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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR FUNCTIONAL EVOLUTION EQUATIONS

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Abstract In this paper we present some results of global existence and attractivity of mild solutions for semilinear evolution equations with infinite delay in a Banach space. The considerations of this paper are based on the Schauder's fixed point theorem and the theory of evolution system.

Keywords Semilinear functional differential evolution equations, mild solution, fixed-point, evolution system, infinite delay, infinite interval, attractivity.

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1. Introduction

Functional evolution equations have a very important role to describe meany phenomena of physics, mechanics, biology etc; For more details on this theory and on its applications we refer to the monographs of Hale & Verduyn Lunel [20], Kolmanovskii & Myshkis [24], and Wu [30] and the reference therein. Recently, meany authors have study the existence of various model of semilinear evolution equations with finite an infinite delay in the Fréchet space, for instance we refer to Baghli & Benchohra [7–9]. In the other hand, different fields of engineering problems which is currently interest in unbounded domains. As a result it has received the attention of the researches see Agarwal & O'Regan [6], Olszowy & Wędrychowicz [27, 28].

Since the qualitative analysis of differential equations is related to both pure and applied mathematics, its applications to various fields such as science, engineering, and ecology have been extensively developed, among the properties that have been considered the boundedness, stability, attractivity, monotonicity and asymptotic behavior and this results are given with various approach as the fixed point theory and measure of noncompactness, for example see Anh & Hieu [5], Banaś & Cabrera [10], Banaś & Dhage [11], Hino & Murakami [21].

Differential equations on infinite intervals frequently occur in mathematical modelling of various applied problems. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium Agarwal & O'Regan [2], Kidder [23], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid Agarwal & O'Regan [3], heat transfer in the radial flow between parallel circular disks Na [26], investigation of the temperature distribution in the problem of phase change of

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solids with temperature dependent thermal conductivity Na [26], as well as numerous problems arising in the study of circular membranes Agarwal & O'Regan [1], Dickey [13, 14], plasma physics Agarwal & O'Regan [3], nonlinear mechanics, and non-Newtonian fluid flows Agarwal & O'Regan [1].

We study in this paper some sufficient conditions for the existence and attractivity of mild solutions to the following semilinear evolution differential equation

$$\begin{cases} y' - A(t)y = f(t, y_t), & t \in J := [0, \infty), \\ y(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t,s)\}_{(t,s) \in J \times J}$ for $0 \leq s \leq t < +\infty$, $f : J \times \mathcal{B} \to E$ be a Carathéodory function and $\mathcal{B}$ is an abstract *phase space* to be specified later, and $\phi \in \mathcal{B}$.

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$: Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$. As far as we know, the literature of the global existence together with the asymptotic behavior of mild solutions of evolution equations is very limited. This paper is filling the gap of such studies.

2. Preliminaries

Let E a Banach space with the norms $|\cdot|$ and $BC(J, E)$ the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

Let $\mathcal{X}$ is the space defined by

$$\mathcal{X} = \{y : \mathbb{R} \to E \text{ such that } y|_J \in BC(J, E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale & Kato [19] and follow the terminology used in [22]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold :

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\, |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1. In the sequel we assume that K and M are bounded on J and

$$\gamma := \max\left\{\sup_{t\in\mathbb{R}_+}\{K(t)\}, \sup_{t\in\mathbb{R}_+}\{M(t)\}\right\}.$$

For other details we refer, for instance to the book by Hino *et al.* [22].

In what follows, we assume that $\{A(t),\ t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.1. A family of bounded linear operators

$$\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : E \to E,\ (t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$$

is called en evolution system if the following properties are satisfied:

1. $U(t,t) = I$ where I is the identity operator in E,
2. $U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,
3. $U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(s,t) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\ y$ is continuous.

More details on evolution systems and their properties could be found on the books of Ahmed [4], Engel & Nagel [17] and Pazy [29].

Lemma 2.1 (Corduneanu, [12])**.** *Let $C \subset BC(J,E)$ be a set satisfying the following conditions:*

(i) *C is bounded in $BC(J,E)$;*

(ii) *the functions belonging to C are equicontinuous on any compact interval of J;*

(iii) *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of J;*

(iv) *the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.*

Then C is relatively compact in $BC(J,E)$.

Theorem 2.1 (Schauder's fixed point theorem, [16])**.** *Let C be a nonempty closed convex bounded subset of a Banach space E. Then every continuous compact mapping $T : C \to C$ has a fixed point.*

3. Main result

Definition 3.1. A function $y \in \mathcal{X}$ is said to be a mild solution of the problem (1.1), if

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0, \\ U(t,0)\phi(0) + \displaystyle\int_0^t U(t,s)f(s,y_s)ds, & \text{if } t \in J. \end{cases} \tag{3.1}$$

We will need to introduce the following hypotheses which are assumed thereafter:

(H_1) The evolution system $\{U(t,s)\}$ is compact for $0 < s \le t < +\infty$, and there exist constants $\widehat{M} \ge 1$ and $\omega > 0$ such that

$$\|U(t,s)\|_{B(E)} \le \widehat{M} e^{-\omega(t-s)} \quad \text{for every } (s,t) \in \Delta.$$

(H_2) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : J \to (0,\infty)$ and such that:

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H_3) $\displaystyle\lim_{t\to+\infty} \int_0^t e^{-w(t-s)} p(s)ds = 0.$

(H_4) There exists a constant $R > 0$ such that:

$$\hat{M}\|p\|_{L^1}\psi(R + (\hat{M}+1)\|\phi\|_{\mathcal{B}}) \le R.$$

Theorem 3.1. *Assume $(H_1)-(H_4)$ are satisfied, then the problem (1.1) admits at least one mild solution.*

Proof. We consider the operator $T : \mathcal{X} \to \mathcal{X}$ defined by:

$$Ty(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ U(t,0)\phi(0) + \displaystyle\int_0^t U(t,s)f(s,y_s)ds, & \text{if } t \in J. \end{cases} \tag{3.2}$$

For $\phi \in \mathcal{B}$, we will define the function $x : (-\infty, +\infty) \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ U(t,0)\phi(0) & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \mathcal{X}$, set

$$y(t) = x(t) + z(t).$$

It is obvious that y is fixed point of the operator T given in (3.2) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s)f(s, z_s + x_s)ds \quad \text{for} \quad t \in J.$$

Let

$$\mathcal{X}_0 = \{z \in \mathcal{X} : \ z_0 = 0\}.$$

The $\mathcal{X}_0$ is a Banach space with norm

$$\|z\|_{\mathcal{X}_0} = \sup_{t\in J} |z(t)| + \|z_0\|_{\mathcal{B}} = \sup_{t\in J} |z(t)|.$$

Now, we consider the operator $\tilde{T} : \mathcal{X}_0 \to \mathcal{X}_0$ defined by

$$\tilde{T}z(t) = \int_0^t U(t,s)f(s, z_s + x_s)ds, \text{ for } t \in J.$$

Obviously the problem (1.1) has a solution is equivalent to $\tilde{T}$ has a fixed point, so it turns to prove that $\tilde{T}$ has a fixed point.

Let $z \in \mathcal{X}_0$, we have

$$\begin{aligned}\|z_t + x_t\|_{\mathcal{B}} &\le \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}}\\ &\le K(t)|z(t)| + K(t)\|U(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} + M(t)\|\phi\|_{\mathcal{B}}\\ &\le \gamma(\|z\|_{\mathcal{X}_0} + \hat{M}e^{-\omega t}\|\phi\|_{\mathcal{B}} + \|\phi\|_{\mathcal{B}})\\ &\le \gamma(\|z\|_{\mathcal{X}_0} + (\hat{M}+1)\|\phi\|_{\mathcal{B}}). \end{aligned} \tag{3.3}$$

Then for $t \in J$, we have

$$\begin{aligned}|(\tilde{T}z)(t)| &\le \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_s + x_s)|ds\\ &\le \hat{M}\int_0^t e^{-\omega(t-s)}p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds\\ &\le \hat{M}\psi(\gamma(\|z\|_{\mathcal{X}_0} + (\hat{M}+1)\|\phi\|_{\mathcal{B}}))\|p\|_{L^1}.\end{aligned}$$

Now, for some R satisfying (H_4) we consider the set

$$D = \{z \in \mathcal{X}_0 : \ \|z\|_{\mathcal{X}_0} \le R\}.$$

Clearly, D is a bounded, convex and closed set of $\mathcal{X}_0$.

Then for $z \in D$, and by (3.3) and (H_4) we have

$$\|\tilde{T}z\|_{\mathcal{X}_0} \le \hat{M}\psi(\delta_R)\|p\|_{L^1} \le R,$$

with $\delta_R := \gamma(R + (\hat{M}+1)\|\phi\|_{\mathcal{B}})$. Consequently, the operator $\tilde{T}$ maps D into itself.

To apply Schauder's theorem, we must prove that $\tilde{T}$ is continuous and compact.

$\tilde{T}$ is continuous: Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{X}_0$ such that $z_k \to z$ in $\mathcal{X}_0$. we get for every $t \in J$

$$\begin{aligned}|\tilde{T}(z_k)(t) - \tilde{T}(z)(t)| &\le \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_{ks} + x_s) - f(s, z_s + x_s)| \ ds\\ &\le \hat{M}\int_0^t e^{-\omega(t-s)}|f(s, z_{ks} + x_s) - f(s, z_s + x_s)| \ ds.\end{aligned}$$

Since f is continuous, we obtain by the Lebesgue dominated convergence theorem

$$\|\tilde{T}z_k - \tilde{T}z\|_{\mathcal{X}_0} \to 0 \quad \text{as} \ \ k \to +\infty.$$

Thus $\tilde{T}$ is continuous.

$\tilde{T}(D)$ is relatively compact: To prove the compactness, we will use Corduneanu's lemma.

Firstly, it is clear that the assumption (i) is holds, then we will demonstrate that $\tilde{T}(D)$ is equicontinuous set for each closed bounded interval $[0, b]$ in J. Let

$s, t \in [0, b]$ with $t > s$ and $z \in D$. Then

$$\begin{aligned}|(\tilde{T}z)(t) - (\tilde{T}z)(s)| =& \left| \int_0^s (U(t,\tau) - U(s,\tau)) f(\tau, z_\tau + x_\tau)\, d\tau \right. \\ &+ \left. \int_s^t U(t,\tau) f(\tau, z_\tau + x_\tau) d\tau \right| \\ \leq& \int_0^s \|U(t,\tau) - U(s,\tau)\|_{B(E)}\ p(\tau)\psi(\|z_\tau + x_\tau\|_{\mathcal{B}}) d\tau \\ &+\ \hat{M} \int_s^t e^{-\omega(t-\tau)}\ p(\tau)\psi(\|z_\tau + x_\tau\|_{\mathcal{B}}) d\tau.\end{aligned}$$

Using the nondecreasing character of ψ and (3.3) we get

$$\begin{aligned}|(\tilde{T}z)(t) - (\tilde{T}z)(s)| \leq& \psi(\delta_R) \int_0^s \|U(t,\tau) - U(s,\tau)\|_{B(E)}\ p(\tau)\ d\tau \\ &+ \hat{M}\psi(\delta_R) \int_s^t p(\tau) d\tau.\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t - s \to 0$, then $\tilde{T}(D)$ is equicontinuous.

Now, we will prove that $D(t) := \{(\tilde{T}z)(t) : \ z \in D\}$ is relatively compact in E. Let $t \in J$ be a fixed and let $0 < \varepsilon < t \leq b$. For $z \in D$ we define

$$\tilde{T}_\varepsilon(z)(t) = U(t, t-\varepsilon) \int_0^{t-\varepsilon} U(t-\varepsilon, s) f(s, z_s + x_s) ds.$$

Since $U(t, s)$ is a compact operator, and the set $D_\varepsilon(t) := \{(\tilde{T}_\varepsilon z)(t) : \ z \in D\}$ is the image of bounded set of E by $U(t, s)$ then $D_\varepsilon(t)$ is precompact in E for every ε, $0 < \varepsilon < t$. Furthermore, for $z \in D$, we have

$$\begin{aligned}|\tilde{T}(z)(t) - \tilde{T}_\varepsilon(z)(t)| \leq& \int_{t-\varepsilon}^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)| ds \\ \leq& \int_{t-\varepsilon}^t \|U(t,s)\|_{B(E)} p(s)\psi(\|z_s + x_s\|_{\mathcal{B}}) ds \\ \leq& \psi(\delta_R)\hat{M} \int_{t-\varepsilon}^t e^{-\omega(t-s)} p(s) ds.\end{aligned}$$

The right-hand side tends to zero as $\varepsilon \to 0$, then $\tilde{T}_\varepsilon(z)$ converge uniformly to $\tilde{T}(z)$ which implies that $D(t)$ is precompact in E.

Finally, it remains to show that $\tilde{T}$ is equiconvergent.

Let $z \in D$, then from (H_1), (H_2) and (3.3) we have

$$|(\tilde{T}z)(t)| \leq \hat{M}\psi(\delta_R) \int_0^t e^{-\omega(t-s)} p(s) ds,$$

it follows immediately that $|(\tilde{T}z)(t)| \longrightarrow 0$ as $t \to +\infty$.

Then

$$\lim_{t \to +\infty} |(\tilde{T}z)(t) - (\tilde{T}z)(+\infty)| = 0,$$

which implies that $\tilde{T}$ is equiconvergent.

Thus by Schauder's fixed point theorem the operator $\tilde{T}$ has at least one fixed point which is a mild solution of problem (1.1).

4. Attractivity of solutions

In this section we study the attractivity of solutions the problem (1.1).

Definition 4.1 ([15]). We say that solutions of (1.1) are locally attractive if there exists a closed ball $\bar{B}(z^*, \rho)$ in the space $\mathcal{X}_0$ for some $z^* \in \mathcal{X}$ such that for arbitrary solutions z and $\tilde{z}$ of (1.1) belonging to $\bar{B}(z^*, \rho)$ we have that

$$\lim_{t \to +\infty} (z(t) - \tilde{z}(t)) = 0.$$

Under the assumption of Section 3, let z^* a solution of (1.1) and $\bar{B}(z^*, \rho)$ the closed ball in $\mathcal{X}_0$ which ρ satisfies the following inequality

$$2\hat{M}\|p\|_{L^1}\psi(\delta_\rho) \leq \rho,$$

with $\delta_\rho := \gamma(\rho + (\hat{M}+1)\|\phi\|_{\mathcal{B}})$.

Then, for $z \in \bar{B}(z^*, \rho)$ by (H_1)-(H_2) and (3.3) we have

$$\begin{aligned}
|(\tilde{T}z)(t) - z^*(t)| &= |(\tilde{T}z)(t) - (\tilde{T}z^*)(t)| \\
&\leq \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s) - f(s, z_s^* + x_s)| ds \\
&\leq \hat{M} \int_0^t e^{-\omega(t-s)} p(s)(\psi(\|z_s + x_s\|_{\mathcal{B}}) + \psi(\|z_s^* + x_s\|_{\mathcal{B}})) ds, \\
&\leq 2\hat{M}\psi(\delta_\rho) \int_0^t p(s) ds \leq \rho.
\end{aligned}$$

Therefore, we get $\tilde{T}(\bar{B}(z^*, \rho)) \subset \bar{B}(z^*, \rho)$.

So, for each $y \in \bar{B}(z^*, \rho)$ solution of problem (1.1) and $t \in J$, we have

$$\begin{aligned}
|z(t) - \tilde{z}(t)| &\leq |\tilde{T}z(t) - \tilde{T}\tilde{z}(t)| \\
&\leq 2\hat{M}\psi(\delta_\rho) \int_0^t e^{-\omega(t-s)} p(s) ds.
\end{aligned}$$

Hence, from (H_3), we conclude that

$$\lim_{t \to \infty} |z(t) - \tilde{z}(t)| = 0.$$

Consequently, the solutions of equation (1.1) are locally attractive. □

5. An Example

To illustrate the above results we consider the following partial functional differential equation.

$$\begin{cases}
\dfrac{\partial w}{\partial t}(t,\xi) = a(t,\xi)\dfrac{\partial^2 w}{\partial \xi^2}(t,\xi) \\
\qquad\qquad + e^{-t} \displaystyle\int_{-\infty}^0 \frac{\exp(w(t+s,\xi))}{1+s^2} ds, & t \geq 0, \xi \in [0,\pi], \\
w(t,0) = w(t,\pi) = 0, & t \geq 0, \\
w(s,\xi) = w_0(s,\xi), & s \in (-\infty, 0], \ \xi \in [0,\pi],
\end{cases} \tag{5.1}$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t and $w_0 : (-\infty, 0] \times [0,\pi] \to \mathbb{R}$ is a given function.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by

$$A(t)v = a(t,\xi)v''$$

with domain

$$D(A) = \{\ v \in E\ :\ v,\ v' \text{ are absolutely continuous},\ v'' \in E,\ v(0) = v(\pi) = 0\ \}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption (H_1) (see Freidman [18], Krein [25]).

For the phase space $\mathcal{B}$, we choose the well known space $BUC((-\infty,0],E)$: the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\| = \sup_{s\leq 0} |\varphi(s)| \quad \text{for} \quad \varphi \in \mathcal{B}.$$

If we put for $\varphi \in BUC(\mathbb{R}_-, E)$ and $\xi \in [0,\pi]$

$$\begin{aligned} &y(t)(\xi) = w(t,\xi),\ t \geq 0,\ \xi \in [0,\pi], \\ &\phi(s)(\xi) = w_0(s,\xi),\ -\infty < s \leq 0,\ \xi \in [0,\pi], \end{aligned}$$

and

$$f(t,\varphi)(\xi) = e^{-t}\int_{-\infty}^{0} \frac{\varphi(s)(\xi)}{1+s^2}ds,\ -\infty < s \leq 0,\ \xi \in [0,\pi],$$

where f satisfies (H_2) with $p(t) = \frac{\pi}{2}e^{-t}$ and $\psi(x) = e^x$ for $x \geq 0$.

Then, problem (5.1) takes the abstract partial functional evolution equation form (1.1). We can easily see that assumptions $(H_1) - (H_4)$ are satisfied, which implies that the mild solution of (5.1) is locally attractive.

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