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THE SHAPE OF LIMIT CYCLES FOR A CLASS OF QUINTIC POLYNOMIAL DIFFERENTIAL SYSTEMS*

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Abstract We consider the problem of finding limit cycles for a class of quintic polynomial differential systems and their global shape in the plane. An answer to this problem can be given using the averaging theory. More precisely, we analyze the global shape of the limit cycles which bifurcate from a Hopf bifurcation and periodic orbits of the linear center $\dot{x} = -y,\ \dot{y} = x$, respectively.

Keywords Limit cycles, quintic differential systems, Abelian equations, averaging methods.

MSC(2000) 34C25, 34C23, 34D10, 34C29.

1. Introduction

It is well known that Hilbert presented a list consisting of 23 mathematical problems in 1900. The second part of the 16th problem appear to be one of the most persistent in that list, second only to the 8th problem, the Riemann conjecture. Limit cycle theory plays the key role in the second part of the 16th problem. The study of limit cycles mainly consists of two aspect: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in the system (e.g. bifurcation). Since the research on the exact number of the limit cycles and relative positions for a polynomial system is difficult, it is still an open problem even for the case $n = 2$. However, for a planar polynomial differential systems the number of limit cycles is finite, see [6, 7, 13]. A classical way for studying the number of limit cycles which bifurcate from the periodic orbits of a period annulus of a center is the *averaging method*, see [1, 5, 11, 17, 18]. This method additionally can give the shape of the bifurcated limit cycles up to any order of the perturbation parameter. Moreover, the averaging method has been applied to solve the center problem, see for instance García & Giné [8]. In the past years the question of the shape of limit cycles of polynomial differential systems has attracted increasing interest. Prohens & Torregrosa [19] studied the shape of limit cycles bifurcating from the period annulus of a class of radial Hamiltonians. Giacomini etc. [9] studied the global shape of the bifurcated limit cycles from an analytic Hamiltonian center when

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it was perturbed by an arbitrary analytic vector field. Then Giacomini etc. [10] continue their study on the shape of the limit cycles which bifurcate from non-Hamiltonian centers under small analytic perturbations. Llibre [15] showed the existence of bifurcating limit cycles for quadratic systems and obtained their global shape. Similar studies for the cubic systems and the quartic systems can be found in [16,2], respectively.

In this paper, we will use the averaging method to consider the global shape of the limit cycles for a class quintic polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= P_1(x,y) + P_5(x,y), \\ \dot{y} &= Q_1(x,y) + Q_5(x,y), \end{aligned} \tag{1.1}$$

where P_n and Q_n denote homogeneous polynomials of degree n. In fact, due to the determinant of coefficients

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & -3 & -1 & 1 & -3 & 5 \\ -10 & -2 & 2 & 2 & -2 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 & -2 & -2 & 2 & 2 & -10 \\ 5 & -3 & 1 & 1 & -3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & -1 & -3 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 & -2 & 2 & 2 & -2 & -10 \\ 10 & 2 & 2 & -2 & -2 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -3 & 1 & 1 & -3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \neq 0,$$

it is clear that the expression of system (1.1) is equivalent to the form of

$$\begin{aligned} \dot{x} = \; & \lambda x - y + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)x^5 + (-5\beta_1 - 3\beta_2 - \beta_3 + \beta_4 \\ & + 3\beta_5 + 5\beta_6)x^4 y + (-10\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_5 - 10\alpha_6)x^3y^2 \\ & + (-10\beta_1 - 2\beta_2 - 2\beta_3 + 2\beta_4 + 2\beta_5 - 10\beta_6)x^2y^3 + (5\alpha_1 - 3\alpha_2 + \alpha_3 \\ & + \alpha_4 - 3\alpha_5 + 5\alpha_6)xy^4 + (-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6)y^5, \\ \dot{y} = \; & x + \lambda y + (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)x^5 + (5\alpha_1 + 3\alpha_2 + \alpha_3 - \alpha_4 \\ & - 3\alpha_5 - 5\alpha_6)x^4 y + (-10\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4 - 2\beta_5 - 10\beta_6)x^3y^2 \\ & + (10\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\alpha_4 - 2\alpha_5 + 10\alpha_6)x^2y^3 + (5\beta_1 - 3\beta_2 + \beta_3 \\ & + \beta_4 - 3\beta_5 + 5\beta_6)xy^4 + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6)y^5 \end{aligned} \tag{1.2}$$

when the origin is a focus. In order to simplify our calculations, we shall study the global shape of the limit cycles which born in Hopf bifurcation at the origin of system (1.2) and also the global shape of the limit cycles of system (1.2) which bifurcate from periodic orbits of the center $\dot{x} = -y,\ \dot{y} = x$.

We can say that the averaging method gives a quantitative relation between the solutions of some non-autonomous differential system and the solutions of the averaged differential system, which is an autonomous one. It is necessary that the system (1.2) is transformed into a particular case of an Abelian differential equation to use this method. We will state the specific results about Abelin equation and Averaging theory in Section 3 and 4, respectively.

2. Statement of the main results

We use the second order approximation of the averaging method to investigate the the periodic solutions which bifurcate from the origin of the quintic system

$$\begin{aligned}\dot{x} = \;& -y + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)x^5 + (-5\beta_1 - 3\beta_2 - \beta_3 + \beta_4 \\ & +3\beta_5 + 5\beta_6)x^4y + (-10\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_5 - 10\alpha_6)x^3y^2 \\ & +(-10\beta_1 - 2\beta_2 - 2\beta_3 + 2\beta_4 + 2\beta_5 - 10\beta_6)x^2y^3 + (5\alpha_1 - 3\alpha_2 + \alpha_3 \\ & +\alpha_4 - 3\alpha_5 + 5\alpha_6)xy^4 + (-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6)y^5 + \varepsilon^2\alpha_0 x, \\ \dot{y} = \;& x + (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)x^5 + (5\alpha_1 + 3\alpha_2 + \alpha_3 - \alpha_4 \\ & -3\alpha_5 - 5\alpha_6)x^4y + (-10\beta_1 - 2\beta_2 + 2\beta_3 + 2\beta_4 - 2\beta_5 - 10\beta_6)x^3y^2 \\ & +(10\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\alpha_4 - 2\alpha_5 + 10\alpha_6)x^2y^3 + (5\beta_1 - 3\beta_2 + \beta_3 v \\ & +\beta_4 - 3\beta_5 + 5\beta_6)xy^4 + (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6)y^5 + \varepsilon^2\alpha_0 y,\end{aligned} \tag{2.1}$$

at $\varepsilon = 0$. By the same method, we study the limit cycles which bifurcate from the periodic orbits of linear center $\dot{x} = -y,\ \dot{y} = x$ when we perturb it inside the quintic systems

$$\begin{aligned}\dot{x} = \;& -y + \varepsilon[(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)x^5 + (-5b_1 - 3b_2 - b_3 + b_4 \\ & +3b_5 + 5b_6)x^4y + (-10a_1 - 2a_2 + 2a_3 + 2a_4 - 2a_5 - 10a_6)x^3y^2 \\ & +(-10b_1 - 2b_2 - 2b_3 + 2b_4 + 2b_5 - 10b_6)x^2y^3 + (5a_1 - 3a_2 + a_3 \\ & +a_4 - 3a_5 + 5a_6)xy^4 + (-b_1 + b_2 - b_3 + b_4 - b_5 + b_6)y^5] + \varepsilon^2\alpha_0 x, \\ \dot{y} = \;& x + \varepsilon[(b_1 + b_2 + b_3 + b_4 + b_5 + b_6)x^5 + (5a_1 + 3a_2 + a_3 - a_4 \\ & -3a_5 - 5a_6)x^4y + (-10b_1 - 2b_2 + 2b_3 + 2b_4 - 2b_5 - 10b_6)x^3y^2 \\ & +(10a_1 + 2a_2 + 2a_3 - 2a_4 - 2a_5 + 10a_6)x^2y^3 + (5b_1 - 3b_2 + b_3 \\ & +b_4 - 3b_5 + 5b_6)xy^4 + (a_1 - a_2 + a_3 - a_4 + a_5 - a_6)y^5] + \varepsilon^2\alpha_0 y.\end{aligned} \tag{2.2}$$

The main results are the following theorems.

Theorem 2.1. *Consider system* (2.1) *with* $\alpha_1 = -\frac{8}{5}\alpha_3$ *and* $\alpha_0 \cdot M > 0$*, where*

$$\begin{aligned}M = \;& -105\alpha_1\beta_1 + 40(\alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_4\beta_1 + \alpha_1\beta_4) + 64(\alpha_3\beta_3 + \alpha_2\beta_4 + \alpha_4\beta_2) \\ & +24(\alpha_1\beta_5 + \alpha_5\beta_1) - 80(\alpha_2\beta_1 + \alpha_1\beta_2),\end{aligned}$$

there is a limit cycle bifurcating from the origin for $\varepsilon = 0$*. Moreover, for any* $\varepsilon > 0$ *sufficiently small, the expression of this limit cycle in polar coordinates* (r, θ) *is given by*

$$\begin{aligned}r(\theta, \varepsilon) = \;& \varepsilon^{1/4}\sigma_0^{1/4} + \tfrac{1}{4}\varepsilon^{5/4}\sigma_0^{5/4}b(\theta) + \tfrac{1}{4}\varepsilon^{5/4}\sigma_0^{5/4}c(\theta) + \tfrac{5}{16}\varepsilon^{9/4}\sigma_0^{9/4}b(\theta)c(\theta) \\ & +\tfrac{5}{32}\varepsilon^{9/4}\sigma_0^{9/4}b^2(\theta) - \tfrac{3}{32}\varepsilon^{9/4}\sigma_0^{9/4}c^2(\theta) + o(\varepsilon)^{13/4},\end{aligned} \tag{2.3}$$

where

$$\sigma_0 = \sqrt{\frac{64\alpha_0}{M}},$$

$$\begin{aligned} b(\theta) = \;& (\tfrac{15}{8}\alpha_1 + \alpha_2 - \alpha_4)\sin(2\theta) + (-\tfrac{5}{8}\beta_1 + \beta_2 + \beta_4)\cos(2\theta) + (\alpha_1 - \alpha_5)\sin(4\theta) \\ & + (-\tfrac{1}{4}\beta_1 + \beta_5)\cos(4\theta) + (-\tfrac{5}{8}\alpha_1 - \alpha_6)\sin(6\theta) + (\tfrac{5}{8}\beta_1 + \beta_6)\cos(6\theta) \\ & + (\tfrac{5}{8}\beta_1 + \beta_3), \\ c(\theta) = \;& (-\tfrac{25}{8}\alpha_1 + \alpha_2 + 3\alpha_4)\sin(2\theta) + (\tfrac{35}{8}\beta_1 + \beta_2 - 3\beta_4)\cos(2\theta) \\ & + (-\tfrac{5}{4}\alpha_1 + 2\alpha_5)\sin(4\theta) + (\tfrac{5}{4}\beta_1 - 2\beta_5)\cos(4\theta) + (\tfrac{25}{24}\alpha_1 + \tfrac{5}{3}\alpha_6)\sin(6\theta) \\ & + (-\tfrac{25}{24}\beta_1 - \tfrac{5}{3}\beta_6)\cos(6\theta). \end{aligned}$$

Theorem 2.2. *Consider system* (2.2) *with* $a_1 = \frac{8}{5}a_3$ *and* $\alpha_0 \cdot N > 0$, *where*

$$\begin{aligned} N = \;& -105a_1b_1 + 40(a_1b_3 + a_3b_1 + a_4b_1 + a_1b_4) + 64(a_3b_3 + a_2b_4 + a_4b_2) \\ & + 24(a_1b_5 + a_5b_1) - 80(a_2b_1 + a_1b_2), \end{aligned}$$

there is a limit cycle, which is bifurcated from a circular periodic solution of linear center with radius $\rho_0 = \sqrt{\frac{64\alpha_0}{N}}$ *of the linear center for* $\varepsilon = 0$. *Moreover, for* $\varepsilon > 0$ *sufficiently small, the expression of this limit cycle in polar coordinates* (r, θ) *is given by*

$$\begin{aligned} r(\theta, \varepsilon) = \;& \rho_0^{1/4} + \tfrac{1}{4}\varepsilon\rho_0^{5/4}b(\theta) + \tfrac{1}{4}\varepsilon\rho_0^{5/4}c(\theta) + \tfrac{5}{16}\varepsilon^2\rho_0^{9/4}b(\theta)c(\theta) \\ & + \tfrac{5}{32}\varepsilon^2\rho_0^{9/4}b^2(\theta) - \tfrac{3}{32}\varepsilon^2\rho_0^{9/4}c^2(\theta) + o(\varepsilon)^3, \end{aligned} \tag{2.4}$$

where $b(\theta)$ *and* $c(\theta)$ *are define as in theorem 2.1 with only formal changes to substitute* α_i *and* β_i *with* a_i *and* b_i, *respectively.*

3. The system in polar coordinates and the Abelian equation

The aim of this section is to present the Abelian equation theory as it was obtained in Li etc. [14]. We deal with the class of real planar polynomial differential systems of the form

$$\dot{x} = \lambda x - y + P_n(x, y), \qquad \dot{y} = x + \lambda y + Q_n(x, y), \tag{3.1}$$

where P_n and Q_n are homogeneous polynomial of degree n. Using polar coordinates (r, θ), system (3.1) becomes

$$\dot{r} = \lambda r + f(\theta)r^n, \qquad \dot{\theta} = 1 + g(\theta)r^{n-1}, \tag{3.2}$$

where

$$\begin{aligned} f(\theta) &= \cos\theta P_n\ (\cos\theta,\ \sin\theta) + \sin\theta Q_n\ (\cos\theta,\ \sin\theta), \\ g(\theta) &= \cos\theta Q_n\ (\cos\theta,\ \sin\theta) - \sin\theta P_n\ (\cos\theta,\ \sin\theta). \end{aligned}$$

We remark that f and g are homogeneous polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. It is clear that the expression of system (3.1) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\lambda r + f(\theta)r^n}{1 + g(\theta)r^{n-1}} \tag{3.3}$$

in the region $S = \{(r, \theta) : 1 + g(\theta)r^{n-1} > 0\}$. Since any periodic orbit surrounding the origin of system (3.1) is in S, but does not intersect the curve $\pounds : \dot{\theta} = 0$ (for more details see the Appendix of Carbonell & Llibre [3]), there exist periodic orbits of equation (3.3).

The change of variables $(r,\theta) \to (\rho,\theta)$ with

$$\rho = \frac{r^{n-1}}{1+g(\theta)r^{n-1}} \tag{3.4}$$

is a diffeomorphism from the region S into its image. As far as we know, the first to use this transformation was Cherkas in [4]. If we express Eq.(3.3) with respect to the variable ρ, we obtain

$$\frac{d\rho}{d\theta} = [(n-1)g(\theta)(\lambda g(\theta) - f(\theta))]\rho^3 + [(n-1)(f(\theta) - 2\lambda g(\theta)) - g'(\theta)]\rho^2 + (n-1)\lambda\rho, \tag{3.5}$$

which is an Abelian differential equation. In fact we have proved the following result.

Lemma 3.1. *The function $r = r(\theta)$ is a periodic solution of system* (3.2) *surrounding the origin if and only if $\rho(\theta) = r(\theta)^{n-1}/(1+g(\theta)r(\theta)^{n-1})$ is a periodic solution of the Abelian differential equation* (3.5).

4. Second-order approximation in general averaging

In this section, we give the necessary results on the theory of averaging that we will apply to prove theorem 2.1 and 2.2. First we need to introduce the following definition.

Definition 4.1. Let $f(t,y)$ be a continuous function in $[0,T]\times\mathbb{D}$, with $\mathbb{D}\subseteq\mathbb{R}$ and T-period in t, then we define the average function of f as

$$f^{(o)} = \frac{1}{T}\int_0^T f(t,y)dt. \tag{4.1}$$

Lemma 4.1. *Consider the initial value problems :*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{4.2}$$

and

$$\dot{y} = \varepsilon f^{(0)}(y) + \varepsilon^2 f^{(10)}(y) + \varepsilon g^{(0)}(y), \quad y(0) = x_0, \tag{4.3}$$

where $x, y,\ x_0 \in \mathbb{D} \subset \mathbb{R}$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, $f(t,x)$, $g(t,x)$ and $h(t,x,\varepsilon)$ are T-periodic in t, and

$$f^1(t,x) = \frac{\partial f}{\partial x}y^1(t,x) - \frac{\partial y^1}{\partial x}f^{(0)}(x),$$

where

$$y^1(t,x) = \int_0^t (f(s,x) - f^{(0)}(x))ds + w(x),$$

with $w(x)$ a $\mathbb{C}^1$ function such that the averaged function of y^1 is zero. $f^{(0)}$, $f^{(10)}$ and $g^{(0)}$ denote the averaged functions of f, f^1 and g, respectively. Furthermore, suppose that:

(a) $\partial f^{(0)}/\partial y$*, g and h are Lypschitz in x and all these functions are defined, continuous;*

(b) $|h(t,x,\varepsilon)|$ *is uniformly bounded by a constant in* $[0,\frac{l}{\varepsilon})\times\mathbb{D}\times(0,\varepsilon_0]$*;*

(c) $y(t)$ *belongs to* $\mathbb{D}$ *on the time-scale* $\frac{1}{\varepsilon}$*.*

Then

(1) $x(t)=y(t)+\varepsilon y^1(t,y(t))+\circ(\varepsilon^2)$ *on the time-scale* $\frac{1}{\varepsilon}$*;*

(2) If $f^{(0)}(y)\equiv 0$ *and p is an equilibrium point of the averaged system* (4.3) *such that*

$$\frac{\partial}{\partial y}(f^{(10)}(y)+g^{(0)}(y))|_{y=p}\neq 0, \tag{4.4}$$

then there exists a T-periodic solution $\phi(t,\varepsilon)$ *of Eq.*(4.2) *which is close to p such that* $\phi(t,\varepsilon)\to p$ *as* $\varepsilon\to 0$*;*

(3) If $f^{(0)}(y)\equiv 0$ *and* (4.4) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)\to p$ *of Eq. (4.2) in the space of* (t,x) *is asymptotically stable for sufficiently small* ε*. If* (4.4) *is positive, then this periodic solution is unstable.*

For a proof see Llibre [15], Sanders etc. [20] and Verhulst [22].

5. The proof of Theorems

Considering the planar polynomial system (1.2), we can rewrite the system into the form:

$$\dot{z}=A_0z+A_1z^5+A_2z^4\overline{z}+A_3z^3\overline{z}^2+A_4z^2\overline{z}^3+A_5z\overline{z}^4+A_6\overline{z}^5, \tag{5.1}$$

where $z=x+yi$, $A_0=\lambda+i$, $A_n=\alpha_n+\beta_n i$, $(n=1,2,\cdots,6)$. Direct computations partially provide the first four Lyapunov constants V_4, V_6, V_8, V_{10} of the quintic system (1.2). These Lyapunov constants are given by

$$V_4=\lambda,\quad V_6=2\alpha_3,\quad V_8=0,\quad V_{10}=-2(\alpha_5\beta_1+\alpha_1\beta_5+\alpha_4\beta_2+\alpha_2\beta_4).$$

The above results are also referred to the formula of [12]. It follows easily from the arguments of [21] that a limit cycle which bifurcates from the origin if $V_8=0$, $|V_6|\ll|V_{10}|$ and $V_6V_{10}<0$. Now, in order to study this Hopf bifurcation, we need to transform the system (1.2) into an Abelian equation. Using polar coordinates (r,θ), system (1.2) becomes

$$\dot{r}=\lambda r+a(\theta)r^5,\quad \dot{\theta}=1+b(\theta)r^4, \tag{5.2}$$

where

$$\begin{aligned}
a(\theta)=\;&(-\tfrac{15}{8}\beta_1-\beta_2+\beta_4)\sin(2\theta)+(-\tfrac{5}{8}\alpha_1+\alpha_2+\alpha_4)\cos(2\theta)\\
&+(-\beta_1+\beta_5)\sin(4\theta)+(-\tfrac{1}{4}\alpha_1+\alpha_5)\cos(4\theta)+(\tfrac{5}{8}\beta_1+\beta_6)\sin(6\theta)\\
&+(\tfrac{5}{8}\alpha_1+\alpha_6)\cos(6\theta)+(\tfrac{5}{8}\alpha_1+\alpha_3),\\
b(\theta)=\;&(\tfrac{15}{8}\alpha_1+\alpha_2-\alpha_4)\sin(2\theta)+(-\tfrac{5}{8}\beta_1+\beta_2+\beta_4)\cos(2\theta)+(\alpha_1-\alpha_5)\sin(4\theta)\\
&+(-\tfrac{1}{4}\beta_1+\beta_5)\cos(4\theta)+(-\tfrac{5}{8}\alpha_1-\alpha_6)\sin(6\theta)+(\tfrac{5}{8}\beta_1+\beta_6)\cos(6\theta)\\
&+(\tfrac{5}{8}\beta_1+\beta_3),
\end{aligned}$$

denote polynomials of degree 6 with respect to the variables $\cos\theta$ and $\sin\theta$. We observe that the system (5.2) is equivalent to

$$\frac{dr}{d\theta} = \frac{\lambda r + a(\theta)r^5}{1 + b(\theta)r^4}. \tag{5.3}$$

Let

$$\rho = \frac{r^4}{1 + b(\theta)r^4},$$

we obtain the following Abelian differential equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= A(\theta)\rho^3 + B(\theta)\rho^2 + 4\lambda\rho \\ &= [4\lambda b(\theta)^2 - 4a(\theta)b(\theta)]\rho^3 + [4a(\theta) - 8\lambda b(\theta) - b'(\theta)]\rho^2 + 4\lambda\rho, \end{aligned} \tag{5.4}$$

with $b'(\theta)$ denote the derivative of b with respect to θ, and $A(\theta)$ and $B(\theta)$ are trigonometric polynomial with respect to $\cos\theta$ and $\sin\theta$ of degree 12 and 6, respectively.

By the second order approximation of the averaging method, we shall study the periodic solutions of the system (5.4) and consequently the periodic orbits surrounding the origin for the quintic system (1.2). Then a good asymptotic estimation of the shape for the limit cycle which bifurcates from the origin is obtained.

Proof of Theorem 2.1. Taking $\lambda = \alpha_0\varepsilon^2$ with $\varepsilon > 0$ sufficiently small, we study the Hopf bifurcation at origin of the system (1.2) given by $V_8 = 0$, $\mid V_6 \mid \ll \mid V_{10} \mid$ and $V_6V_{10} < 0$. This way leads to system (2.1) with associated Abelian differential equation (5.4). In addition, the perturbation of system (1.2) is relevant to ε^2, so we apply the second order approximation of averaging method to Abelian equation (5.4). Now, using the change of variables $\rho = \sigma\varepsilon$, Eq.(5.4) becomes

$$\frac{d\sigma}{d\theta} = \varepsilon f(\theta,\sigma) + \varepsilon^2 g(\theta,\sigma) + \varepsilon^3 h(\theta,\sigma,\varepsilon), \tag{5.5}$$

where $f(\theta,\sigma) = (4a(\theta) - b'(\theta))\sigma^2$, $g(\theta,\sigma) = 4(\alpha_0 - a(\theta)b(\theta)\sigma^2)\sigma$, $h(\theta,\sigma,\varepsilon) = 4(-2 + b(\theta)\varepsilon\sigma)\alpha_0 b(\theta)\sigma^2$. The functions f, g and h satisfy all the assumptions of Lemma (4.1) with $T = 2\pi$. According to $\alpha_1 = -\frac{8}{5}\alpha_3$, we compute and obtain these following integrals using Maple

$$\begin{aligned} &f^{(0)}(\sigma) = 0, \\ &y^1(\theta,\sigma) = [(-\tfrac{25}{8}\alpha_1 + \alpha_2 + 3\alpha_4)\sin(2\theta) + (\tfrac{35}{8}\beta_1 + \beta_2 - 3\beta_4)\cos(2\theta) \\ &\qquad + (-\tfrac{5}{4}\alpha_1 + 2\alpha_5)\sin(4\theta) + (\tfrac{5}{4}\beta_1 - 2\beta_5)\cos(4\theta) + (\tfrac{25}{24}\alpha_1 + \tfrac{5}{3}\alpha_6)\sin(6\theta) \\ &\qquad + (-\tfrac{25}{24}\beta_1 - \tfrac{5}{3}\beta_6)\cos(6\theta)]\sigma^2, \\ &f^{(10)}(\sigma) = 0, \\ &g^{(0)}(\sigma) = \tfrac{1}{16}(64\alpha_0 - M\sigma^2)\sigma, \end{aligned}$$

with

$$\begin{aligned} M = \; & -105\alpha_1\beta_1 + 40(\alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_4\beta_1 + \alpha_1\beta_4) + 64(\alpha_3\beta_3 + \alpha_2\beta_4 + \alpha_4\beta_2) \\ & +24(\alpha_1\beta_5 + \alpha_5\beta_1) - 80(\alpha_2\beta_1 + \alpha_1\beta_2), \end{aligned}$$

where we have used the notation of Section 4. Since $\alpha_0 \cdot M > 0$, Lemma (4.1) implies that Eq. (5.5) has a periodic solution $\sigma(\theta,\varepsilon)$ near to $\sigma_0 = \sqrt{\frac{64\alpha_0}{M}}$, and satisfying $\sigma(\theta,\varepsilon) \to \sigma_0$ as $\varepsilon \to 0$, where σ_0 is an equilibrium point of the averaged equation of Eq. (5.5). In fact, using Lemma (4.1), we have

$$\sigma(\theta,\varepsilon) = \sigma_0 + \varepsilon y^1(\theta,\sigma_0) + o(\varepsilon^2).$$

We infer that Eq. (5.4) has a 2π-periodic solution near to $\rho(\theta,\varepsilon) = \sigma(\theta,\varepsilon)\varepsilon$ such that $\rho_0(\theta,\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, returning to the Eq. (5.3), we see that it has the 2π-periodic solution

$$r(\theta,\varepsilon) = \left(\frac{\varepsilon\sigma(\theta,\varepsilon)}{1-\varepsilon b(\theta)\sigma(\theta,\varepsilon)}\right)^{1/4}, \tag{5.6}$$

with $r(\theta,\varepsilon) \to 0$ as $\varepsilon \to 0$. So this periodic solution is produced by a Hopf's bifurcation at the origin of system (1.2) when $\varepsilon = 0$. And expression (2.3) is obtained when we expend (5.6) in power series of ε. The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. Now, we investigate limit cycles which bifurcate from the linear center $\dot{x} = -y,\ \dot{y} = x$, when it is perturbed inside the class of quintic systems (1.2). Let $\lambda = \alpha_0\varepsilon^2$, $\alpha_i = \varepsilon a_i$ and $\beta_i = \varepsilon b_i$, we obtain the quintic system (2.2). Then the corresponding Abelian equation (5.4) becomes

$$\frac{d\rho}{d\theta} = \varepsilon f(\theta,\rho) + \varepsilon^2 g(\theta,\rho) + \varepsilon^3 h(\theta,\rho,\varepsilon), \tag{5.7}$$

where $f(\theta,\rho) = (4\bar{a}(\theta) - \bar{b}'(\theta))\rho^2$, $g(\theta,\rho) = 4(\alpha_0 - \bar{a}(\theta)\bar{b}(\theta)\rho^2)\rho$ and $h(\theta,\rho,\varepsilon) = 4(-2+\bar{b}(\theta)\rho\varphi)\alpha_0 b(\theta)\rho^2$, $\bar{a}(\theta) = a(\theta)/\varepsilon$ and $\bar{b}(\theta) = b(\theta)/\varepsilon$. Note that Eq. (5.5) and (5.7) are exactly the same with only formal changed to substitute σ, a and b by ρ, $\bar{a}$ and $\bar{b}$, respectively. Taking into account these changes, we see that the results obtained for σ remain also valid for ρ. Hence

$$\rho(\theta,\varepsilon) = \rho_0 + \varepsilon y^1(\theta,\rho_0) + o(\varepsilon^2),$$

where

$$\rho_0 = \sqrt{\frac{64\alpha_0}{N}},$$

with

$$\begin{aligned} N = &-105a_1b_1 + 40(a_1b_3 + a_3b_1 + a_4b_1 + a_1b_4) + 64(a_3b_3 + a_2b_4 + a_4b_2) \\ &+ 24(a_1b_5 + a_5b_1) - 80(a_2b_1 + a_1b_2), \\ y^1(\theta,\rho_0) = &[(-\tfrac{25}{8}a_1 + a_2 + 3a_4)\sin(2\theta) + (\tfrac{35}{8}b_1 + b_2 - 3b_4)\cos(2\theta) \\ &+ (-\tfrac{5}{4}a_1 + 2a_5)\sin(4\theta) + (\tfrac{5}{4}b_1 - 2b_5)\cos(4\theta) + (\tfrac{25}{24}a_1 + \tfrac{5}{3}a_6)\sin(6\theta) \\ &+ (-\tfrac{25}{24}b_1 - \tfrac{5}{3}b_6)\cos(6\theta)]\rho_0^2. \end{aligned}$$

Returning to the equation in polar coordinate (5.3), it has the 2π-periodic solution

$$r(\theta,\varepsilon) = \left(\frac{\rho(\theta,\varepsilon)}{1-\varepsilon\bar{b}(\theta)\rho(\theta,\varepsilon)}\right)^{1/4}, \tag{5.8}$$

such that $r(\theta,\varepsilon) \to \sqrt[4]{\rho_0}$ as $\varepsilon \to 0$. So this periodic solution is produced by a bifurcation of the circular orbit of system (2.2) when $\varepsilon = 0$. Expanding Eq.(5.8) in power series of ε, we obtain expression (2.4). The Theorem 2.2 is proved. □

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