, \tag{3.26}$$

$$-(\xi_{p,t}, q) + a(q, \xi_p) = (\xi_y, q) + (\eta_{p,t}, q), \qquad \forall\, q \in W,\ t \in J. \tag{3.27}$$

Lemma 3.3. *Let ξ_y and ξ_p satisfy (3.26)-(3.27). Then the following estimates hold:*

$$\|\xi_y\|_{L^\infty(L^2)} \leq C(\|\eta_{y,t}\|_{L^2(L^2)} + \|\eta_y(0)\|), \tag{3.28}$$

$$\|\xi_p\|_{L^\infty(L^2)} \leq C(\|\eta_{p,t}\|_{L^2(L^2)} + \|\eta_{y,t}\|_{L^2(L^2)} + \|\eta_y(0)\|). \tag{3.29}$$

Proof. By choosing $w = \xi_y$ in (3.26), we obtain

$$(\xi_{y,t}, \xi_y) + a(\xi_y, \xi_y) = (\eta_{y,t}, \xi_y). \tag{3.30}$$

From Hölder's inequality and Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}(\|\xi_y\|^2) + c\|\xi_y\|_1^2 \leq C(\delta)\|\eta_{y,t}\|^2 + \delta\|\xi_y\|^2. \tag{3.31}$$

Integrating (3.31) with respect to time from 0 to t and using Gronwall's lemma, we derive

$$\|\xi_y\|^2_{L^\infty(L^2)} \leq C(\delta)\left(\|\eta_{y,t}\|^2_{L^2(L^2)} + \|\eta_y(0)\|^2\right). \tag{3.32}$$

By selecting $q = \xi_p$ in (3.27), we have

$$-(\xi_{p,t}, \xi_p) + a(\xi_p, \xi_p) = (\xi_y, \xi_p) + (\eta_{p,t}, \xi_p). \tag{3.33}$$

From Hölder's inequality and Young's inequality, we obtain

$$-\frac{1}{2}\frac{d}{dt}(\|\xi_p\|^2) + c\|\xi_p\|_1^2 \leq C(\delta)(\|\xi_y\|^2 + \|\eta_{p,t}\|^2) + \delta\|\xi_p\|^2. \tag{3.34}$$

Note that $\xi_p(T) = 0$, on integrating (3.34) with respect to time from t to T and using Gronwall's lemma, we derive

$$\|\xi_p\|^2_{L^\infty(L^2)} \leq C(\delta)\left(\|\xi_y\|^2_{L^2(L^2)} + \|\eta_{p,t}\|^2_{L^2(L^2)}\right). \tag{3.35}$$

Then (3.28)-(3.29) follows from (3.32) and (3.35). □

From (3.5)-(3.6), we derive the error equations:

$$a(\eta_y, w_h) = 0, \qquad \forall\, w_h \in W_h, \tag{3.36}$$

$$a(q_h, \eta_p) = (\eta_y, q_h), \qquad \forall\, q_h \in W_h. \tag{3.37}$$

Lemma 3.4. *Let* (y_h, p_h, u_h) *and* $(\tilde{y}, \tilde{p})$ *satisfy (2.9)-(2.11) and (3.5)-(3.6), respectively. There exists a positive constant* C *which depends only on the coefficient matrix* A*, the domain* Ω*, the shape regularity of the elements and polynomial degree* l *such that*

$$\begin{aligned}\|\eta_y\|^2 \leq & C\Big(\|h^{1+\min\{1,l\}}(y_{h,t} - div(A\nabla y_h) - f - u_h)\|^2 \\ & + \min_{w_h\in W_h} \|h(\nabla y_h - \nabla_h w_h)\|^2\Big), \qquad (3.38)\\ \|\eta_{y,t}\|^2 \leq & C\Big(\|h^{1+\min\{1,l\}}(y_{h,t} - div(A\nabla y_h) - f - u_h)_t\|^2 \\ & + \min_{w_h\in W_h} \|h(\nabla y_h - \nabla_h w_h)\|^2\Big), \qquad (3.39)\\ \|\eta_p\|^2 \leq & C\Big(\|h^{1+\min\{1,l\}}(p_{h,t} - div(A^*\nabla p_h) - y_h + y_d)\|^2 + \|\eta_y\|^2 \\ & + \min_{w_h\in W_h} \|h(\nabla p_h - \nabla_h w_h)\|^2\Big), \qquad (3.40)\\ \|\eta_{p,t}\|^2 \leq & C\Big(\|h^{1+\min\{1,l\}}(p_{h,t} - div(A^*\nabla p_h) - y_h + y_d)_t\|^2 \\ & + \|\eta_{y,t}\|^2 + \min_{w_h\in W_h} \|h(\nabla p_h - \nabla_h w_h)\|^2\Big), \qquad (3.41)\end{aligned}$$

where A^* *is the adjoint matrix of* A.

Proof. Set $w = \tilde{y} - y_h$ in (3.5), we have

$$a(\tilde{y} - y_h, \tilde{y} - y_h) = -(y_{h,t} - f - u_h, \tilde{y} - y_h) - a(y_h, \tilde{y} - y_h).$$

Similar to [2, 17], by using embedding theorem and Cauchy' inequality, we can obtain (3.38). Similarly, it is easy to prove (3.39)-(3.41). □

4. A posteriori error estimates

We now derive a posteriori error estimates for the semidiscrete finite element approximation of the parabolic optimal control problem. By collecting Lemmas 3.1-3.4, we finally derive the following results:

Theorem 4.1. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of (2.2)-(2.4) and (2.9)-(2.11), respectively. Assume that all the conditions in Lemmas 3.1-3.4 are valid. Then the following a posteriori error estimates hold:*

$$\|u - u_h\|_{L^2(L^2)} \leq C\left(\eta_1 + \|\eta_{y,t}\|_{L^2(L^2)} + \|\eta_{p,t}\|_{L^2(L^2)} + \|y_0^h - y_0\|\right), \quad (4.1)$$

$$\|y - y_h\|_{L^\infty(L^2)} \leq C\left(\|u - u_h\|_{L^2(L^2)} + \|\eta_y\|_{L^2(L^2)}\right), \quad (4.2)$$

$$\|p - p_h\|_{L^\infty(L^2)} \leq C\left(\|u - u_h\|_{L^2(L^2)} + \|\eta_y\|_{L^2(L^2)} + \|\eta_p\|_{L^2(L^2)}\right), \quad (4.3)$$

where η_1 *is defined in Lemma 3.2 and the estimates for* η_y*,* $\eta_{y,t}$*,* η_p *and* $\eta_{p,t}$ *are define in Lemma 3.4.*

Theorem 4.2. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of (2.2)-(2.4) and (2.9)-(2.11), respectively. Assume that all the conditions in Theorem 4.1 are valid. There exists a positive constant* C *independent of* h *such that*

$$\begin{aligned}&\|u - u_h\|_{L^\infty(L^2)} \\ \leq & C\left(\eta_1 + \|\eta_{y,t}\|_{L^2(L^2)} + \|\eta_y\|_{L^2(L^2)} + \|\eta_{p,t}\|_{L^2(L^2)} + \|\eta_p\|_{L^2(L^2)} + \|y_0^h - y_0\|\right).\end{aligned} \quad (4.4)$$

Proof. From Lemmas 2.1-2.2, we have

$$\|u - u_h\|_{L^\infty(L^2)} \leq C\|p - p_h\|_{L^\infty(L^2)}. \tag{4.5}$$

Then (4.4) follows from (4.1), (4.3) and (4.5). $\square$

5. Numerical experiment

In this section, we present a numerical example to illustrate our theoretic results. The optimal control problem was dealt numerically with codes developed based on AFEPack. The package is freely available and the details can be found at [11].

We solve the following parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\|y(x,t) - y_d(x,t)\|^2 + \|u(x,t) - u_d(x,t)\|^2\right) dt, \\ y_t(x,t) - \mathrm{div}(A(x)\nabla y(x,t)) = f(x,t) + u(x,t), \quad \text{in}\,\Omega \times (0,T], \\ y(x,t) = 0, \quad \text{on}\,\partial\Omega \times (0,T], \\ y(x,0) = y_0(x), \quad \text{in}\,\Omega. \end{cases}$$

The partial derivative of time is approximated by the backward Euler method. For ease of exposition, we take a small time size $\Delta t = 10^{-2}$, $N = T/\Delta t \in \mathbb{Z}^+$, $t_n = n\Delta t, n = 0, 1, \cdots, N$, $\phi^n = \phi(x, t_n)$ and set the discrete time-dependent norm

$$|||\phi||| = \left(\sum_{n=1}^{N} \Delta t \|\phi^n\|^2\right)^{\frac{1}{2}}.$$

Example 5.1. The data are as follows:

$$\begin{aligned} &T = 1,\ \Omega = [0,1] \times [0,1], \\ &A(x) = \begin{cases} 2 \cdot E, & x_1 + x_2 \leq 1, \\ E, & x_1 + x_2 > 1, \end{cases} \\ &y(x,t) = \begin{cases} \sin(\pi x_1)\sin(\pi x_2)\sin(\pi t), & x_1 + x_2 \leq 1, \\ 2\sin(\pi x_1)\sin(\pi x_2)\sin(\pi t), & x_1 + x_2 > 1, \end{cases} \\ &p(x,t) = y(x,t), \\ &u(x,t) = \max(0, \overline{p(x,t)}) - p(x,t), \\ &f(x,t) = y_t(x,t) - \mathrm{div}(A(x)\nabla y(x,t)) - u(x,t), \\ &y_d(x,t) = y(x,t) + p_t(x,t) + \mathrm{div}(A^*(x)\nabla p(x,t)). \end{aligned}$$

Numerical results based on a sequence of uniformly refined meshes and adaptive meshes are listed in Table 1. It is clear that the adaptive meshes generated via the error estimators η_1, η_y, $\eta_{y,t}$, η_p and $\eta_{p,t}$ are able to save substantial computational work, in comparison with the uniform meshes. In Figure 1, it is easy to see that the mesh adapts very well to the neighborhood of the discontinuous line $x_1 + x_2 = 1$, and a higher density of node points are indeed distributed along the line.

Mesh	nodes	sides	elements	$\|\|\|u-u_h\|\|\|$	$\|\|\|y-y_h\|\|\|$	$\|\|\|p-p_h\|\|\|$
uniform mesh	2065	6032	3968	5.46e-02	3.51e-02	3.52e-02
adaptive mesh	667	1856	1190	543e-02	3.54e-02	3.53e-02

Table 1. Numerical results, Example 5.1.

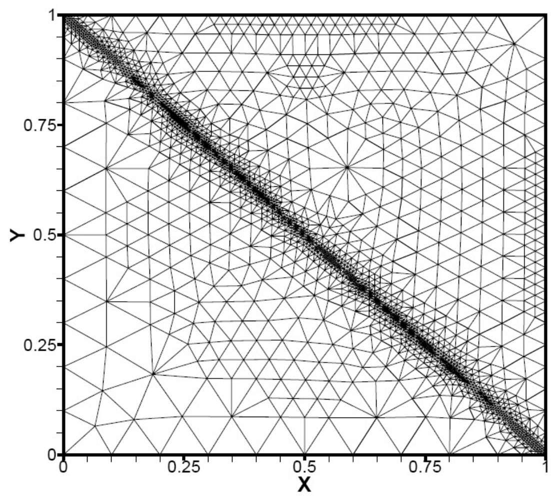

Figure 1. The adaptive mesh.

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