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EXISTENCE OF SOLUTIONS FOR A ONE-DIMENSIONAL ALLEN-CAHN EQUATION

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Abstract Our aim in this paper is to prove the existence and uniqueness of solutions for a one-dimensional Allen-Cahn type equation based on a modification of the Ginzburg-Landau free energy proposed in [10]. In particular, the free energy contains an additional term called Willmore regularization and takes into account anisotropy effects.

Keywords Allen-Cahn equation, Willmore regularization, anisotropy effects, well-posedness.

MSC(2000) 35B45, 35K55.

1. Introduction

The Allen-Cahn equation,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = 0, \tag{1.1}$$

where u is the order parameter and $f(s) = s^3 - s$, describes important processes related with phase separation in binary alloys, namely, the ordering of atoms in a lattice (see [1]). This equation is obtained by considering the Ginzburg-Landau free energy,

$$\Psi_{\mathrm{GL}} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u))\, dx, \tag{1.2}$$

where Ω is the domain occupied by the material and $F(s) = \frac{1}{4}(s^2-1)^2$. Assuming a relaxation dynamics, i.e., writing

$$\frac{\partial u}{\partial t} = -\frac{D\Psi_{\mathrm{GL}}}{Du}, \tag{1.3}$$

where $\frac{D}{Du}$ denotes a variational derivative, we obtain (1.1).

In [10] (see also [2]), the authors introduced the following modification of the Ginzburg-Landau free energy:

$$\Psi_{\mathrm{AGL}} = \int_\Omega (\delta(\frac{\nabla u}{|\nabla u|})(\frac{1}{2}|\nabla u|^2 + F(u)) + \frac{\beta}{2}\omega^2)\, dx,\ \beta > 0, \tag{1.4}$$

$$\omega = -\Delta u + f(u), \tag{1.5}$$

where $G(u) = \frac{1}{2}\omega^2$ is called nonlinear Willmore regularization, β is a small regularization parameter and the function δ accounts for anisotropy effects. The Willmore

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regularization is relevant, e.g., in determining the equilibrium shape of a crystal in its own liquid matrix, when anisotropy effects are strong. Indeed, in that case, the equilibrium interface may not be a smooth curve, but may present facets and corners with slope discontinuities (see, e.g., [8]), which can lead to an ill-posed problem and requires regularization.

The Allen-Cahn equation associated with (1.4) has been studied in [5] in the particular cases $\delta \equiv 1$ (isotropic case) and $\delta \equiv -1$ (in that case, Ψ_{AGL} is also called functionalized Cahn-Hilliard energy in [7]). In particular, well-posedness results have been obtained. The Cahn-Hilliard equation associated with (1.4) (obtained by writing $\frac{\partial u}{\partial t} = \Delta \frac{D\Psi_{\mathrm{AGL}}}{Du}$) has been studied in [4], again, in the isotropic case $\delta \equiv 1$; we also refer the reader to [2] and [11] for numerical studies.

In one space dimension, i.e., taking $\Omega = (0, L)$, and setting β equal to one, (1.4) reads

$$\Psi_{\mathrm{AGL}} = \int_0^L (\delta(\frac{\frac{\partial u}{\partial x}}{|\frac{\partial u}{\partial x}|})(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u)) + \frac{1}{2}\omega^2)\, dx. \tag{1.6}$$

We actually consider the following natural regularization of Ψ_{AGL}:

$$\Psi_{\mathrm{RAGL}} = \int_0^L (\delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u)) + \frac{1}{2}\omega^2)\, dx,\ \ \epsilon > 0. \tag{1.7}$$

In that case, we have, formally,

$$\begin{aligned}
&D\Psi_{\mathrm{RAGL}}\\
= &\int_0^L (\delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})(\frac{\partial u}{\partial x}\frac{\partial Du}{\partial x} + f(u)Du) + \omega D\omega)\, dx\\
&+\epsilon \int_0^L \delta'(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{1}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u))\frac{\partial Du}{\partial x}\, dx\\
= &\int_0^L (\delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})(\frac{\partial u}{\partial x}\frac{\partial Du}{\partial x} + f(u)Du) + \omega f'(u)Du - \omega\frac{\partial^2 Du}{\partial x^2})\, dx\\
&+\epsilon \int_0^L \delta'(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{1}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u))\frac{\partial Du}{\partial x}\, dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{D\Psi_{\mathrm{RAGL}}}{Du} = \ &-\frac{\partial}{\partial x}(\delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{\partial u}{\partial x}) + \delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})f(u)\\
&-\epsilon\frac{\partial}{\partial x}(\delta'(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{1}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u)))\\
&+\omega f'(u) - \frac{\partial^2 \omega}{\partial x^2}.
\end{aligned} \tag{1.8}$$

In this paper, we will consider the simplest case $\delta(s) = s$ (note that $\frac{\frac{\partial u}{\partial x}}{|\frac{\partial u}{\partial x}|} = 1$ if

$\frac{\partial u}{\partial x} > 0$ and $\frac{\frac{\partial u}{\partial x}}{|\frac{\partial u}{\partial x}|} = -1$ if $\frac{\partial u}{\partial x} < 0$), hence,

$$\begin{aligned} \frac{D\Psi_{\mathrm{RAGL}}}{Du} = \; & -\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u) \\ & -\epsilon\frac{\partial}{\partial x}(\frac{1}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u))) + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2}. \end{aligned} \tag{1.9}$$

Assuming again a relaxation dynamics,

$$\frac{\partial u}{\partial t} = -\frac{D\Psi_{RAGL}}{Du},$$

we finally obtain the following (regularized) anisotropic Allen-Cahn system:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u)$$

$$-\epsilon\frac{\partial}{\partial x}(\frac{1}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}(\frac{1}{2}(\frac{\partial u}{\partial x})^2 + F(u))) + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2} = 0, \tag{1.10}$$

$$\omega = -\frac{\partial^2 u}{\partial x^2} + f(u). \tag{1.11}$$

Our aim in this paper is to prove the existence and uniqueness of solutions to (1.10)-(1.11).

2. A priori estimates

We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u) - \frac{\epsilon}{2}\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}$$

$$-\epsilon\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2} = 0, \tag{2.1}$$

$$\omega = -\frac{\partial^2 u}{\partial x^2} + f(u), \tag{2.2}$$

$$u(0) = u(L) = \omega(0) = \omega(L) = 0, \tag{2.3}$$

$$u|_{t=0} = u_0, \tag{2.4}$$

where

$$f(s) = s^3 - s, \;\; F(s) = \frac{1}{4}(s^2-1)^2. \tag{2.5}$$

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$, and we denote by $\|\cdot\|_X$ the norm in the Banach space X.

Throughout the paper, the same letter c (and, sometimes, c') denotes constants which may vary from line to line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

We multiply (2.1) by u and have, integrating over $(0, L)$ and by parts and owing to (2.2),

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + ((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}, \frac{\partial u}{\partial x})) + ((\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u), u))$$

$$+\frac{\epsilon}{2}((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial u}{\partial x})) + \epsilon((\frac{F(u)}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial u}{\partial x})) \tag{2.6}$$

$$+\|\omega\|^2 + \int_0^L (uf'(u)f(u) - f(u)^2)\,dx + ((uf''(u)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})) = 0.$$

We note that

$$\int_0^L (uf'(u)f(u) - f(u)^2)\,dx \geq c_0\|f(u)\|^2 - c_1,\ c_0 > 0, \tag{2.7}$$

and

$$uf''(u) \geq 0. \tag{2.8}$$

Furthermore,

$$|((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}, \frac{\partial u}{\partial x}))| \leq \|\frac{\partial u}{\partial x}\|^2, \tag{2.9}$$

$$|((\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u), u))| \leq \|f(u)\|\|u\| \leq \frac{c_0}{2}\|f(u)\|^2 + c\|u\|^2, \tag{2.10}$$

$$\frac{\epsilon}{2}|((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial u}{\partial x}))| \leq \frac{\epsilon}{2} \tag{2.11}$$

and

$$\epsilon|((\frac{F(u)}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial u}{\partial x}))| \leq \int_\Omega |F(u)|\,dx \leq \frac{c_0}{2}\|f(u)\|^2 + c. \tag{2.12}$$

We thus deduce from (2.6)-(2.12) that

$$\frac{d}{dt}\|u\|^2 + 2\|\omega\|^2 \leq c\|u\|^2_{H^1(0,L)} + c'. \tag{2.13}$$

We then note that

$$f' \geq -c_2,\ c_2 \geq 0, \tag{2.14}$$

which yields

$$\|\omega\|^2 \geq \|\frac{\partial^2 u}{\partial x^2}\|^2 + \|f(u)\|^2 - 2c_2\|\frac{\partial u}{\partial x}\|^2. \tag{2.15}$$

We thus obtain

$$\frac{d}{dt}\|u\|^2 + 2\|\frac{\partial^2 u}{\partial x^2}\|^2 + 2\|f(u)\|^2 \leq c\|u\|^2_{H^1(0,L)} + c'. \tag{2.16}$$

Employing the interpolation inequality

$$\|u\|_{H^1(0,L)} \leq c\|u\|^{\frac{1}{2}}\|\frac{\partial^2 u}{\partial x^2}\|^{\frac{1}{2}}, \tag{2.17}$$

we finally find

$$\frac{d}{dt}\|u\|^2+\|\frac{\partial^2 u}{\partial x^2}\|^2+\|f(u)\|^2\le c\|u\|^2+c'. \tag{2.18}$$

We then multiply multiply (2.1) by $\frac{\partial u}{\partial t}$ and obtain, owing to (2.2),

$$\|\frac{\partial u}{\partial t}\|^2-((\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac12}},\frac{\partial u}{\partial t}))+((\frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac12}}f(u),\frac{\partial u}{\partial t}))$$

$$-\frac{\epsilon}{2}((\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}},\frac{\partial u}{\partial t}))-\epsilon((\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}},\frac{\partial u}{\partial t}))+\frac12\frac{d}{dt}\|\omega\|^2=0. \tag{2.19}$$

We have

$$\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac12}}=\frac{2\epsilon\frac{\partial u}{\partial x}+(\frac{\partial u}{\partial x})^3}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}}\frac{\partial^2 u}{\partial x^2},$$

so that

$$|((\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac12}},\frac{\partial u}{\partial t}))|\le 3\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\le\frac{1}{16}\|\frac{\partial u}{\partial t}\|^2+c\|\frac{\partial^2 u}{\partial x^2}\|^2. \tag{2.20}$$

Furthermore,

$$|((\frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac12}}f(u),\frac{\partial u}{\partial t}))|\le\|f(u)\|\|\frac{\partial u}{\partial t}\|\le\frac{1}{16}\|\frac{\partial u}{\partial t}\|^2+c\|f(u)\|^2. \tag{2.21}$$

Then,

$$\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}}=(\frac{2\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}}-\frac{3(\frac{\partial u}{\partial x})^3}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac52}})\frac{\partial^2 u}{\partial x^2},$$

which yields

$$\frac{\epsilon}{2}|((\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}},\frac{\partial u}{\partial t}))|\le\frac52\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\le\frac18\|\frac{\partial u}{\partial t}\|^2+c\|\frac{\partial^2 u}{\partial x^2}\|^2. \tag{2.22}$$

Finally,

$$\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}}=\frac{f(u)\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}}-\frac{3F(u)\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac52}}\frac{\partial^2 u}{\partial x^2},$$

hence, owing to Agmon's inequality (see, e.g., [9]) and (2.17),

$$\begin{aligned}
&\epsilon|((\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac32}},\frac{\partial u}{\partial t}))|\\
\le\;& \|f(u)\|\|\frac{\partial u}{\partial t}\|+3\epsilon^{-1}\|F(u)\|_{L^\infty(0,L)}\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\\
\le\;& \|f(u)\|\|\frac{\partial u}{\partial t}\|+c\epsilon^{-1}(\|u\|^4_{L^\infty(0,L)}+1)\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\\
\le\;& \|f(u)\|\|\frac{\partial u}{\partial t}\|+c\epsilon^{-1}(\|u\|^2\|\frac{\partial u}{\partial x}\|^2+1)\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\\
\le\;& \|f(u)\|\|\frac{\partial u}{\partial t}\|+c\epsilon^{-1}(\|u\|^3\|\frac{\partial^2 u}{\partial x^2}\|+1)\|\frac{\partial^2 u}{\partial x^2}\|\|\frac{\partial u}{\partial t}\|\\
\le\;& \frac14\|\frac{\partial u}{\partial t}\|^2+c\epsilon^{-2}(\|u\|^6\|\frac{\partial^2 u}{\partial x^2}\|^2+1)\|\frac{\partial^2 u}{\partial x^2}\|^2+c'\|f(u)\|^2.
\end{aligned} \tag{2.23}$$

It thus follows from (2.19)-(2.23) that

$$\frac{d}{dt}\|\omega\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leq c\epsilon^{-2}(\|u\|^6\|\frac{\partial^2 u}{\partial x^2}\|^2 + 1)(\|\frac{\partial^2 u}{\partial x^2}\|^2 + \|f(u)\|^2). \tag{2.24}$$

Furthermore, as above and employing (2.17),

$$\begin{aligned}\|\omega\|^2 &\geq \|\frac{\partial^2 u}{\partial x^2}\|^2 + \|f(u)\|^2 - 2c_0\|\frac{\partial u}{\partial x}\|^2 \\ &\geq \frac{1}{2}(\|\frac{\partial^2 u}{\partial x^2}\|^2 + \|f(u)\|^2) - c\|u\|^2.\end{aligned} \tag{2.25}$$

3. Existence and uniqueness of solutions

We have the

Theorem 3.1. *We assume that* $u_0 \in H^2(0,L) \cap H_0^1(0,L)$. *Then,* (2.1)-(2.4) *possesses a unique solution* u *such that* $u \in L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L))$, $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(0,L))$ *and* $f(u) \in L^\infty(0,T; L^2(0,L))$, $\forall T > 0$.

Proof. a) Existence:

The proof of existence is based on a standard Galerkin scheme and the a priori estimates derived in the previous section.

A weak (variational) formulation for (2.1)-(2.4) reads

$$\begin{aligned}&\frac{d}{dt}((u,v)) + ((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}, \frac{\partial v}{\partial x})) + ((\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u), v)) \\ &+\frac{\epsilon}{2}((\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial v}{\partial x})) + \epsilon((\frac{F(u)}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial v}{\partial x})) \\ &+((\omega f'(u), v)) - ((\omega, \frac{\partial^2 v}{\partial x^2})) = 0, \ \forall v \in H^2(0,L) \cap H_0^1(0,L),\end{aligned} \tag{3.1}$$

$$((u,w)) = ((f(u),w)) + ((\omega, \frac{\partial^2 w}{\partial x^2})), \ \forall w \in H^2(0,L) \cap H_0^1(0,L), \tag{3.2}$$

$$u|_{t=0} = u_0. \tag{3.3}$$

Let v_1, v_2, ... be an orthonormal (in $L^2(0,L)$) and orthogonal (in $H_0^1(0,L)$) family associated with the eigenvalues $0 < \lambda_1 \leq \lambda_2 ...$ of the operator $-\frac{\partial^2}{\partial x^2}$ associated with Dirichlet boundary conditions. We set $V_m = \text{Span}(v_1, ..., v_m)$ and consider the approximated problem

Find $(u_m, \omega_m) : [0,T] \to V_m \times V_m$ such that

$$\begin{aligned}&\frac{d}{dt}((u_m,v)) + ((\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}}, \frac{\partial v}{\partial x})) + ((\frac{\frac{\partial u_m}{\partial x}}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}} f(u_m), v)) \\ &+\frac{\epsilon}{2}((\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial v}{\partial x})) + \epsilon((\frac{F(u_m)}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}}, \frac{\partial v}{\partial x})) \\ &+((\omega_m f'(u_m), v)) - ((\omega_m, \frac{\partial^2 v}{\partial x^2})) = 0, \ \forall v \in V_m,\end{aligned} \tag{3.4}$$

$$((u_m,w)) = ((f(u_m),w)) + ((\omega_m, \frac{\partial^2 w}{\partial x^2})), \ \forall w \in V_m, \tag{3.5}$$

$$u_m|_{t=0} = u_{0,m}, \tag{3.6}$$

where $u_{0,m} = P_m u$, P_m being the orthogonal projector from $L^2(0, L)$ onto V_m (for the L^2-norm).

The existence of a local (in time) solution is standard, as we have to solve a (continuous) finite system of ODE's. It then follows from the a priori estimates derived in the previous section that this solution is global.

In particular, it follows from (2.18) (which holds at the approximated level) that u_m is bounded in $L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$, independently of m. Having this, it follows from (2.24)-(2.25) that u_m is bounded in $L^\infty(0, T; H^2(0, L))$, $f(u_m)$ is bounded in $L^\infty(0, T; L^2(0, L))$ and $\frac{\partial u_m}{\partial t}$ is bounded in $L^2(0, T; L^2(0, L))$.

It then follows from classical Aubin-Lions compactness results that, up to a subsequence which we do not relabel (also note that $\frac{\partial}{\partial t}\frac{\partial u_m}{\partial x}$ is bounded in $L^2(0, T; H^{-1}(0, L))$),

$$u_m \to u \text{ in } L^\infty(0, T; H^2(0, L))\text{weak star, } L^2(0, T; L^2(0, L))\text{and a.e.,}$$

$$f(u_m) \to f(u)\text{in } L^2(0, T; L^2(0, L))\text{and a.e.}$$

(indeed, $\|f(u_m) - f(u)\| \le c(\|u_m\|^2_{H^1(0,L)} + \|u\|^2_{H^1(0,L)} + 1)\|u_m - u\|$) and

$$\frac{\partial u_m}{\partial x} \to \frac{\partial u}{\partial x} \text{ in } L^\infty(0, T; H^1(0, L)) \text{ weak star, } L^2(0, T; L^2(0, L)) \text{ and a.e..}$$

We then need to pass to the limit in the nonlinear terms. We have

$$|\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}}| \le |\frac{\partial u_m}{\partial x}|.$$

Therefore, since $\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon+(\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}} \to \frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}$ a.e. and $|\frac{\partial u_m}{\partial x}| \le g \in L^2((0, L) \times (0, T))$ a.e. (up again to a subsequence which we do not relabel), we deduce from Lebesgue's theorem that $\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon+(\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}} \to \frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}$ in $L^2(0, T; L^2(0, L))$ (here, we have used the fact that $L^2(0, T; L^2(0, L))$ is isometric to $L^2((0, L) \times (0, T))$). Similarly,

$$|\frac{\frac{\partial u_m}{\partial x}}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}} f(u_m)| \le |f(u_m)|,$$

which yields that $\frac{\frac{\partial u_m}{\partial x}}{(\epsilon+(\frac{\partial u_m}{\partial x})^2)^{\frac{1}{2}}} f(u_m) \to \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u)$ in $L^2(0, T; L^2(0, L))$, and

$$|\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}}| \le \epsilon^{-\frac{1}{2}},$$

so that $\frac{(\frac{\partial u_m}{\partial x})^2}{(\epsilon+(\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}} \to \frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}$ in $L^2(0, T; L^2(0, L))$. Furthermore,

$$|\frac{F(u_m)}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}}| \le \epsilon^{-\frac{3}{2}}|F(u_m)| \le c\epsilon^{-\frac{3}{2}}(|u_m|^4 + 1),$$

so that

$$|\frac{F(u_m)}{(\epsilon + (\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}}| \le c\epsilon^{-\frac{3}{2}}(|f(u_m)|^{\frac{4}{3}} + 1),$$

hence $\frac{F(u_m)}{(\epsilon+(\frac{\partial u_m}{\partial x})^2)^{\frac{3}{2}}} \to \frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}$ in $L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(0,L))$. Finally, noting that $\omega_m \to \omega$ in $L^2(0,T;L^2(\Omega))$ weak, we have, for $\varphi \in \mathcal{C}([0,L]\times[0,T])$,

$$\begin{aligned}
&|\int_0^T \int_0^L (\omega_m f'(u_m) - \omega f'(u))\varphi\,dx\,dt| \\
\leq\ & |\int_0^T \int_0^L (\omega_m - \omega) f'(u)\varphi\,dx\,dt| + |\int_0^T \int_0^L \omega_m (f'(u_m) - f'(u))\varphi\,dx\,dt| \\
\leq\ & |\int_0^T \int_0^L (\omega_m - \omega) f'(u)\varphi\,dx\,dt| + c\|u_m - u\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

which finishes the proof of the passage to the limit, hence the existence of a solution.

b) Uniqueness:

Let u_1 and u_2 be two solutions to (2.1)-(2.3) (ω_1 and ω_2 being defined as in (2.2)) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. Then, setting $u = u_1 - u_2$, $\omega = \omega_1 - \omega_2$ and $u_0 = u_{0,1} - u_{0,2}$, we have

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\varphi_1(\frac{\partial u_1}{\partial x}) - \varphi_1(\frac{\partial u_2}{\partial x})) + \varphi_2(\frac{\partial u_1}{\partial x})f(u_1) - \varphi_2(\frac{\partial u_1}{\partial x})f(u_2) \\
&-\frac{\epsilon}{2}\frac{\partial}{\partial x}(\varphi_3(\frac{\partial u_1}{\partial x}) - \varphi_3(\frac{\partial u_2}{\partial x})) - \epsilon\frac{\partial}{\partial x}(\varphi_4(\frac{\partial u_1}{\partial x})F(u_1) - \varphi_4(\frac{\partial u_2}{\partial x})F(u_2)) \\
&+\omega_1 f'(u_1) - \omega_2 f'(u_2) - \frac{\partial^2 \omega}{\partial x^2} = 0, \qquad (3.7)
\end{aligned}$$

$$\omega = -\frac{\partial^2 u}{\partial x^2} + f(u_1) - f(u_2), \qquad (3.8)$$

$$u(0) = u(L) = \omega(0) = \omega(L) = 0, \qquad (3.9)$$

$$u|_{t=0} = u_0, \qquad (3.10)$$

where

$$\varphi_1(s) = \frac{s^2}{(\epsilon+s^2)^{\frac{1}{2}}},\ \varphi_2(s) = \frac{s}{(\epsilon+s^2)^{\frac{1}{2}}},$$

$$\varphi_3(s) = \frac{s^2}{(\epsilon+s^2)^{\frac{3}{2}}},\ \varphi_4(s) = \frac{1}{(\epsilon+s^2)^{\frac{3}{2}}}.$$

We multiply (3.7) by u and obtain, owing to (3.8),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2 + ((\varphi_1(\frac{\partial u_1}{\partial x}) - \varphi_1(\frac{\partial u_2}{\partial x}), \frac{\partial u}{\partial x})) + ((\varphi_2(\frac{\partial u_1}{\partial x})f(u_1) - \varphi_2(\frac{\partial u_2}{\partial x})f(u_2), u)) \\
&+\frac{\epsilon}{2}((\varphi_3(\frac{\partial u_1}{\partial x}) - \varphi_3(\frac{\partial u_2}{\partial x}), \frac{\partial u}{\partial x})) + \epsilon((\varphi_4(\frac{\partial u_1}{\partial x})F(u_1) - \varphi_4(\frac{\partial u_2}{\partial x})F(u_2), \frac{\partial u}{\partial x})) \\
&+((\omega_1 f'(u_1) - \omega_2 f'(u_2), u)) + \|\frac{\partial^2 u}{\partial x^2}\|^2 = 0. \qquad (3.11)
\end{aligned}$$

We have

$$\begin{aligned}
&|((\varphi_1(\frac{\partial u_1}{\partial x}) - \varphi_1(\frac{\partial u_2}{\partial x}), \frac{\partial u}{\partial x}))| \\
\leq\ & \int_0^L \int_0^1 |\varphi_1'(\tau\frac{\partial u_1}{\partial x} + (1-\tau)\frac{\partial u_2}{\partial x}|\,d\tau|\frac{\partial u}{\partial x}|^2\,dx,
\end{aligned}$$

where

$$\varphi_1'(s) = \frac{2\epsilon s + s^3}{(\epsilon + s^2)^{\frac{3}{2}}}$$

satisfies

$$|\varphi_1'(s)| \leq 3, \;\; s \in \mathbb{R},$$

so that

$$|((\varphi_1(\frac{\partial u_1}{\partial x}) - \varphi_1(\frac{\partial u_2}{\partial x}), \frac{\partial u}{\partial x}))| \leq 3\|\frac{\partial u}{\partial x}\|^2. \tag{3.12}$$

Furthermore,

$$\begin{aligned} & |((\varphi_2(\frac{\partial u_1}{\partial x})f(u_1) - \varphi_2(\frac{\partial u_2}{\partial x})f(u_2), u))| \\ \leq \;& |(((\varphi_2(\frac{\partial u_1}{\partial x}) - \varphi_2(\frac{\partial u_2}{\partial x}))f(u_1), u))| + |((\varphi_2(\frac{\partial u_2}{\partial x})(f(u_1) - f(u_2)), u))| \\ \leq \;& \int_0^L \int_0^1 |\varphi_2'(\tau\frac{\partial u_1}{\partial x} + (1-\tau)\frac{\partial u_2}{\partial x}|\, d\tau |f(u_1)||\frac{\partial u}{\partial x}||u|\, dx \\ & + |((\varphi_2(\frac{\partial u_2}{\partial x})(f(u_1) - f(u_2)), u))|. \end{aligned}$$

Noting that

$$|\varphi_2(s)| \leq 1, \;\; s \in \mathbb{R},$$

and that

$$\varphi_2'(s) = \frac{\epsilon}{(\epsilon + s^2)^{\frac{3}{2}}},$$

so that

$$|\varphi_2'(s)| \leq \epsilon^{-\frac{1}{2}}, \;\; s \in \mathbb{R},$$

it follows that

$$\begin{aligned} & |((\varphi_2(\frac{\partial u_1}{\partial x})f(u_1) - \varphi_2(\frac{\partial u_2}{\partial x})f(u_2), u))| \\ \leq \;& \epsilon^{-\frac{1}{2}} Q(T, \|u_{0,1}\|_{H^2(0,L)}, \|u_{0,2}\|_{H^2(0,L)})\|\frac{\partial u}{\partial x}\|^2. \end{aligned} \tag{3.13}$$

Here, we have used the fact, owing to the continuous embedding $H^1(0,L) \subset \mathcal{C}([0,L])$, $\|f^{(i)}(w)\|_{L^\infty(0,L)} \leq Q(\|w\|_{H^1(0,L)}) \leq Q(\|w\|_{H^2(0,L)})$, $i = 0,\ 1$, $\forall w \in H^2(0,L)$. Similarly,

$$\begin{aligned} & \frac{\epsilon}{2}|((\varphi_3(\frac{\partial u_1}{\partial x}) - \varphi_3(\frac{\partial u_2}{\partial x}), \frac{\partial u}{\partial x}))| \\ \leq \;& \frac{\epsilon}{2}\int_0^L \int_0^1 |\varphi_3'(\tau\frac{\partial u_1}{\partial x} + (1-\tau)\frac{\partial u_2}{\partial x}|\, d\tau |\frac{\partial u}{\partial x}|^2\, dx, \end{aligned}$$

where

$$\varphi_3'(s) = \frac{2s}{(\epsilon + s^2)^{\frac{3}{2}}} - \frac{3s^3}{(\epsilon + s^2)^{\frac{5}{2}}}$$

satisfies

$$|\varphi_3'(s)| \leq 5\epsilon^{-1}, \;\; s \in \mathbb{R},$$

so that

$$\frac{\epsilon}{2}|((\varphi_3(\frac{\partial u_1}{\partial x})-\varphi_3(\frac{\partial u_2}{\partial x}),\frac{\partial u}{\partial x}))|\le\frac{5}{2}\|\frac{\partial u}{\partial x}\|^2. \tag{3.14}$$

Then,

$$\begin{aligned}&\epsilon|((\varphi_4(\frac{\partial u_1}{\partial x})F(u_1)-\varphi_4(\frac{\partial u_2}{\partial x})F(u_2),\frac{\partial u}{\partial x}))|\\ \le\;&\epsilon\int_0^L\int_0^1|\varphi_4'(\tau\frac{\partial u_1}{\partial x}+(1-\tau)\frac{\partial u_2}{\partial x}|\,d\tau|F(u_1)||\frac{\partial u}{\partial x}|^2\,dx\\ &+\epsilon|((\varphi_4(\frac{\partial u_2}{\partial x})(F(u_1)-F(u_2)),\frac{\partial u}{\partial x}))|.\end{aligned}$$

Noting that

$$|\varphi_4(s)|\le\epsilon^{-\frac{3}{2}},\;\; s\in\mathbb{R},$$

and that

$$\varphi_4'(s)=-\frac{3s}{(\epsilon+s^2)^3},$$

so that

$$|\varphi_4'(s)|\le 3\epsilon^{-\frac{5}{2}},\;\; s\in\mathbb{R},$$

we find, proceeding as in (3.13),

$$\begin{aligned}&\epsilon|((\varphi_4(\frac{\partial u_1}{\partial x})F(u_1)-\varphi_4(\frac{\partial u_2}{\partial x})F(u_2),\frac{\partial u}{\partial x}))|\\ \le\;&\epsilon^{-\frac{3}{2}}Q(T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})\|\frac{\partial u}{\partial x}\|^2.\end{aligned} \tag{3.15}$$

Now,

$$\begin{aligned}&|((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))|\\ \le\;&|((\omega f'(u_1),u))|+|((\omega_2(f'(u_1)-f'(u_2)),u))|\\ \le\;&Q(T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})(\|\omega\|\|u\|+\|\omega_2\|\|u\|^2_{L^4(0,L)})\\ \le\;&Q(T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})(\|\frac{\partial^2 u}{\partial x^2}\|\|u\|+\|\frac{\partial u}{\partial x}\|^2)\\ \le\;&\frac{1}{4}\|\frac{\partial^2 u}{\partial x^2}\|^2+Q(T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})\|\frac{\partial u}{\partial x}\|^2.\end{aligned} \tag{3.16}$$

Finally,

$$\begin{aligned}&|((f(u_1)-f(u_2),\frac{\partial^2 u}{\partial x^2}))|\\ \le\;&\frac{1}{4}\|\frac{\partial^2 u}{\partial x^2}\|^2+Q(T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})\|\frac{\partial u}{\partial x}\|^2.\end{aligned} \tag{3.17}$$

We thus deduce from (3.11)-(3.17) that

$$\frac{d}{dt}\|u\|^2+\|\frac{\partial^2 u}{\partial x^2}\|^2\le Q(\epsilon^{-1},T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})\|\frac{\partial u}{\partial x}\|^2,$$

which yields, employing the interpolation inequality (2.17),

$$\frac{d}{dt}\|u\|^2\le Q(\epsilon^{-1},T,\|u_{0,1}\|_{H^2(0,L)},\|u_{0,2}\|_{H^2(0,L)})\|u\|^2, \tag{3.18}$$

hence the uniqueness, as well as the continuous dependence with respect to the L^2-norm. □

Remark 3.1. We can more generally consider the free energy (1.7), i.e., the Allen-Cahn system

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{\partial u}{\partial x}) + \delta(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})f(u)$$
$$-\frac{\epsilon}{2}\frac{\partial}{\partial x}(\delta'(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}}) - \epsilon\frac{\partial}{\partial x}(\delta'(\frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}})\frac{F(u)}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}})$$
$$+\omega f'(u) - \frac{\partial^2 \omega}{\partial x^2} = 0, \tag{3.19}$$
$$\omega = -\frac{\partial^2 u}{\partial x^2} + f(u). \tag{3.20}$$

Assuming that δ is of class $\mathcal{C}^1$ and noting that $|\frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}| \leq 1$, we can proceed exactly as above to prove the existence of a solution. Furthermore, assuming that δ is of class $\mathcal{C}^2$, we can easily adapt the proof of uniqueness and deduce the existence and uniqueness of solutions.

Remark 3.2. We can note that our estimates are not independent of ϵ, so that we cannot pass to the limit as ϵ goes to 0. This is not surprising, as the problem formally obtained by taking $\epsilon = 0$ cannot correspond to the (Allen-Cahn) problem associated with the free energy (1.6) (see also [2] and [10]). Actually, this is related with a proper functional setting for the limit problem and, more precisely, for the Allen-Cahn system associated with (1.6) and will be studied elsewhere. We can note that anisotropic versions of the Allen-Cahn equation have been studied in [3] and the references therein, based on viscosity solutions. Such an approach is not straightforward here, as there is no maximum/comparison principle for fourth-order in space parabolic equations.

Remark 3.3. It is also important to study the Cahn-Hilliard system associated with (1.7) (for $\delta(s) = s$), namely,

$$\frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}(-\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}}f(u)$$
$$-\frac{\epsilon}{2}\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} - \epsilon\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon + (\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} + \omega f'(u) - \frac{\partial^2 \omega}{\partial x^2}) = 0, \tag{3.21}$$
$$\omega = -\frac{\partial^2 u}{\partial x^2} + f(u). \tag{3.22}$$

Taking, for simplicity, Dirichlet boundary conditions,

$$u(0) = u(L) = \frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial x^2}(L) = \omega(0) = \omega(L) = \frac{\partial^2 \omega}{\partial x^2}(0) = \frac{\partial^2 \omega}{\partial x^2}(L) = 0,$$

we can rewrite (3.21) as

$$
\begin{aligned}
&(-\frac{\partial^2}{\partial x^2})^{-1}\frac{\partial u}{\partial t} - -\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u) \\
&-\frac{\epsilon}{2}\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} - \epsilon\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2} = 0.
\end{aligned} \tag{3.23}
$$

We thus have an equation which bears some resemblance with (2.1), except that we have less regularity on $\frac{\partial u}{\partial t}$, which prevents us from proceeding as in the proof of Theorem 3.1. However, if we consider the viscous Cahn-Hilliard equation (introduced in [6] for the usual Cahn-Hilliard equation),

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \alpha\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}(-\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u) \\
&-\frac{\epsilon}{2}\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} - \epsilon\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2}) = 0,\ \alpha > 0,
\end{aligned} \tag{3.24}
$$

or, equivalently,

$$
\begin{aligned}
&(-\frac{\partial^2}{\partial x^2})^{-1}\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial t} - -\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} + \frac{\frac{\partial u}{\partial x}}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{1}{2}}} f(u) \\
&-\frac{\epsilon}{2}\frac{\partial}{\partial x}\frac{(\frac{\partial u}{\partial x})^2}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} - \epsilon\frac{\partial}{\partial x}\frac{F(u)}{(\epsilon+(\frac{\partial u}{\partial x})^2)^{\frac{3}{2}}} + \omega f'(u) - \frac{\partial^2\omega}{\partial x^2} = 0,
\end{aligned} \tag{3.25}
$$

then, proceeding as in the proof of Theorem 3.1, we have the existence and uniqueness of solutions.

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