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GLOBAL EXISTENCE AND UNIQUENESS OF SOLUTION FOR A SYSTEM OF SEMILINEAR DIFFUSION-REACTION EQUATIONS

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Abstract In this paper, we consider a system of highly nonlinear multi-species diffusion-reaction equations with homogeneous Neumann boundary condition. All reactions are reversible (see (1.1)). For this system, the existence and uniqueness of the weak solution are proved on the interval $[0, T)$ for any $T > 0$. We obtain, global in time, L^∞- estimates of the solution with the help of a Lyapunov functional. For the existence of the solution, we use Schaefer's fixed point theorem, maximal regularity and Lyapunov type arguments.

Keywords Global solution, nonlinear parabolic equation, reversible reactions, Lyapunov functionals, maximal regularity.

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1. Introduction

Recently Kräutle (cf. [9], [10]) has shown the global existence and uniqueness of the solution in $[H^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);H^{2,p}(\Omega))]^I$ of a system of diffusion-reaction equations for a multi-species reactive transport problem, where Ω is the given porous medium, I is the number of chemical species and $p > n+1$. He showed that with the help of a *Lyapunov functional*, we can be able to obtain some *a-priori* estimates which are global in time and these estimates will help us to show the existence of a unique weak solution on the time interval $S := [0,T)$ for any $T > 0$. But to our knowledge, it seems that this idea has not been fully excavated to its full strength when the solution $u(t)$ has derivative only upto the first order, i.e., if only $u(t) \in H^{1,p}(\Omega)$. In this paper, we show the global existence and uniqueness of the weak solution of a system of nonlinear multi-species diffusion-reaction equations under appropriate initial and boundary conditions (see equations (1.2) - (1.8)) in $[H^{1,p}(0,T;H^{1,q}(\Omega)^*) \cap L^p(0,T;H^{1,p}(\Omega))]^I$ for $p > n+2$, where $\frac{1}{p} + \frac{1}{q} = 1$. The lower regularity of the data involved gives more freedom for applications and it seems that this is an appropriate setting for dealing with homogenization problems (cf. [14]). The ingredients for the existence of the solution are a Lyapunov functional, Schaefer's fixed point theorem and a result from [17] which is based on the maximal regularity of differential operators. We investigate the following model:[a]

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let I be the number of mobile species present in the carrier substance (e.g.

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[a]The proposed mathematical model is inspired from the work of Kräutle, see pages 13-16, 66, 74 in [9].

water) in the domain Ω. These species diffuse and react with each other. The reactions are reversible and given by

$$\tau_{1j}X_1 + \tau_{2j}X_2 + ... + \tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1 + \nu_{2j}X_2 + ... + \nu_{Ij}X_I, \text{ for } 1 \le j \le J, \quad (1.1)$$

where X_i, $1 \le i \le I$, denotes the chemical species involved in J reactions. The stoichiometric coefficients $-\tau_{ij} \in \mathbb{Z}_0^-$ and $\nu_{ij} \in \mathbb{Z}_0^+$ respectively. Let u_i denote the concentration of X_i and set $u := (u_1, u_2, ..., u_I)$. Then the system of diffusion-reaction equations of these species is given by

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u = SR(u) \quad \text{in } (0,T)\times\Omega, \quad (1.2)$$

$$-D\,\nabla u \cdot \vec{n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \quad (1.3)$$

$$u(0,x) = u_0(x) \quad \text{in } \Omega. \quad (1.4)$$

where $SR(u)$ is the reaction term and $D > 0$ is constant. Here S is the $I \times J$-*th* order stoichiometric matrix with entries $s_{ij} = \nu_{ij} - \tau_{ij}$ for $1 \le i \le I$ and $1 \le j \le J$, and $R = (R_j)_{1\le j\le J}$ is the J-*th* order reaction rate vector whose j-*th* component is given as

$$R_j(u) = R_j^f(u) - R_j^b(u), \quad (1.5)$$

where

$$R_j^f(u) = \text{ forward reaction rate } = k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} \quad (1.6)$$

and

$$R_j^b(u) = \text{ backward reaction rate } = k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}, \quad (1.7)$$

where k_j^f and $k_j^b > 0$ are the forward and backward reaction rate factors respectively. Therefore the reaction rate term for the i-*th* species is given by

$$\begin{aligned}(SR(u))_i &= \sum_{j=1}^{J} s_{ij}R_j(u)\\ &= \sum_{j=1}^{J} s_{ij}\left(R_j^f(u) - R_j^b(u)\right)\\ &= \sum_{j=1}^{J} s_{ij}\left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}}\right). \quad (1.8)\end{aligned}$$

We denote the problem (1.2) - (1.8) by (P).

2. Notations and some preliminaries

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. As usual, $L^p(\Omega)$ is the space of real-valued measurable functions u such that $|u(\cdot)|^p$ is Lebesgue integrable with the usual modification for $p = \infty$ and the corresponding norm is given by

$$||u||_{L^p(\Omega)} = \begin{cases} \left[\int_\Omega |u(x)|^p \, dx\right]^{\frac{1}{p}} & \text{for} \quad 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| & \text{for} \quad p = \infty. \end{cases} \tag{2.1}$$

$H^{1,p}(\Omega)$ is the usual Sobolev space w.r.t. the norm

$$||u||_{H^{1,p}(\Omega)} = \begin{cases} \left[||u||^p_{L^p(\Omega)} + ||\nabla u||^p_{L^p(\Omega)}\right]^{\frac{1}{p}} & \text{for} \quad 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} [|u(x)| + |\nabla u(x)|] & \text{for} \quad p = \infty. \end{cases} \tag{2.2}$$

$H^{1,q}(\Omega)^*$ denotes the dual of $H^{1,q}(\Omega)$. We define a continuous embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ as

$$\langle f, v\rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} = \langle f, v\rangle_{L^p(\Omega)\times L^q(\Omega)} \text{ for } f \in L^p(\Omega),\ v \in H^{1,q}(\Omega). \tag{2.3}$$

For $k \in \mathbb{Z}_0^+$, $C^k(\bar{\Omega})$ denotes the Banach space of all k-times continuously differentiable functions w.r.t. the norm

$$||u||_{C^k(\bar{\Omega})} = \sum_{|\alpha|\le k} \sup_{x\in\bar{\Omega}} |D^\alpha u(x)| . \tag{2.4}$$

Suppose that $0 < \gamma \le 1$. $C^\gamma(\bar{\Omega})$ consists of all functions $u \in C(\bar{\Omega})$ such that

$$||u||_{C^\gamma(\bar{\Omega})} = ||u||_{C(\bar{\Omega})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{\frac{|u(x) - u(y)|}{|x - y|^\gamma}\right\} < \infty. \tag{2.5}$$

We introduce the *Sobolev-Bochner space* as

$$\begin{aligned} F := F^p(\Omega) &:= \left\{u \in L^p((0,T); H^{1,p}(\Omega)) : \frac{du}{dt} \in L^p((0,T); H^{1,q}(\Omega)^*)\right\} \\ &= H^{1,p}((0,T); H^{1,q}(\Omega)^*) \cap L^p((0,T); H^{1,p}(\Omega)), \end{aligned} \tag{2.6}$$

where $\frac{du}{dt}$ is the distributional time derivative of u and for $u \in F$,

$$||u||_F := ||u||_{L^p((0,T);H^{1,p}(\Omega))} + ||u||_{L^p((0,T);H^{1,q}(\Omega)^*)} + \left|\left|\frac{du}{dt}\right|\right|_{L^p((0,T);H^{1,q}(\Omega)^*)} . \tag{2.7}$$

For $0 < \theta < 1$, let

$$\left(H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\right)_{\theta,p} \text{ -- the real-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega) \tag{2.8}$$

and

$$\left[H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\right]_\theta \text{ -- the complex-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega), \tag{2.9}$$

endowed with one of their usual equivalent norms (cf. [3], [18], [12]). Finally, C denotes a generic positive constant which is not same all the time.

Lemma 2.1. $F \hookrightarrow C([0,T];(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p})$.

Proof. See theorem 4.10.2 in [1], or proposition 1.2.10 and remark 1.2.11 in [12]. □

Lemma 2.2. *Let $p > n+2$, then $F \hookrightarrow\hookrightarrow L^\infty((0,T)\times\Omega)$.*

Proof. **Step 1.** We notice that

$$\begin{aligned}
||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*} &= \left|\left|\int_{t_0}^t v^{'}(s)\,ds\right|\right|_{H^{1,q}(\Omega)^*} \\
&\le \int_{t_0}^t \left|\left|v^{'}(s)\right|\right|_{H^{1,q}(\Omega)^*} ds \\
&\le \left[\int_{t_0}^t \left|\left|v^{'}(s)\right|\right|^p_{H^{1,q}(\Omega)^*} ds\right]^{\frac{1}{p}} \left[\int_{t_0}^t ds\right]^{\frac{1}{q}} \\
&\le ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)} |t-t_0|^{\frac{1}{q}} \\
\implies \quad \frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\frac{1}{q}}} &\le ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)}\,. \qquad (2.10)
\end{aligned}$$

This implies $H^{1,p}((0,T);H^{1,q}(\Omega)^*) \hookrightarrow C^\delta([0,T];H^{1,q}(\Omega)^*)$, where $\delta = \frac{1}{q} = 1-\frac{1}{p}$.

Step 2. The condition $p > n+2$ implies $\frac{1}{2}+\frac{n}{2p} < 1-\frac{1}{p}$. Choose $\lambda \in \left(\left(\frac{1}{2}+\frac{n}{2p}\right)\left(1-\frac{1}{p}\right)^{-1}, 1\right)$ and set $\eta := \lambda(1-\frac{1}{p})$. Then by the reiteration theorem on real-interpolation

$$\begin{aligned}
&\frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\eta,1}}}{|t-t_0|^{\delta(1-\lambda)}} \\
=&\frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{\lambda(1-\frac{1}{p}),1}}}{|t-t_0|^{\delta(1-\lambda)}} \\
=&\frac{||v(t)-v(t_0)||_{(H^{1,q}(\Omega)^*,(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{(1-\frac{1}{p}),p})_{\lambda,1}}}{|t-t_0|^{\delta(1-\lambda)}} \\
\le& C\frac{||v(t)-v(t_0)||^{1-\lambda}_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta(1-\lambda)}} \times ||v(t)-v(t_0)||^{\lambda}_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \\
\le& C\left(\frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta}}\right)^{1-\lambda} \times 2\sup_{t\in(0,T)} ||v(t)||^{\lambda}_{(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \\
\le& C\left(\frac{||v(t)-v(t_0)||_{H^{1,q}(\Omega)^*}}{|t-t_0|^{\delta}}\right)^{1-\lambda} \times ||v||^{\lambda}_{C([0,T];(H^{1,q}(\Omega)^*,H^{1,p}(\Omega))_{1-\frac{1}{p},p})}\,. \qquad (2.11)
\end{aligned}$$

Therefore, by step 1 and lemma 2.1, it follows that $F \hookrightarrow C^\beta([0,T];(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\eta,1})$, where $\frac{1}{2}+\frac{n}{2p} < \eta < 1-\frac{1}{p}$ and $\beta = \delta(1-\lambda)$.

Step 3. We have the following embeddings (cf. theorem 1.3.3.d in [18] and corollary

5.28 in [13])

$$(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\eta,1} \hookrightarrow (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\eta,p} \hookrightarrow H^{2\eta-1,p}(\Omega) \hookrightarrow C^{\alpha}(\bar{\Omega}),$$

where $\alpha = 2\eta - 1 - \frac{n}{p} > 0$. Therefore combining the steps 2 and 3, we obtain

$$F \hookrightarrow C^{\beta}([0,T]; C^{\alpha}(\bar{\Omega})) \hookrightarrow C^{\sigma}([0,T] \times \bar{\Omega}) \hookrightarrow\hookrightarrow L^{\infty}((0,T) \times \Omega),$$

where $\sigma = \min(\alpha, \beta)$. □

Lemma 2.3. *Let $p > n+2$. Then $(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p} \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$.*

Proof. Let us denote $E_0 := H^{1,q}(\Omega)^*$, $E_1 := H^{1,p}(\Omega)$ and $E_{1-\frac{1}{p},p} := (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$. By lemma 3.4 in [8]: $[E_0, E_1]_{\frac{1}{2}} \hookrightarrow L^p(\Omega)$. From this and the reiteration theorem on real-interpolation, we obtain

$$E_{1-\frac{1}{p},p} := ([E_0, E_1]_{\frac{1}{2}}, [E_0, E_1]_1)_{1-\frac{2}{p},p} \hookrightarrow (L^p(\Omega), H^{1,p}(\Omega))_{1-\frac{2}{p},p} = H^{1-\frac{2}{p},p}(\Omega).$$

There exists a $t > 0$ such that $p > n+2 \Rightarrow 1 - \frac{n+2}{p} > t > 0 \Rightarrow 1 - \frac{2}{p} > t + \frac{n}{p}$. From theorem 4.6.1 (e) in [18]: $H^{1-\frac{2}{p},p}(\Omega) \hookrightarrow C^t(\bar{\Omega})$. Since $C^t(\bar{\Omega}) \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$, $H^{1-\frac{2}{p},p}(\Omega) \hookrightarrow C^t(\bar{\Omega}) \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$. Therefore $(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p} \hookrightarrow\hookrightarrow L^{\infty}(\Omega)$. □

Lemma 2.4 (Schaefer's fixed point theorem)**.** *Let X be Banach space. Assume that $Z : X \to X$ is a continuous and compact mapping and the solution of the set*

$$\{u \in X | \exists \lambda \in [0,1] : u = \lambda Z(u)\}$$

is bounded. Then Z has a fixed point.

Proof. See theorem 4 in section 9.2.2 in [7]. □

Next we introduce the norms on the vector-valued function spaces. Let $I \in \mathbb{N}$ and $u : \Omega \to \mathbb{R}^I$, and set

$$[L^p(\Omega)]^I := \underbrace{L^p(\Omega) \times L^p(\Omega) \times ... \times L^p(\Omega)}_{I\text{-times}}. \tag{2.12}$$

For $u \in [L^p(\Omega)]^I$ the corresponding norm is given as

$$|||u|||_{[L^p(\Omega)]^I} := \left[\sum_{i=1}^{I} ||u_i||^p_{L^p(\Omega)} \right]^{\frac{1}{p}}. \tag{2.13}$$

Similarly,

$$|||u|||_{[L^{\infty}(\Omega)]^I} := \max_{1 \le i \le I} ||u_i||_{L^{\infty}(\Omega)}, \tag{2.14}$$

$$|||u|||_{[H^{1,p}(\Omega)]^I} := \left[\sum_{i=1}^{I} ||u_i||^p_{H^{1,p}(\Omega)} \right]^{\frac{1}{p}}, \tag{2.15}$$

$$|||u|||_{[H^{1,\infty}(\Omega)]^I} := \max_{1 \le i \le I} ||u_i||_{H^{1,\infty}(\Omega)}, \tag{2.16}$$

$$|||u|||_{[H^{1,q}(\Omega)^*]^I} := \left[\sum_{i=1}^{I} ||u_i||^p_{H^{1,q}(\Omega)^*} \right]^{\frac{1}{p}}. \tag{2.17}$$

We define

$$\mathcal{F}_p^u := \left[H^{1,p}((0,T);H^{1,q}(\Omega)^*) \cap L^p((0,T);H^{1,p}(\Omega))\right]^I \tag{2.18}$$

and

$$\mathcal{X}_p^u := \left[\left(H^{1,q}(\Omega)^*, H^{1,p}(\Omega)\right)_{1-\frac{1}{p},p}\right]^I. \tag{2.19}$$

For $u \in \mathcal{F}_p^u$

$$|||u|||_{\mathcal{F}_p^u} := \left[\sum_{i=1}^I ||u_i||_F^p\right]^{\frac{1}{p}} \tag{2.20}$$

and for $v \in \mathcal{X}_p^u$

$$|||v|||_{\mathcal{X}_p^u} := \left[\sum_{i=1}^I ||v_i||^p_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}\right]^{\frac{1}{p}}. \tag{2.21}$$

Definition 2.1 (Weak formulation)**.** A function $u \in \mathcal{F}_p^u$ is said to be a weak solution of the problem (P) if it satisfies

$$(i)\quad \left\langle \frac{\partial u(t)}{\partial t}, \phi \right\rangle_{[H^{1,q}(\Omega)^*]^I \times [H^{1,q}(\Omega)]^I} + \int_\Omega \langle D\nabla u(t,x), \nabla\phi(x)\rangle_I \, dx$$
$$= \langle SR(u(t)), \phi\rangle_{[H^{1,q}(\Omega)^*]^I \times [H^{1,q}(\Omega)]^I}$$
$$\text{for every } \phi \in [H^{1,q}(\Omega)]^I \text{ and for a.e. } t. \tag{2.22}$$

$$(ii)\quad u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{2.23}$$

Let the following assumptions hold:

(i) $p > n+2,$ (2.24)

(ii) $u_0 \geq 0$, i.e., $u_{0_i} \geq 0$ for all $i = 1,2,...,I,$ (2.25)

(iii) $u_{0_i} \in (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}$ for $i = 1,2,...,I,$ (2.26)

(iv) All reactions are linearly independent such that the stoichiometric matrix $S = (s_{ij})_{\substack{1\leq i\leq I\\ 1\leq j\leq J}}$ has maximal column rank, i.e., $rank(S) = J.$ (2.27)

Our main result reads as

Theorem 2.1 (Existence theorem)**.** *Let the assumptions (2.24) - (2.27) be satisfied, then there exists a unique positive global weak solution $u \in \mathcal{F}_p^u$ of the problem (P).*

3. Maximal regularity

Definition 3.1. Let $1 < p < \infty$, X be a Banach space and $A : X \to X$ be a closed, not necessarily bounded, operator, where the domain $D(A)$ of A is dense in X. A

is said to have the maximal L^p-regularity if for every $f \in L^p((0,T);X)$ there exists a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of the problem

$$\frac{du(t)}{dt} + Au(t) = f(t) \quad \text{for } t > 0, \tag{3.1}$$

$$u(0) = 0. \tag{3.2}$$

which satisfies

$$||u||_{L^p((0,T);X)} + ||u_t||_{L^p((0,T);X)} + ||u||_{L^p((0,T);D(A))} \leq C\,||f||_{L^p((0,T);X)}, \tag{3.3}$$

where $C > 0$ is a constant independent of f.

For a detailed overview on maximal regularity, we refer to [2, 5, 11, 15, 16] and references therein. Now we set $D(A) := H^{1,p}(\Omega)$ and $X := H^{1,q}(\Omega)^*$. Clearly, $D(A) \overset{d}{\subseteq} X$.* Let $\mu = (\mu_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}$ be a positive definite symmetric matrix-field, where $\mu_{ij} \in C(\bar{\Omega})$ and there is a constant $C > 0$

$$\sum_{i,j=1}^{n} \mu_{ij}(x)\zeta_i\zeta_j \geq C|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{3.4}$$

We define a sesquilinear form $a(u,v) : H^{1,p}(\Omega) \times H^{1,q}(\Omega) \to \mathbb{R}$ by

$$a(u,v) := \int_\Omega \mu\nabla u \cdot \nabla v\,dx + \kappa \int_\Omega uv\,dx \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega), \tag{3.5}$$

where $\kappa > 0$. We further define an operator $A : H^{1,p}(\Omega) \to H^{1,q}(\Omega)^*$ associated with the form $a(u,v)$ by

$$\langle Au, v\rangle := a(u,v) \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega). \tag{3.6}$$

In [4] and [5], it is shown that: (i) $\left|\left|A^{is}\right|\right|_{L(X)} \leq Ke^{\theta|s|}$ for some $0 < \theta < \frac{\pi}{2}$, $s \in \mathbb{R}$, where $K > 0$ and (ii) $(-\infty, 0] \subset \rho(A)$ (resolvent of A) and $\left|\left|(\lambda + A)^{-1}\right|\right|_{L(X)} \leq \frac{C}{1+|\lambda|}$ for every $\lambda \in [0,\infty)$, where $C > 0$. By a theorem of Dore and Venni (cf. [6]), A has maximal L^p-regularity on $H^{1,q}(\Omega)^*$.

4. Proof of theorem 2.1

Strategy of the proof: We modify some parts of the methodology of Kräutle (cf. [10]) in order to prove the positivity, existence and uniqueness of the global solution of the problem (P). Before dealing with (P), we consider a slightly modified problem and introduce the rate function $\bar{R} : \mathbb{R}^I \to \mathbb{R}^J$ as

$$\bar{R}(u) := R(u^+), \tag{4.1}$$

where u^+ is the positive part of u defined componentwise as

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*$A \overset{d}{\subseteq} B$ means that A is dense in B.

$$\left.\begin{aligned} & u_i^+ := \max(u_i, 0), \\ & u_i^- := \max(-u_i, 0) = -\min(u_i, 0) \\ \text{and} \qquad & u_i = u_i^+ - u_i^-. \end{aligned}\right\} \tag{4.2}$$

Replacing R by $\bar{R}$ in (1.2), we get

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u = S\bar{R}(u) \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.3}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \Omega, \tag{4.4}$$

$$-D\nabla u \cdot \vec{n} = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega. \tag{4.5}$$

Let us denote this problem by (P^+). We will prove the existence of a global solution of (P^+). Since we show that the solution of (P^+) is non-negative, it solves (P). We conclude this section by proving the uniqueness of the solution of (P). We commence our investigation with the positivity of the solution of (P^+).

Lemma 4.1. *Let (2.24)-(2.27) hold and a function $u \in \mathcal{F}_p^u$ be the solution of (P^+). Then $u_i \geq 0$ on $(0,T)\times\Omega$ for all i.*

Proof. The proof follows exactly as the one for lemma 2 given in [10]: Let $\Omega_i^{p^-}(t)$ be the support of $u_i^-(t)$. We multiply the *i-th* PDE of (4.3) by $-u_i^-(t)$ and integrate over $\Omega_i^{p^-}(t)$. The rest follows by Gronwall's inequality. □

Now we show the existence of a global weak solution of (P^+). For technical reasons, we add an extra term on both sides of (P^+), i.e., for a constant $\kappa > 0$ we have

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u + \kappa u = S\bar{R}(u) + \kappa u \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.6}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \Omega, \tag{4.7}$$

$$-D\nabla u \cdot \vec{n} = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega. \tag{4.8}$$

We denote (4.6) - (4.8) by (P_M^+). We see that the solution of (P_M^+) is also the solution of (P^+). We prove the global existence of a weak solution of (P_M^+).

4.1. Fixed point operator

Let us define a fixed point operator $Z_1 : \mathcal{F}_p^u \to \mathcal{F}_p^u$ via

$$Z_1(v) = u, \tag{4.9}$$

where u is the solution of the linear problem

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u + \kappa u = S\bar{R}(v) + \kappa v \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.10}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \Omega, \tag{4.11}$$

$$-D\nabla u_i \cdot \vec{n} = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega \tag{4.12}$$

for $i = 1, 2, ..., I$.

Remark 4.1. The reformulation of (4.10)-(4.12) is given by

$$\frac{du}{dt} + Au = f(v), \tag{4.13}$$

$$u(0) = u_0, \tag{4.14}$$

where $f(v) = S\bar{R}(v) + \kappa v$ and the operator $A : H^{1,p}(\Omega)^I \to [H^{1,q}(\Omega)^*]^I$ is defined as $Au := (A_1 u_1, A_2 u_2, ..., A_I u_I)$ such that for $1 \leq i \leq I$,

$$\langle A_i u_i, w_i \rangle := \int_\Omega D\nabla u_i(x) \cdot \nabla w_i(x)\, dx + \kappa \int_\Omega u_i(x) w_i(x)\, dx \quad \text{for } u_i \in H^{1,p}(\Omega) \text{ and } w_i \in H^{1,q}(\Omega).$$

The assumption (2.24) guarantees $u_0 \in \mathcal{X}_p^u$. By lemma 2.2: Since $v \in \mathcal{F}_p^u$, $v \in L^\infty((0,T)\times\Omega)^I$. This shows that $f(v) = S\bar{R}(v) + \kappa v \in [L^p((0,T); H^{1,q}(\Omega)^*)]^I$.[†] Moreover section 3 ensures the maximal regularity of A on $[H^{1,q}(\Omega)^*]^I$.[‡] Therefore theorem 2.5 in [17] gives the existence of a unique solution $u \in \mathcal{F}_p^u$ of the problem (4.13) - (4.14). Thus the operator Z_1 is well-defined.

Remark 4.2. Every fixed point of Z_1 is a solution of the problem (P_M^+).

In order to use Schaefer's fixed point theorem, we need to verify the following conditions:

(i) The operator Z_1 is continuous and compact.

(ii) The set $\{u \in \mathcal{F}_p^u | \exists \lambda \in [0,1] : u = \lambda Z_1(u)\}$ is bounded, i.e., there exists a constant $C > 0$ such that any arbitrary solution $u \in \mathcal{F}_p^u$ of the equation

$$u = \lambda Z_1(u) \tag{4.15}$$

satisfies

$$|||u|||_{\mathcal{F}_p^u} \leq C, \tag{4.16}$$

where C is independent of λ, u and t. Equations (4.10)-(4.12) and (4.15) imply

$$\frac{\partial u}{\partial t} - \nabla \cdot D\nabla u + \kappa u = \lambda S\bar{R}(u) + \lambda\kappa u \qquad \text{in} \quad (0,T)\times\Omega, \tag{4.17}$$

$$u(0,x) = \lambda u_0(x) \qquad \text{in} \quad \Omega, \tag{4.18}$$

$$-D\nabla u \cdot \vec{n} = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega. \tag{4.19}$$

We denote the problem (4.17)-(4.19) as $(P_{M_\lambda}^+)$.

4.1.1. Introduction of the Lyapunov functions

Let $\mu^0 \in \mathbb{R}^I$ be a solution of the linear system

$$S^T \mu^0 = -\log K, \tag{4.20}$$

[†]We have used $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$.

[‡]The operator A is said to have maximal regularity on $[H^{1,q}(\Omega)^*]^I$ if each A_i has maximal regualrity on $H^{1,q}(\Omega)^*$.

where $K \in \mathbb{R}^J$ is the vector of equilibrium constants $K_j = \frac{k_j^f}{k_j^b}$ related to the J kinetic reactions. Due to assumption (2.25), the system (4.20) has a solution μ^0. Following [10], we define the following functions:

Let $g_i : \mathbb{R}_0^+ \to \mathbb{R}$ and $g : {\mathbb{R}_0^+}^I \to \mathbb{R}$ be defined as[§]

$$g_i(u_i) = (\mu_i^0 - 1 + \log u_i)u_i + e^{(1-\mu_i^0)} \text{ for each } i = 1, 2, ..., I \tag{4.21}$$

and

$$g(u) = \sum_{i=1}^{I} g_i(u_i). \tag{4.22}$$

For $r \in \mathbb{N}$, we define $f_r : {\mathbb{R}_0^+}^I \to \mathbb{R}$ and $F_r : L_+^\infty(\Omega)^I \to \mathbb{R}$ as

$$f_r(u) = [g(u)]^r \tag{4.23}$$

and

$$F_r(u) = \int_\Omega f_r(u(x))\, dx. \tag{4.24}$$

Proposition 4.1. *For all $i = 1, 2, ..., I$,*

$$g(u) \geq g_i(u_i) \geq u_i \geq 0 \tag{4.25}$$

and

$$F_r(u) \geq ||u_i||_{L^r(\Omega)}^r . \tag{4.26}$$

Proof. It can be easily seen that the minimum of the function $\psi(u_i) := g_i(u_i) - u_i$ is zero which implies inequality (4.25). For (4.26) see that

$$F_r(u) = \int_\Omega f_r(u(x))\, dx = \int_\Omega [g(u(x))]^r \; dx \geq \int_\Omega |u_i(x)|^r \; dx.$$

□

Proposition 4.2. *Let $\alpha > 0$. Then the following inequalities hold*

$$g_i(u_i) \leq C\ (1 + u_i^{1+\alpha}) \quad \text{for all } i, \tag{4.27}$$

$$g(u) \leq C\ (1 + |u|_I^{1+\alpha}) \tag{4.28}$$

and

$$f_r(u) \leq C\ (1 + |u|_I^{r(1+\alpha)}), \tag{4.29}$$

where $C > 0$ is a constant depending on α and μ_i but is independent of u_i.

Proof. The proof follows directly from the definitions of g_i, g and f_r. □

From (4.26) it is clear that the L^r - norm of u_i will be finite if we can obtain an upper bound of $F_r(u)$. Thus obtaining an upper bound for $F_r(u)$ is the main concern of the following theorem:

Theorem 4.1. *Let $r \in \mathbb{N}$ $(r \geq 2)$, $0 \leq t \leq T$, $p > n + 2$ and $0 \leq \lambda \leq 1$. Further assume that $u \in \mathcal{F}_p^u$ is a solution of $(P_{M_\lambda}^+)$. Then the following inequality holds:*

$$F_r(u(t)) \leq\ e^{Ir\kappa(e(e-1))^{-1}t}\ F_r(u(0)) \quad \text{ for all } r \text{ and for a.e. } t. \tag{4.30}$$

[§]Here we have considered the natural logarithm, i.e. $\log_e \cdots$.

To prove this theorem, we need the following lemmas as basic ingredients. For $p > n+1$ and $\zeta \in [H^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);H^{2,p}(\Omega))]^I$, these lemmas have been proved in [10], but they can be adapted for the functions in $\mathcal{F}_p^u$ with $p > n+2$.

Lemma 4.2. *The map $F_r : L_+^\infty(\Omega)^I \to \mathbb{R}$ is continuous.*

Proof. The proof is analogous to the proof of the lemma 3.4 in [9]. □

Let us consider the derivative (in the classical sense) of $f_r : \mathbb{R}_0^{+I} \to \mathbb{R}^I$ which is given as

$$\begin{aligned}\partial f_r(v) &= \nabla_v f_r(v)\\ &= r[g(v)]^{r-1}\nabla_v g(v)\\ &= r f_{r-1}(v)\left(\mu^0 + \log v\right).\end{aligned}$$

We see that $\partial f_r(v)$ is undefined for $v = 0$ whereas $f_{r-1}(v)$ is defined for all $v \geq 0$. Since we only know the nonnegativity of v, we define, for any $\delta > 0$,

$$v_\delta := v + \delta. \tag{4.31}$$

Clearly, $v_\delta \geq \delta > 0$ and $v_\delta \in \mathcal{F}_p^u$. From here on we work with the function v_δ unless stated otherwise. We aim to prove that for $v_\delta \in \mathcal{F}_p^u$,

$$\partial f_r(v_\delta) \in L^q((0,T);H^{1,q}(\Omega))^I. \tag{4.32}$$

To prove (4.32), our point of departure is the following lemma which deals with the continuity of ∂f_r.

Lemma 4.3. *Let $p > n+2$ and $\delta > 0$, then the map*

$$v_\delta \mapsto \partial f_r(v_\delta), \text{ i.e., } \partial f_r : \mathcal{F}_p^u \to L^\infty((0,T)\times\Omega)^I$$

is continuous.

Proof. Let $v_\delta \in \mathcal{F}_p^u$. For $p > n+2$, from lemma 2.2 it follows that $v_{\varepsilon_\delta} \in [L^\infty((0,T)\times\Omega)]^I$. The rest follows as in the proof of lemma 3.6 in [9]. □

Lemma 4.4. *(Derivative of the vector function $x \longmapsto \partial f_r(v_\delta(t,x))$ w.r.t. $x \in \Omega$) Let $p > n+2$, $r \in \mathbb{N}$ $(r \geq 2)$ and $v_\delta \in \mathcal{F}_p^u$. We define the map $w(v_\delta) : (0,T)\times\Omega \to \mathbb{R}^{I\times n}$ by*

$$w(v_\delta)(t,x) := \{r(r-1)f_{r-2}(v_\delta)M_\mu(v_\delta) + r f_{r-1}(v_\delta)\Lambda_{\frac{1}{v_\delta}}\}\nabla_x v_\delta(t,x), \tag{4.33}$$

where $M_\mu(v_\delta)$ is the $I\times I$-th order symmetric matrix with entries $(\mu_i^0 + \log v_{\delta_i})(\mu_j^0 + \log v_{\delta_j})$ and $\Lambda_{\frac{1}{v_\delta}}$ is the $I \times I$-th order diagonal matrix with entries $\frac{1}{v_{\delta_i}}$. Then

$$\nabla_x(\partial f_r(v_\delta)) = w(v_\delta) \in L^q((0,T);L^q(\Omega))^{I\times n}, \tag{4.34}$$

$$\text{i.e.,} \quad \partial f_r(v_\delta) \in L^q((0,T);H^{1,q}(\Omega))^I. \tag{4.35}$$

Proof. Let $v_\delta \in \mathcal{F}_p^u$. For $p > n+2$, lemma 2.2 implies $v_\delta \in L^\infty((0,T)\times\Omega)^I$. Since $v_\delta \geq \delta$, from the definitions of $f_r(v_\delta)$, $M_\mu(v_\delta)$ and $\Lambda_{\frac{1}{v_\delta}}$, we have

$$r(r-1)f_{r-2}(v_\delta)M_\mu(v_\delta) + r f_{r-1}(v_\delta)\Lambda_{\frac{1}{v_\delta}} \in L^\infty((0,T)\times\Omega)^{I\times I}. \tag{4.36}$$

Also note that for $p > n+2$ and $v_\delta \in \mathcal{F}_p^u$, $\nabla_x v_\delta \in L^q((0,T);L^q(\Omega))^{I\times n}$. Therefore $w(v_\delta) \in L^q((0,T);L^q(\Omega))^{I\times n}$. Next we prove that $\nabla_x(\partial f_r(v_\delta)) = w(v_\delta)$. This follows from the density of $C^\infty([0,T]\times\bar{\Omega})^I$ in $\mathcal{F}_p^u$ (for details cf. lemma 3.6 in [9], e.g.). □

Lemma 4.5. *Let $u \in \mathcal{F}_p^u$ be the solution of the problem $(P_{M_\lambda}^+)$ and $\delta > 0$ be such that $u_\delta := u + \delta$. Then we have the following inequality*

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \le Ir\kappa\,(e(e-1))^{-1}\int_0^t F_r(u_\delta(\tau))\,d\tau + l(t,u_\delta,\delta) + h(t,\delta,u_\delta) \text{ for a.e. } t, \tag{4.37}$$

where $h(t,\delta,u_\delta)$ and $l(t,u_\delta,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$ for a.e. t.

Proof. As in the proof of lemma 4.1 it can be shown that the solution of $(P_{M_\lambda}^+)$ is also nonnegative, set

$$u_\delta := u + \delta \ge \delta. \tag{4.38}$$

Clearly, $u_\delta \in \mathcal{F}_p^u$. By lemma 4.4, $\partial f_r(u_\delta) \in L^q((0,T);H^{1,q}(\Omega))^I$. Using $\partial f_r(u_\delta)$ as test function for the weak formulation of $(P_{M_\lambda}^+)$, we obtain

$$\begin{aligned}
&\int_0^t \left\langle \tfrac{\partial u}{\partial \tau}, \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \\
&- \int_0^t \langle \nabla D\nabla u, \partial f_r(u_\delta)\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau + \kappa \int_0^t\int_\Omega \langle u, \partial f_r(u_\delta(\tau))\rangle_I\,dx\,d\tau \\
= \;& \lambda\int_0^t \left\langle S\bar{R}(u), \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} dt + \lambda\kappa\int_0^t\int_\Omega \langle u, \partial f_r(u_\delta(\tau))\rangle_I\,dx\,d\tau,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_0^t \left\langle \tfrac{\partial u}{\partial \tau}, \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \\
= \;& -\int_0^t \langle D\nabla u_\delta, \nabla_x(\partial f_r(u_\delta))\rangle_{[L^p(\Omega)]^{I\times n}\times[L^q(\Omega)]^{I\times n}} d\tau \\
&+\lambda\int_0^t \left\langle S\bar{R}(u), \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \\
&-(1-\lambda)\kappa\int_0^t\int_\Omega \langle u, \partial f_r(u_\delta(\tau))\rangle_I\,dx\,d\tau,
\end{aligned}$$

i.e.,

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \quad =: \; I_{diff}^{(t)} + I_{reac}^{(t)} + I_{Ex}^{(t)} \quad \text{for a.e. } t, \tag{4.39}$$

where

$$I_{diff}^{(t)} := -\sum_{k=1}^n \int_0^t\int_\Omega \left\langle D\frac{\partial}{\partial x_k}u_\delta, \frac{\partial}{\partial x_k}(\partial f_r(u_\delta))\right\rangle_I dx\,d\tau, \tag{4.40}$$

$$I_{reac}^{(t)} := \lambda\int_0^t \left\langle S\bar{R}(u), \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau \tag{4.41}$$

and

$$I_{Ex}^{(t)} := -(1-\lambda)\kappa\int_0^t\int_\Omega \langle u, \partial f_r(u_\delta(\tau))\rangle_I\,dx\,d\tau. \tag{4.42}$$

Now we simplify the terms on the r.h.s. of (4.39) one by one. [¶]

$$\begin{aligned} I_{reac}^{(t)} &= \lambda \int_0^t \int_\Omega \left\langle S\bar{R}(u), \partial f_r(u_\delta) \right\rangle_I dx\, d\tau \\ &= \lambda \int_0^t \int_\Omega \left\langle r f_{r-1}(u_\delta) \left(\mu^0 + \log u_\delta\right), S\bar{R}(u) \right\rangle_I\, dx\, d\tau \\ &= \lambda r \int_0^t \int_\Omega f_{r-1}(u_\delta) \left\langle \mu^0 + \log u_\delta, S\bar{R}(u) \right\rangle_I\, dx\, d\tau. \end{aligned} \tag{4.43}$$

Following the steps of lemma 5 in [10], we can estimate the integral on the r.h.s. of (4.43), i.e.,

$$\begin{aligned} I_{reac}^{(t)} \le \lambda\, r\, C\, \sum_{i=1}^{I} \Big(&\int_0^t \int_\Omega \left(\delta |\mu_i^0| + T|\Omega| \delta |\log \delta|\right)\, dx\, d\tau \\ &+ \delta \int_0^t \int_\Omega (u_i + \delta)\, dx\, d\tau \Big) =: h(t, \delta, u_\delta) \quad \text{for a.e. } t, \end{aligned}$$

where C is independent of λ and u_δ, and all the other factors of $h(t,\delta,u_\delta)$ are bounded and tending to zero as $\delta \longrightarrow 0$ for a.e. t, i.e.,

$$I_{reac}^{(t)} \le h(t,\delta,u_\delta) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0 \quad \text{for a.e. } t. \tag{4.44}$$

From lemma 5 in [10] we get

$$\begin{aligned} I_{diff}^{(t)} =& -\sum_{k=1}^{n} \int_0^t \int_\Omega \left\langle D \frac{\partial}{\partial x_k} u_\delta, \frac{\partial}{\partial x_k} \left(\partial f_r(u_\delta)\right) \right\rangle_I dx\, d\tau \\ =& - r(r-1) D \int_0^t \int_\Omega f_{r-2}(u_\delta) \sum_{k=1}^{n} \left\langle \mu^0 + \log u_\delta, \partial_{x_k} u_\delta \right\rangle_I^2 dx\, d\tau \\ & - rD \int_0^t \int_\Omega f_{r-1}(u_\delta) \sum_{i=1}^{I} \sum_{k=1}^{n} \frac{1}{u_{\delta_i}} \left(\frac{\partial u_{\delta_i}}{\partial x_k} \right)^2 dx\, d\tau \quad \text{for a.e. } t. \end{aligned} \tag{4.45}$$

Both the terms of (4.45) are nonpositive, hence

$$I_{diff}^{(t)} \le 0 \quad \text{for a.e. } t. \tag{4.46}$$

[¶] $p > n+2$ and $u \in \mathcal{F}_p^u$ imply $u \in L^\infty((0,T)\times\Omega)^I$. This gives $SR(u) \in L^p((0,T);L^p(\Omega))^I \hookrightarrow L^p((0,T);H^{1,q}(\Omega)^*)^I$. Recall the definition (2.3) for the continuous embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$ as

$$\langle f, \zeta \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} = \langle f, \zeta \rangle_{L^p(\Omega)\times L^q(\Omega)}, \text{ for } f \in L^p(\Omega) \text{ and } \zeta \in H^{1,q}(\Omega).$$

$$\begin{aligned}
I_{Ex}^{(t)} &= -\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega u_i\partial f_r(u_\delta)_i\,dx\,d\tau\\
&= \kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega r(\delta-u_{\delta_i})f_{r-1}(u_\delta)(\mu_i^0+\log u_{\delta_i})\,dx\,d\tau \quad \text{since } u_{\delta_i}=u_i+\delta\\
&= \delta\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega r(\mu_i^0+\log u_{\delta_i})f_{r-1}(u_\delta)\,dx\,d\tau\\
&\quad + r\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega -u_{\delta_i}(\mu_i^0+\log u_{\delta_i})f_{r-1}(u_\delta)\,dx\,d\tau. \qquad (4.47)
\end{aligned}$$

It can be shown that

$$-u_{\delta_i}(\mu_i^0+\log u_{\delta_i}) \le e^{-(1+\mu_i^0)} \qquad \forall i. \qquad (4.48)$$

We have $\log u_{\delta_i} \le u_{\delta_i} \le g_i(u_{\delta_i})$ and $g_i(u_{\delta_i}) \ge (e-1)e^{-\mu_i^0}$. Choosing a constant $C = \max_{1\le i\le I}\left(1+\left|\mu_i^0\right|e^{-\mu_i^0}(e-1)\right)$, we obtain

$$\mu_i^0+\log u_{\delta_i} \le \mu_i^0+g_i(u_{\delta_i}) \le \left|\mu_i^0\right|+g_i(u_{\delta_i}) \le C\, g_i(u_{\delta_i}). \qquad (4.49)$$

Combining (4.47), (4.48) and (4.49), we get

$$\begin{aligned}
I_{Ex}^{(t)} &\le r\delta\kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega Cg_i(u_{\delta_i})f_{r-1}(u_\delta)\,dx\,d\tau\\
&\quad + \kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega re^{-(1+\mu_i^0)}f_{r-1}(u_\delta)\,dx\,d\tau\\
&\le r\delta\kappa(1-\lambda)C\sum_{i=1}^{I}\int_0^t\int_\Omega g(u_\delta)f_{r-1}(u_\delta)\,dx\,d\tau\\
&\quad + \kappa(1-\lambda)\sum_{i=1}^{I}\int_0^t\int_\Omega r(e(e-1))^{-1}g(u_\delta)f_{r-1}(u_\delta)\,dx\,d\tau, \text{ since } g_i(u_{\delta_i}) \le g(u_\delta)\\
&\le r\,\delta\,\kappa\,I\,C\int_0^t\int_\Omega f_r(u_\delta)\,dx\,d\tau + \kappa\,I\,r\,(e(e-1))^{-1}\int_0^t\int_\Omega f_r(u_\delta)\,dx\,d\tau,\\
&\quad \text{since } 0\le\lambda\le 1 \text{ and } f_r = f_{r-1}g \quad \text{for a.e. } t. \qquad (4.50)
\end{aligned}$$

As $\delta \to 0$, $f_r(u_\delta)$ is bounded in $L^1((0,T)\times\Omega)$. Therefore for a.e. t the first term on the right hand side in (4.50) tends to zero as $\delta \to 0$. Denote the first term by $l(t,u_\delta,\delta)$, then

$$I_{Ex}^{(t)} \le l(t,u_\delta,\delta) + Irk(e(e-1))^{-1}\int_0^t\int_\Omega f_r(u_\delta)\,dx\,d\tau \quad \text{for a.e. } t. \qquad (4.51)$$

Therefore combining (4.39), (4.44), (4.46) and (4.51) we obtain

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u}{\partial\tau}, \partial f_r(u_\delta)\right\rangle_{[H^{1,q}(\Omega)^*]^I\times[H^{1,q}(\Omega)]^I} d\tau\\
&\le h(t,u_\delta,\delta) + l(t,u_\delta,\delta) + Ir\kappa(e(e-1))^{-1}\int_0^t F_r(u_\delta)\,d\tau \quad \text{for a.e. } t,
\end{aligned}$$

where $h(t,\delta,u_\delta)$ and $l(t,u_\delta,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$ for a.e. t. □

Proof of theorem 4.1. Let $u \in \mathcal{F}_p^u$ be the solution of $(P_{M_\lambda}^+)$. Due to lemma 4.1 we know that $u \geq 0$. For any fixed $\delta > 0$, let

$$u_\delta := u + \delta. \tag{4.52}$$

Let $\eta > 0$ and choose a smooth function $\bar{u}_\delta \in C^\infty([0,T] \times \bar{\Omega})^I$ sufficiently close to u_δ such that

$$\bar{u}_\delta \geq \frac{\delta}{2}, \tag{4.53}$$

$$||\partial_t \bar{u}_\delta - \partial_t u_\delta||_{L^p((0,T);H^{1,q}(\Omega)^*)^I} \leq \eta, \tag{4.54}$$

$$|[F_r(u_\delta(t)) - F_r(u_\delta(0))] - [F_r(\bar{u}_\delta(t)) - F_r(\bar{u}_\delta(0))]| \leq \delta, \tag{4.55}$$

$$||\partial f_r(u_\delta) - \partial f_r(\bar{u}_\delta)||_{L^\infty((0,T)\times\Omega)^I} \leq \eta, \tag{4.56}$$

and

$$\eta\, ||\partial f_r(u_\delta)||_{L^q((0,T);H^{1,q}(\Omega))^I} + \eta\, ||\partial_t \bar{u}_\delta||_{L^1((0,T)\times\Omega)^I} \leq \delta. \tag{4.57}$$

Then for a.e. t

$$\begin{aligned}
&\Big| \int_0^t \langle \partial f_r(u_\delta), \partial_\tau u_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I}\, d\tau \\
&- \int_0^t \langle \partial f_r(\bar{u}_\delta), \partial_\tau \bar{u}_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I}\, d\tau \Big| \\
\leq\ & \sum_{i=1}^I \int_0^T \left| \langle \partial f_r(u_\delta)_i - \partial f_r(\bar{u}_\delta)_i, \partial_\tau \bar{u}_{\delta_i} \rangle_{H^{1,q}(\Omega) \times H^{1,q}(\Omega)^*} \right| d\tau \\
&+ \sum_{i=1}^I \int_0^T \left| \langle \partial_\tau u_{\delta_i} - \partial_\tau \bar{u}_{\delta_i}, \partial f_r(u_\delta)_i \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} \right| d\tau \\
\leq\ & \sum_{i=1}^I \int_0^T \left| \langle \partial f_r(u_\delta)_i - \partial f_r(\bar{u}_\delta)_i, \partial_\tau \bar{u}_{\delta_i} \rangle_{L^q(\Omega) \times L^p(\Omega)} \right| d\tau \\
&+ \sum_{i=1}^I \int_0^T \left| \langle \partial_\tau u_{\delta_i} - \partial_\tau \bar{u}_{\delta_i}, \partial f_r(u_\delta)_i \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} \right| d\tau \\
\leq\ & \sum_{i=1}^I \Big[||\partial f_r(u_\delta)_i - \partial f_r(\bar{u}_\delta)_i||_{L^\infty((0,T)\times\Omega)}\, ||\partial_t \bar{u}_{\delta_i}||_{L^1((0,T)\times\Omega)} \\
&+ ||\partial_t u_{\delta_i} - \partial_t \bar{u}_{\delta_i}||_{L^p((0,T);H^{1,q}(\Omega)^*)}\, ||\partial f_r(u_\delta)_i||_{L^q((0,T);H^{1,q}(\Omega))} \Big] \\
\leq\ & \sum_{i=1}^I \Big[\eta\, ||\partial_t \bar{u}_{\delta_i}||_{L^1((0,T)\times\Omega)} + \eta\, ||\partial f_r(u_\delta)_i||_{L^q((0,T);H^{1,q}(\Omega))} \Big] \\
\leq\ & \sum_{i=1}^I \delta \ = \ \delta I \quad \text{by (4.57).}
\end{aligned} \tag{4.58}$$

For the smooth function $\bar{u}_\delta$, we have

$$\begin{aligned} F_r(\bar{u}_\delta(t)) - F_r(\bar{u}_\delta(0)) &= \int_0^t \frac{d}{d\tau}\left(F_r(\bar{u}_\delta(\tau))\right) d\tau \\ &= \int_0^t \int_\Omega \frac{\partial}{\partial \tau} f_r(\bar{u}_\delta)\, dx\, d\tau \\ &= \int_0^t \left\langle \partial f_r(\bar{u}_\delta), \frac{\partial \bar{u}_\delta}{\partial \tau} \right\rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I} d\tau. \end{aligned} \tag{4.59}$$

This implies

$$\begin{aligned} & \left| F_r(u_\delta(t)) - F_r(u_\delta(0)) - \int_0^t \langle \partial f_r(u_\delta), \partial_\tau u_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I} \; d\tau \right| \\ \leq \quad & |[F_r(u_\delta(t)) - F_r(u_\delta(0))] - [F_r(\bar{u}_\delta(t)) - F_r(\bar{u}_\delta(0))]| \\ & + \left| \int_0^t \langle \partial f_r(\bar{u}_\delta), \partial_\tau \bar{u}_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I} \; d\tau \right. \\ & \left. - \int_0^t \langle \partial f_r(u_\delta), \partial_\tau u_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I} \; d\tau \right| \quad \text{by (4.59)} \\ \leq \quad & \delta + \delta I \quad \text{by} \quad (4.55) \quad \text{and} \quad (4.58) \\ = \quad & (I+1)\delta. \end{aligned}$$

This gives

$$\begin{aligned} & |F_r(u_\delta(t)) - F_r(u_\delta(0))| \\ & \leq (I+1)\delta + \int_0^t \langle \partial f_r(u_\delta), \partial_\tau u_\delta \rangle_{[H^{1,q}(\Omega)]^I \times [H^{1,q}(\Omega)^*]^I} \; d\tau \\ & \leq (I+1)\delta + h(t, u_\delta, \delta) + l(t, u_\delta, \delta) \\ & \quad + I \; r\kappa \; (e(e-1))^{-1} \int_0^t F_r(u_\delta)\, d\tau \quad \text{ by lemma 4.5 and for a.e. } t, \end{aligned}$$

where $h(t, u_\delta, \delta)$ and $l(t, u_\delta, \delta)$ as $\delta \to 0$ for a.e. t. Therefore from the continuity of F_r, we obtain

$$F_r(u(t)) \leq F_r(u(0)) + Ir\kappa(e(e-1))^{-1} \int_0^t F_r(u)\, d\tau \quad \text{for a.e. } t.$$

Gronwall's inequality gives

$$F_r(u(t)) \leq e^{Ir\kappa(e(e-1))^{-1}t} F_r(u(0)) \qquad \text{for all } r \text{ and for a.e. } t.$$

This completes the proof. □

An immediate consequence of theorem 4.1 is the following corollary which gives the *a-priori* estimates (global in time) of the solution of $(P^+_{M_\lambda})$.

Corollary 4.1. *Let $p > n+2, r \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. Suppose that $u \in \mathcal{F}^u_p$ is the solution of the problem $(P^+_{M_\lambda})$, then the following estimates hold:*

$$|||u(t)|||_{L^r(\Omega)^I} \leq C < \infty \quad \text{for all } r \text{ and for a.e. } t, \tag{4.60}$$

$$\textit{and} \quad |||u(t)|||_{L^\infty(\Omega)^I} \leq C < \infty \quad \text{for a.e. } t, \tag{4.61}$$

where C is independent of i, u and t.

Proof. From lemma 2.3, it follows that for $p > n+2$, $u_0 \in L^\infty(\Omega)^I$. For the problem $(P^+_{M_\lambda})$, $u(0) = \lambda u_0$. Therefore from theorem 4.1, for a.e. $0 \le t \le T$, we have

$$F_r(u(t)) \le e^{Ir\kappa(e(e-1))^{-1}\,t}\, F_r(u(0)) \qquad \text{for all } r \text{ and for a.e. } t.$$
$$\implies \int_\Omega u_i^r(t,x)\,dx \le e^{Ir\kappa(e(e-1))^{-1}t} \int_\Omega f_r(\lambda u_0(x))\,dx \qquad \text{for all } r \text{ and for a.e. } t. \tag{4.62}$$

From proposition 4.2, we have

$$f_r(\lambda u_0) \le C\left(1 + |\lambda u_0|_I^{r(1+\alpha)}\right), \tag{4.63}$$

where $\alpha > 0$ and C are independent of δ, λ and u_i. Combining (4.62) and (4.63), we obtain

$$\begin{aligned}||u_i(t)||^r_{L^r(\Omega)} &\le C\, e^{Ir\kappa(e(e-1))^{-1}t} \int_\Omega (1 + |u_0|_I^{r(1+\alpha)})\,dx \quad \text{since} \quad 0 \le \lambda \le 1\\ &\le C \int_\Omega \left(1 + \left(I^{\frac{1}{2}} ||u_0||_{L^\infty(\Omega)^I}\right)^{r(1+\alpha)}\right) dx \quad \text{for a.e. } t,\end{aligned}$$

i.e.,

$$\sum_{i=1}^{I} ||u_i(t)||^r_{L^r(\Omega)} \le I\, C \int_\Omega \left(1 + \left(I^{\frac{1}{2}} ||u_0||_{L^\infty(\Omega)^I}\right)^{r(1+\alpha)}\right) dx < \infty \quad \text{for a.e. } t.$$

This gives (4.60). Note that C in (4.60) depends on r. Again from theorem 4.1, for a.e. $0 \le t \le T$, we have

$$F_r(u(t)) \le e^{Ir\kappa(e(e-1))^{-1}t} F_r(\lambda u_0) \quad \text{for all } r \text{ and for a.e. } t.$$

Proceeding as above, we obtain

$$\begin{aligned}&||u_i(t)||^r_{L^r(\Omega)} \le C\, e^{Ir\kappa(e(e-1))^{-1}t} \left|\left|1 + |u_0|_I^{(1+\alpha)}\right|\right|^r_{L^r(\Omega)}\\ \implies &||u_i(t)||_{L^r(\Omega)} \le \left(C\, e^{Ir\kappa(e(e-1))^{-1}t}\right)^{\frac{1}{r}} \left|\left|1 + |u_0|_I^{(1+\alpha)}\right|\right|_{L^r(\Omega)}\\ &\qquad \le \sup_{r\in\mathbb{N}} \left(C\, e^{Ir\kappa(e(e-1))^{-1}t}\right)^{\frac{1}{r}} \left|\left|1 + |u_0|_I^{(1+\alpha)}\right|\right|_{L^r(\Omega)}\\ &\qquad \forall\, i \text{ and } r, \text{ and for a.e. } t.\end{aligned}$$

Taking limit sup as $r \to \infty$ on both sides, we obtain

$$\begin{aligned}||u_i(t)||_{L^\infty(\Omega)} &\le C \left|\left|1 + |u_0|_I^{(1+\alpha)}\right|\right|_{L^\infty(\Omega)}\\ &\le C\left(1 + \left(I^{\frac{1}{2}}\, |||u_0|||_{L^\infty(\Omega)^I}\right)^{(1+\alpha)}\right) < \infty \;\; \forall i \text{ and for a.e. } t.\end{aligned} \tag{4.64}$$

By (4.64), $|||u(t)|||_{L^\infty(\Omega)^I} = \max_{1\le i\le I} ||u_i(t)||_{L^\infty(\Omega)} < \infty$ for a.e. t which is (4.61). $\square$

Corollary 4.2. *Let $p > n+2$, $r \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. Then there exists a positive constant C (depending only on r, T, $|\Omega|$ and I but independent of λ and u) such that any arbitrary solution $u \in \mathcal{F}_p^u$ of the problem $(P_{M_\lambda}^+)$ satisfies*

$$|||u|||_{\mathcal{F}_p^u} \leq C.$$

Proof. Choosing $r \in \mathbb{N}$ sufficiently large in corollary 4.1 and application of Hölder's inequality shows the r.h.s., $\lambda S\bar{R}(u) + \lambda\kappa u$, of $(P_{M_\lambda}^+)$ is in $L^p((0,T);L^p(\Omega))^I$. Since $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$, $\lambda S\bar{R}(u) + \lambda\kappa u \in L^p((0,T);H^{1,q}(\Omega)^*)^I$.

The reformulation of (4.17)-(4.19) is given by

$$\frac{du(t)}{dt} + Au(t) = f(t), \tag{4.65}$$

$$u(0) = \lambda u_0, \tag{4.66}$$

where $f(t) = \lambda S\bar{R}(u(t)) + \lambda\kappa u(t)$ and $\kappa > 0$. f is in $L^p((0,T);H^{1,q}(\Omega)^*)^I$. The operator A is defined as in remark 4.1. Moreover, by assumption (2.24) $u_0 \in \mathcal{X}_p^u$. The operator A has the *maximal parabolic regularity* on $[H^{1,q}(\Omega)^*]^I$. Therefore from the theory of linear evolution equation (cf. theorem 2.5 [17]), there exists a $\tilde{C} > 0$ such that[‖]

$$\begin{aligned} |||u|||_{\mathcal{F}_p^u} &\leq \tilde{C}\left(||\lambda u_0||_{\mathcal{X}_p^u} + \left|\left|\lambda S\bar{R}(u) + \lambda\kappa u\right|\right|_{L^p((0,T);H^{1,q}(\Omega)^*)^I}\right) \\ &\leq \tilde{C}\ \left(||u_0||_{\mathcal{X}_p^u} + \left|\left|S\bar{R}(u)\right|\right|_{L^p((0,T);H^{1,q}(\Omega)^*)^I} + \kappa\,||u||_{L^p((0,T);H^{1,q}(\Omega)^*)^I}\right) \\ &=:\ C < \infty, \end{aligned}$$

where C is independent of λ and u. □

4.1.2. Compactness and continuity of Z_1

Lemma 4.6. *The fixed point operator Z_1 is continuous and compact.*

Proof. Here we will only show the continuity of Z_1 as the compactness follows with similar arguments. Let $(v_n)_{n\geq 1}$ be a sequence in $\mathcal{F}_p^u$ converging to a limit $v \in \mathcal{F}_p^u$. From lemma 2.2, $(v_n)_{n\geq 1}$ is convergent to v in $[L^\infty((0,T)\times\Omega)]^I$. This implies that $(SR(v_n) + \kappa v_n)_{n\geq 1}$ is convergent to $SR(v) + \kappa v$ in $[L^p((0,T)\times\Omega)]^I$. Due to the continuous embedding $L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*$, $(SR(v_n) + \kappa v_n)_{n\geq 1}$ is convergent to $SR(v) + \kappa v$ in $[L^p((0,T);H^{1,q}(\Omega)^*)]^I$. From the linear theory of evolution equation (cf. theorem 2.5 in [17]), we conclude that the map Z_1 is continuous. □

Proof of theorem 2.1. Applying Schaefer's fixed point theorem, thanks to corollary 4.2 and lemma 4.6, we get the existence of at least one fixed point, i.e., existence of at least one solution of the problem (P_M^+) and this solution solves (P^+). Due to lemma 4.1, this solution is also a solution of (P). Now we prove the uniqueness of the solution of (P). Let u_1 and $u_2 \in \mathcal{F}_p^u$ be two solutions of the problem (P), where $u_1 \neq u_2$. Then we have

$$\frac{\partial u_k}{\partial t} - \nabla\cdot D\nabla u_k = SR(u_k) \qquad \text{in } (0,T)\times\Omega, \tag{4.67}$$

$$u_k(0,x) = u_0(x) \qquad \text{in } \Omega, \tag{4.68}$$

$$-D\nabla u_k \cdot \vec{n} = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{4.69}$$

[‖]Note that $0 \leq \lambda \leq 1$.

for $k = 1, 2$. Set $\bar{u} = u_1 - u_2$. Taking the difference and using $\bar{u}_i$ as the test function in the $i-th$ PDE, we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^t \frac{d}{d\tau}\,||\bar{u}_i(\tau)||^2_{L^2(\Omega)}\;d\tau + D\int_0^t ||\nabla\bar{u}_i(\tau)||^2_{L^2(\Omega)}\;d\tau\\ \leq&\frac{1}{2}\int_0^t\Big[||SR(u_1(\tau))_i - SR(u_2(\tau))_i||^2_{L^2(\Omega)} + ||\bar{u}_i(\tau)||^2_{L^2(\Omega)}\Big]\;d\tau.\end{aligned}$$

Expanding the term $R_j(u_1)-R_j(u_2)$, each term in $R_j(u_1)-R_j(u_2)$ contains a factor of the type $u_{1_l} - u_{2_l}$, whereas all the other factors are bounded in $L^\infty((0,T)\times\Omega)$, therefore we obtain

$$||\bar{u}_i(t)||^2_{L^2(\Omega)} \leq C\int_0^t \sum_{i=1}^{I} ||\bar{u}_i(\tau)||^2_{L^2(\Omega)}\;d\tau$$

$$\text{i.e.,}\qquad |||\bar{u}(t)|||^2_{L^2(\Omega)^I} \leq C\int_0^t |||\bar{u}(\tau)|||^2_{L^2(\Omega)^I}\;d\tau \quad \text{for a.e. } t.$$

Gronwall's inequality gives

$$\begin{aligned}&|||\bar{u}(t)|||^2_{L^2(\Omega)^I} = 0 \quad \text{for a.e. } t,\\ \Longrightarrow\qquad &u_1 = u_2.\end{aligned}$$

Hence the solution exists uniquely. □

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