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OPTIMAL CONTROL FOR SYSTEMS DESCRIBED BY HYPERBOLIC EQUATION WITH STRONG NONLINEARITY

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Abstract The optimization control problem for a hyperbolic equation is considered. The system is nonlinear with respect to the state derivative. The regularization technique for the state equation is applied. The necessary conditions of optimality for the regularized control problem are proved. It uses the extended differentiability of the control-state mapping for the regularized equation. The convergence of the regularization method is proved. Therefore the optimal control for the regularized problem with small enough regularization parameter can be chosen as an approximate solution of the initial optimization problem.

Keywords Optimization, nonlinear hyperbolic equation, regularization, extended derivative, variational inequality.

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1. Introduction

Optimization control problems for the systems described by Goursat – Darboux problems for nonlinear hyperbolic equations are well known (see for example [1, 5, 12–14]). The case of the standard boundary problems is not well enough researched. Some results are obtained in [4, 6, 12, 15, 18, 19]. Besides, the corresponding equations are nonlinear as a rule with respect to the state functions but not to its derivatives. Necessary conditions of optimality for nonlinear hyperbolic equations including nonlinearity with respect to the time derivative of the state function were considered by D. Tiba [19]. However he has an integral nonsmooth nonlinear term (see [19], equality (3.8), p.42). Thus the control systems for hyperbolic equations with standard strong nonlinearities are not researched in really as yet. We consider an optimization problem for the system described by the first boundary problem for the second order hyperbolic equation. It is nonlinear with respect to the time derivative of the state function. The partial case of this problem was considered in [16].

There are two classes of the analysis of optimization control problems. The state equation is a mean for the implicit determination of the control-state mapping for the first methods class. Then the minimization problem of the given functional on the admissible controls set can be solved with using of standard optimality conditions, for example, variational inequalities, see [9]. However the state equation

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is interpreted as a constraint for the second methods class. So the state function is not depending from the control here. Then the state functional is minimized on the set of admissible pairs "control-state". Thus this problem can be solved by means of standard constraints optimization methods, for example, Lagrange multipliers method [4] or penalty method [6].

The first approach uses special peculiarities of the considered equations. The corresponding optimization problem is easy enough. However it is difficult to prove the differentiability of the control-state mapping because it is defined implicitly by state equation. On the contrary evidently known operators only are differentiated for the second approach. However the properties of the state equation do not used, and the constraints are more difficult.

The mentioned preferences and shortcomings take into consideration for the choice of the suitable method for the concrete problem. The first approach is better if there is the advanced theory of the given equations. However the existence and the uniqueness of solutions are absent for the singular systems. So we do not know properties of the state function after the variation of the control. Then the first approach can be inapplicable in this case, and the second one can be better [4, 10]. The second approach is preferable also for problems with state constraints because the state function is not depending from the control here. It can be changed directly. Thus the using of this means is reasonable whenever we have singular systems or state constraints. The theory of monotone operators is applicable for the given system. Besides we do not have any state constraints. Therefore we prefer to use the first methods class. However its practical application is difficult enough because of the strong nonlinearity of the equation.

We cannot any possibilities to use here the known methods (see [4, 6, 15, 18, 19]) because of the difficulty for passing to limits for the justification of the necessary optimality conditions. The analogical difficulties are overcome by means of the extended derivatives for optimization control problem with weak nonlinearity [17]. However the corresponding adjoint equation does not have necessary a priori estimates because of the strong form of the nonlinearity. This obstacle was overcome in [16] by means of a regularization method for the partial case of the considered problem. Unfortunately this method is inapplicable for our problem because we permit more large class of the minimizing functional. Then the properties of the adjoint equation are not strong enough for passing to the limit and obtaining necessary conditions of optimality for the initial optimization problem. So we will prove the extended differentiability of the control-state mapping with using of the idea of [17]. But the properties of the state function are weak enough too. Therefore we cannot to pass to the limit in the necessary optimality conditions of the regularized problem for obtaining conditions of optimality for initial one. However the sequence of optimal controls for the regularized problem is the minimizing sequence for the given one. Hence we will find the approximate solution of the initial optimization problem as an optimal control for the regularized problem.

2. Problem statement

Let Ω be n-dimensional bounded set with boundary S, $Q = \Omega \times (0,T)$, $\Sigma = S \times (0,T)$. Consider the the equation

$$y'' - \Delta y + |y'|^{\rho} y' = v,\ (x,t) \in Q \tag{2.1}$$

with homogeneous boundary conditions

$$y \,=\, 0,\ (x,t) \in \Sigma, \tag{2.2}$$

$$y(x,0) \,=\, 0,\ y'(x,0) \,=\, 0,\ x \in \Omega, \tag{2.3}$$

where $\rho > 0$, y' and y'' are time derivatives.

Denote by Y the space of functions that equal to zero with its first time derivatives for $t = 0$ and satisfy the inclusions

$$y \in L_\infty(0,T;H_0^1(\Omega)),\quad y' \in L_\infty(0,T;L_2(\Omega)) \cap L_q(Q),\quad y'' \in L_{q'}(0,T;Z),$$

where $q = \rho + 2$, $Z = H^{-1}(\Omega) + L_{q'}(\Omega)$, $1/q + 1/q' = 1$. Let the operator A maps each element $y \in Y$ to the left side of the equality (2.1). So boundary problem is transformed to $Ay = v$. Define the bilinear continuous form by the equality

$$[\varphi,\psi] \,=\, \int\limits_Q \varphi\psi' dQ, \quad \forall\varphi,\psi \in Y.$$

So we obtain the inequality

$$[A\varphi - A\psi, \varphi - \psi] \geq 0, \quad \forall\varphi,\psi \in Y.$$

Thus the operator A is monotone in this sense. Therefore for any v from the space $V = L_2(Q)$ there exists a unique solution $y = y[v]$ from Y because of the monotone operators theory (see [3], Chapter 2, Theorem 6.1); besides the mapping $y[\cdot] : V \to Y$ is *-weakly continuous.

Consider the functional

$$I(v) \,=\, \int\limits_Q F\big(x,t;y[v](x,t),v(x,t)\big)dQ,$$

where the function F is known. Let U by a convex closed bounded subset of the space V. We have the following optimization control problem.

Problem P. *Find the control $u \in U$ that minimizes the functional I on the set U.*

The analogical problem was considered in [16] for the case of the simple quadratic functional.

Theorem 2.1. *Let F be a Caratheodory function on the set $Q \times R^2$, besides $F(x,t;\varphi,\cdot)$ is convex for all $(x,t) \in Q$ and $\varphi \in R$, and suppose the existence of an increasing convex lower semicontinuous function $\eta : R_+ \to R_+$, such that $\eta(\sigma)/\sigma \to \infty$ as $\sigma \to \infty$, and $F(x,t;\varphi,\psi) \geq \eta(|\psi|)$ for all $\psi \in R$. Then the Problem P is solvable.*

Proof. If $\{u_k\}$ is a minimizing sequence, then there exists its subsequence with initial designation such that $u_k \to u$ weakly in V because of the boundedness of U. Then $y[u_k] \to y[u]$ *-weakly in Y. This convergence is true after extracting a subsequence in the sense of the strong topology of $L_2(Q)$ and a.e. on Q by Rellich-Kondrashov Theorem. Then $I(u) \leq \underline{\inf} I(u_k)$ because of the semicontinuity functional theorem (see [3], Chapter VIII, Theorem 2.1). So the control u is optimal. □

3. Regularization method

The direct obtaining of the necessary optimality conditions by means of known methods (see, for example, [1, 4–6, 12–15, 18, 19]) is difficult enough because of the strong form of the nonlinearity for the state equation. Particularly we do not necessary a priory estimates of the solutions of the corresponding linearized equation and adjoint one. Then we use the regularization (see [16]) for obtaining an additional a priory estimate. Consider the regularized equation

$$-\varepsilon_k \Delta y_k' + y_k'' - \Delta y_k + |y_k'|^\rho y_k' = v, (x,t) \in Q \tag{3.1}$$

with boundary conditions

$$y_k = 0, \ (x,t) \in \Sigma, \tag{3.2}$$

$$y_k(x,0) = 0, \ y_k'(x,0) = 0, \ x \in \Omega, \tag{3.3}$$

where $\varepsilon_k > 0$, besides $\varepsilon_k \to 0$ as $k \to \infty$. Denote by Y_1 the subspace of functions $y \in Y$ such that $y' \in S$, where $S = L_2(0,T; H_0^1(\Omega))$.

Lemma 3.1. *For any $k = 1, 2, ...$ and $v \in V$ there exists a unique solution $y_k = y_k[v]$ from Y_1 of the boundary problem* (3.1), (3.2) *and* (3.3), *besides the map $y_k[\cdot] : V \to Y_1$ is *-weakly continuous.*

Proof. Denote the operator A_k such that $A_k y = -\varepsilon_k \Delta y' + Ay$. Then problem (3.1), (3.2) and (3.3) is transformed to $A_k y_k = v$. Operator A_k is monotone. Multiplying equality (3.1) by y_k' , we get after integration

$$\int_\Omega \Big(-\varepsilon_k \Delta y_k' + y_k'' - \Delta y_k + |y_k'|^\rho y_k' \Big) y_k' dx = \int_\Omega v y_k' dx. \tag{3.4}$$

Now we have

$$-\int_\Omega \Delta y_k' y_k' dx = \int_\Omega \left|\nabla y_k'\right|^2 dx = \left\|y_k'(t)\right\|^2,$$

$$\int_\Omega y_k'' y_k' dx = \frac{1}{2}\frac{d}{dt}\left\|y_k'(t)\right\|_{2,\Omega}^2 ,$$

$$-\int_\Omega \Delta y_k y_k' dx = \int_\Omega \nabla y_k \nabla y_k' dx = \frac{1}{2}\frac{d}{dt}\left\|y_k(t)\right\|^2,$$

where $\|\cdot\|$ and $\|\cdot\|_{p,\Theta}$ are the norms of the spaces $H_0^1(\Omega)$ and $L_p(\Theta)$, and $\varphi(t)$ is the function $\varphi = \varphi(x,t)$ with fixed value t. Integrating equality (3.4) with respect to t from 0 to t, we see that

$$\varepsilon_k \int_0^t \left\|y_k'(\tau)\right\|^2 d\tau + \frac{1}{2}\left\|y_k'(t)\right\|_{2,\Omega}^2 + \frac{1}{2}\left\|y_k(t)\right\|^2 + \int_0^t \left\|y_k'(\tau)\right\|_{q,\Omega}^q d\tau$$

$$\le \frac{1}{2}\|v\|_V^2 + \frac{1}{2}\int_0^t \left\|y_k'(\tau)\right\|_{q,\Omega}^q d\tau.$$

because of homogeneousness of the initial conditions. By Gronwall Lemma we get

$$\sqrt{\varepsilon_k}\big\|y_k'\big\|_S \le c\|v\|_V,$$
$$\big\|y_k'\big\|_{L_\infty(0,T;L_2(\Omega))} \le c\|v\|_V,$$
$$\big\|y_k\big\|_{L_\infty(0,T;H_0^1(\Omega))} \le c\|v\|_V,$$
$$\big\|y_k'\big\|_{q,Q} \le c\|v\|_V^{2/q},$$

where different positive constants are denoted by c. Besides it does not depend from k. Using (3.1) we obtain

$$\begin{aligned}\big\|y_k''\big\|_{L_{q'}(0,T;Z)} &\le c\Big[\varepsilon_k\|\Delta y_k'\|_{S'} + \|\Delta y_k\|_{L_\infty(0,T;H^{-1}(\Omega))} + \big\||y_k'|^\rho y_k'\big\|_{q',Q} + \|v\|_V\Big] \\ &\le c\Big[\sqrt{\varepsilon_k}\|y_k'\|_S + \big\|y_k\big\|_{L_\infty(0,T;H_0^1(\Omega))} + \big\|y_k'\big\|_{q,Q}^{\rho+1} + \|v\|_V\Big],\end{aligned}$$

where S' is the corresponding adjoint space. Hence we get the estimate of the derivative y_k'' in the space $L_{q'}(0,T;Z)$. Then the solution of the equation (3.1) is bounded in the space Y_1. The proof can be finished by known technique (see [3], Chapter 2, Theorem 6.1) because of the monotony of the operator A_k. □

The convergence of the regularization is guarantee by the following result.

Lemma 3.2. *$y_k[v] \to y[v]$ *-weakly in Y uniformly with respect to $v \in U$ as $k \to \infty$.*

Proof. By a priory estimates we get $y_k[v] \to y$ *-weakly in Y and $\sqrt{\varepsilon_k}y_k' \to \varphi$ weakly in S uniformly with respect to $v \in U$. Integrating (3.4) with respect to t, we obtain

$$\varepsilon_k\|y_k'\|_S^2 + [Ay_k, y_k] = [v, y_k].$$

Hence

$$\overline{\lim_{k\to\infty}}[Ay_k, y_k] \le \lim_{k\to\infty}[v, y_k] = [v, y].$$

We have the convergence $[A\lambda, y_k] \to [A\lambda, y]$ for all $\lambda \in Y$. By (3.1) we get

$$[Ay_k, \lambda] = \big[\varepsilon_k\Delta y_k' + v, \lambda\big] = \sqrt{\varepsilon_k}\int\limits_Q \sqrt{\varepsilon_k}\Delta y_k'\lambda' dQ + [v, \lambda],$$

so $[Ay_k, \lambda] \to [v, \lambda]$.

Thus we obtain

$$\begin{aligned}&[v - A\lambda, y - \lambda] = [v, y] - [v, \lambda] - [A\lambda, y - \lambda] \\ &\ge \overline{\lim_{k\to\infty}}\{[Ay_k, y_k] - [Ay_k, \lambda] - [A\lambda, y_k - \lambda]\} = \overline{\lim_{k\to\infty}}\big[Ay_k - A\lambda, y_k - \lambda\big].\end{aligned}$$

By the monotony of the operator A we get the inequality $[v - A\lambda, y - \lambda] \ge 0$. Define $\lambda = y - \sigma\xi$, where $\xi \in Y$, $\sigma > 0$. Divide by σ and pass to the limit as $\sigma \to 0$. For any $\xi \in Y$ we get $[v - A\lambda, \xi] \ge 0$, so $Ay = v$. Thus $y = y[v]$. Then we have $y_k[v] \to y[v]$ *-weakly in Y uniformly with respect to $v \in U$ by uniqueness of the boundary problem solution for equation (2.1). This completes the proof of Lemma 3.2. □

Consider the functional

$$I_k(v) = \int\limits_Q F(x, t; y_k[v](x, t), v(x, t))dQ.$$

We have the following regularized problem.

Problem P_k. *Find the control that minimizes the functional* I_k *on the set* U.

We can prove its solvability with using of the technique of Theorem 2.1. The necessary optimality conditions of the initial problem was obtained in [16] by means of the passing to the limit in the conditions of optimality for the regularized problem. However the adjoint equation in our case is more difficult because of properties of the given functional. So we cannot any possibility now to pass to the limit in the conditions of optimality for the regularized problem. But the convergence of the regularization method can be proved in the other sense.

Theorem 3.1. *Suppose the function* F *satisfies the assumptions of Theorem 2.1, besides* $|F(x,t;\varphi,\psi)| \le a(x,t) + b(\varphi^2+\psi^2)$ *for all* $(x,t) \in Q$ *and* $\varphi,\psi \in R$, *where* $a \in L_1(Q)$, $b > 0$; *then the sequence* $\{u_k\}$ *of solutions of Problem* P_k *is minimizing for Problem* P. *Suppose the additional equality* $F(x,t,\varphi,\psi) = \Phi(x,t,\varphi) + \chi\psi^2$, *where* Φ *is Carathéodory function on the set* $Q \times R$, *besides* $|\Phi(x,t,\varphi)| \le b\varphi^2$; *then it is true the convergence* $u_k \to u$ *strongly in* V *after extracting a subsequence, where* u *is a solution of Problem* P.

Proof. Let u be a solution of Problem P. We get

$$I(u) \;=\; \min I(U) \;\le\; I(u_k) \;\le\; |I(u_k) - I_k(u_k)| \;+\; I_k(u_k), \tag{3.5}$$

besides

$$|I(u_k) - I_k(u_k)| \;\le\; \sup_{v\in U} |I(v) - I_k(v)|.$$

By Lemma 3.2 we obtain the convergence $y_k[v] \to y[v]$ *-weakly in Y with respect to $v \in U$. Using Rellich-Kondrashov Theorem we get $y_k[v] \to y[v]$ strongly in $L_2(Q)$. Then $F_k[v] \to F[v]$ in $L_1(Q)$ by Krasnosel'skiy Theorem (see [8], p.312), where $F_k[v](x,t) = F(x,t;y_k[v](x,t),v(x,t))$, $F[v](x,t) = F(x,t;y[v](x,t),v(x,t))$. So $I_k(v) \to I(v)$ uniformly with respect to $v \in U$. Then we get

$$0 \;\le\; \lim_{k\to\infty} |I(u_k) - I_k(u_k)| \;\le\; \lim_{k\to\infty} \sup_{v\in U} |I(v) - I_k(v)| \;=\; 0. \tag{3.6}$$

Using

$$I_k(u_k) \;=\; \min I(U) \;\le\; I_k(u) \;\le\; |I_k(u) - I(u)| \;+\; I(u)$$

and passing to the limit, we obtain $\lim\limits_{k\to\infty} I_k(u_k) \le I(u)$. By (3.5) and (3.6) we have $I(u_k) \to I(u)$.

We proved in really that a subsequence of solutions of Problem P_k is minimizing for Problem P. Suppose there exists a subsequence of $\{I(u_k)\}$, which is not has $\inf I(U)$ as a limit point. Then we can repeat the previous analysis and extract its subsequence, which convergences to $\inf I(U)$. So this value is the limit of the whole sequence $\{I(u_k)\}$. Hence $\{u_k\}$ is the minimizing sequence for the given problem.

Prove now the strong convergence of solutions of the regularized problem for the given partial case. The sequence $\{u_k\}$ is bounded. So we have $u_k \to v$ weakly in V after extracting of a subsequence. Then $y[u_k] \to y[v]$ *-weakly in Y and strongly in $L_2(Q)$. Therefore $\Phi\{y[u_k]\} \to \Phi\{y[v]\}$ in $L_1(Q)$ because of Krasnosel'skiy Theorem. So we get

$$\int_Q \Phi\Big(x,t;y[u_k](x,t)\Big)dQ \to \int_Q \Phi\Big(x,t;y[v](x,t)\Big)dQ.$$

Using the lower semicontinuity of the norm, we obtain

$$\int_Q v^2 dQ \leq \inf \lim_{k\to\infty} \int_Q u_k^2 dQ. \tag{3.7}$$

Then

$$I(v) \leq \inf \lim_{k\to\infty} I(u_k) = \inf I(U). \tag{3.8}$$

We have enclosure $v \in V$ because of the convexity of the set U. So v is a solution of Problem P because of (3.8). We can denote it by u.

Suppose that it is true the strong inequality (3.7) for the subsequence, which conforms to the lower limit. Then the inequality (3.8) becomes strong too. But it contradicts the enclosure $v \in V$. So we have the convergence $\|u_k\|_V \to \|v\|_V$ for the noted subsequence. Using the weak convergence $u_k \to v$ in V, we get its strong convergence. This completes the proof of Theorem 3.1. □

We note that the solution of our problem can be no unique. So we can obtain different limits for different subsequences of $\{u_k\}$. But our analysis is true for each subsequence. So each limit point of the sequence $\{u_k\}$ is a solution of Problem P. However it is possible that some solutions cannot obtain by means of this technique.

Thus the value of the functional I at the solution of the regularization problem is close enough to its minimum on the admissible controls set for a large enough number k. So the optimal control for the regularized problem can be chosen as an approximate solution of the initial optimization problem. Besides it is close enough to the exact solution of Problem P for the partial case. Now we obtain a solution of the regularized problem.

4. Solving of the regularized problem

The necessary condition of optimality for Gateaux differentiable functional J on the convex set U is the variational inequality

$$\langle J'(u), v - u\rangle \geq 0 \; \forall v \in U, \tag{4.1}$$

where $J'(u)$ is the derivative of the functional at the point u, and $\langle \varphi, v\rangle$ is the value of the linear continuous functional φ at the point v. It is necessary to prove the differentiability of the regularized functional for using this result in our case. This functional depends from control by means of the control-state mapping $y_k[\cdot] : V \to Y_1$ of equation (3.1). We cannot any possibilities to prove its Gateaux differentiability because of strong nonlinearity of the equation. However we can obtain more weak property (see [17]).

Definition 4.1. Let L and W be Banach spaces. An operator $L : V \to W$ is called $(V_0, W_0; V_*, W_*)$**-extended differentiable** at the point $u \in V$, if the following conditions hold:

i) there exists Banach spaces V_0, W_0, V_*, W_* such that the embeddings $V_* \subset V_0 \subset V$ and $W \subset W_* \subset W_0$ are continuous;

ii) there exists a linear continuous operator $D : V_0 \to W_0$ such that $[L(u+\sigma h) - Lu]/\sigma \to Dh$ in W_* as $\sigma \to 0$ for all $h \in V_*$.

The $(V,W;V,W)$-extended derivative is Gateaux derivative. But the extended derivative is definite on the narrower set and has values from a larger set than classical one in the general case. Besides we can guarantee the corresponding convergence only in the sense of a weaker topology and for more narrow class of directions h. So this notion is a generalization of the Gateaux derivative. An example of the extended differentiable operator without Gateaux derivative is given in [17]. We note that all four spaces from the extended derivative definition do not equal to the given spaces in the general case. It can depend also on the point u (see [17]).

We claim that the map $y_k[\cdot] : V \to Y_1$ is extended differentiable at the arbitrary point $u \in V$. Indeed, subtracting equality (3.1) for the control u from this equality for the control $u + \sigma h$, we have

$$-\varepsilon_k \Delta\eta'_\sigma[h] + \eta''_\sigma[h] - \Delta\eta_\sigma[h] + (g_\sigma[h])^2\eta'_\sigma[h] = h, \tag{4.2}$$

where

$$\eta_\sigma[h] = (y_k[u+\sigma h] - y_k[u])/\sigma,$$
$$\big(g_\sigma[h]\big)^2 = (\rho+1)\big|y'_k[u] + \delta(y'_k[u+\sigma h] - y'_k[u])\big|^\rho, \quad \delta \in [0,1].$$

Remark 4.1. We explain the definition of the last term in the left side of the equality (4.2). Determine power operator by equality $Fz = |z|^\rho z$ on the space $L_q(Q)$. It is the partial case of Nemytsky's operator (see [8], p.312). Using Lagrange formula we obtain $Fz_1 - Fz_2 = (\rho+1)\big|z_2 + \delta(z_1 - z_2)\big|^\rho (z_1 - z_2)$, where $\delta \in [0,1]$. In our case we have $z_1 = y'_k[u+\sigma h]$, $z_2 = y'_k[u]$.

Multiply (4.2) by a smooth enough function λ such that equals to zero with its time derivative for $t = T$ and on the lateral surface of Q. Integrating this result over Q, we get

$$\int_Q \Big\{ -\varepsilon_k \Delta\eta'_\sigma[h] + \eta''_\sigma[h] - \Delta\eta_\sigma[h] + (g_\sigma[h])^2\eta'_\sigma[h]\Big\}\lambda dQ = \int_Q h\lambda dQ.$$

We have the equalities

$$\int_Q \Delta\eta'_\sigma[h]\lambda dQ = \int_Q \eta'_\sigma[h]\Delta\lambda dQ,$$
$$\int_Q \eta''_\sigma[h]\lambda dQ = -\int_Q \eta'_\sigma[h]\lambda' dQ,$$
$$\int_Q \Delta\eta_\sigma[h]\lambda dQ = \int_Q \eta_\sigma[h]\Delta\lambda dQ = -\int_Q \eta'_\sigma[h]\Delta\Pi_t\lambda dQ,$$

where

$$\Pi_t\lambda = \int_t^T \lambda dt.$$

Then the previous equality transform to

$$\int_Q \Big\{ -\varepsilon_k \Delta\lambda - \lambda' - \Delta\Pi_t\lambda + (g_\sigma[h])^2\lambda\Big\}\eta'_\sigma[h] dQ = \int_Q h\lambda dQ. \tag{4.3}$$

Consider the equation

$$-\varepsilon_k \Delta p - p' - \Delta \Pi_t p + (g_\sigma[h])^2 p \;=\; \mu \tag{4.4}$$

with homogeneous conditions for $t = T$ and on the lateral surface of Q. Particularly this equality transform to

$$-\varepsilon_k \Delta p - p' - \Delta \Pi_t p + (\rho+1)|y_k'[u]|^\rho p \;=\; \mu \tag{4.5}$$

for $\sigma = 0$.

Let P be the space of functions p that equals to zero with its first time derivative for $t = T$ and satisfy the inclusions $p \in L_\infty(0,T;L_2(\Omega))$ and $\Pi_t p \in L_\infty(0,T;H_0^1(\Omega))$. Denote by P_* the space $\{p|\, p \in P \cap S,\, p' \in L_{q'}(0,T;Z)\}$. Consider the space $R_\sigma = \{p|\, p \in S,\, g_\sigma[h]p \in L_2(Q)\}$. It is Hilbert space with scalar product

$$(\varphi,\psi) \;=\; \int\limits_Q \nabla\varphi\nabla\psi dQ \;+\; \int\limits_Q (g_\sigma[h])^2\varphi\psi dQ.$$

It has the adjoint space

$$R_\sigma' = \Big\{\mu|\, \mu = \chi + g_\sigma[h]\eta,\, \chi \in S',\, \eta \in L_2(Q)\Big\}.$$

Particularly we obtain

$$R_0 = \Big\{p|\, p \in S,\, |y_k'[u]|^{\rho/2}p \in L_2(Q)\Big\},$$
$$R_0' = \Big\{\mu|\, \mu = \chi + |y_k'[u]|^{\rho/2}\eta,\, \chi \in S',\, \eta \in L_2(Q)\Big\}.$$

Consider also the space

$$P_\sigma = \Big\{p|\, p \in R_\sigma \cap L_\infty(0,T;L_2(\Omega)),\, p' \in R_\sigma'\Big\}$$

and the set $M \;=\; \big\{\mu \in S'|\, \|\mu\|_{S'} = 1\big\}$.

Lemma 4.1. *For any $k = 1,2,...,$ $\sigma > 0,$ $\mu \in R_\sigma'$ the equation* (4.4) *has a unique solution $p = p_{k\sigma}^h[\mu]$ from the space P_σ, besides $p_{k\sigma}^h[\mu] \to p_k[\mu]$ *-weakly in P_* uniformly with respect to $\mu \in M$ as $\sigma \to 0$, where $p_k[\mu]$ is a solution of equation* (4.5).

Proof. 1. If $\mu \in R_\sigma'$ then there exists functions $\chi \in S'$, $\eta \in L_2(Q)$ such that $\mu = \chi + g_\sigma[h]\eta$. Multiplying (4.2) by p and integrating the result, we get

$$\begin{aligned}
&\varepsilon_k\|p(t)\|^2 - \frac{1}{2}\frac{d}{dt}\Big[\|p(t)\|_{2,\Omega}^2 + \|\Pi_t p(t)\|^2\Big] + \|(g_\sigma[h]p)(t)\|_{2,\Omega}^2\\
\le\;& \|\chi(t)\|_*\|p(t)\| \;+\; \|(g_\sigma[h]p)(t)\|_{2,\Omega}\|\|\eta(t)\|_{2,\Omega}\\
\le\;& \frac{\varepsilon_k}{2}\|p(t)\|^2 + \frac{1}{2\varepsilon_k}\|\chi(t)\|_*^2 + \frac{1}{2}\|(g_\sigma[h]p)(t)\|_{2,\Omega}^2 + \frac{1}{2}\|\eta(t)\|_{2,\Omega}^2,
\end{aligned}$$

where $\|\cdot\|_*$ is the norm of the space $H^{-1}(\Omega)$. Integrating this inequality with respect to t, we obtain

$$\begin{aligned}
&\varepsilon_k\int\limits_t^T \|p(t)\|^2dt + \|p(t)\|_{2,\Omega}^2 + \|\Pi_t p(t)\|^2 + \int\limits_t^T \|(g_\sigma[h]p)(t)\|_{2,\Omega}^2 dt\\
\le\;& \frac{1}{\varepsilon_k}\|\chi(t)\|_{S'} + \|\eta(t)\|_{2,Q}^2 \le c.
\end{aligned}$$

From (4.4) it follows that

$$\|p'\|_{R'_\sigma} \le \varepsilon_k\|\Delta p\|_{S'} + \|\Delta\Pi_t p\|_{S'} + \|(g_\sigma[h])^2 p\|_{q',Q} + \|\mu\|_{R'_\sigma}$$
$$\le c[\varepsilon_k\|p\|_S + \|\Pi_t p\|_S + \|g_\sigma[h]\|_{2q/\rho,Q}\|g_\sigma[h]p\|_{2,Q} + \|\mu\|_{R'_\sigma}] \le c.$$

Thus the solution of equation (4.4) has an estimate in the space P_σ. Differentiate this equation in t, we get a standard linear second order hyperbolic equation with additional condition for the final time. It transforms to initial condition by change of variable t to $T-t$. By classical linear hyperbolic equations theory we obtain that the corresponding homogeneous boundary problem has a unique solution $p = p^h_{k\sigma}[\mu]$ in the space P_σ.

2. If $\sigma \to 0$, then $(u+\sigma h) \to u$ in V for all $h \in V$. Using the technique from the proof of Lemma 3.1, we obtain $y_k[u+\sigma h] \to y[u]$ *-weakly in Y_1 for all number k and $h \in V$. Then $y'_k[u+\sigma h] \to y'[u]$ *-weakly in the space

$$Y_2 = \{\varphi|\,\varphi \in S \cap L_q(Q),\, \varphi' \in L_2(0,T;Z)\}.$$

So the set $\{g_\sigma[h]\}$ is bounded in the space $L_{2q/\rho}(Q)$. Choosing $\mu \in M$, we get the inequality

$$\varepsilon_k\|p(t)\|^2 - \frac{1}{2}\frac{d}{dt}[\|p(t)\|^2_{2,\Omega} + \|\Pi_t p(t)\|^2] + \|(g_\sigma[h]p)(t)\|^2_{2,\Omega}$$
$$\le \frac{\varepsilon_k}{2}\|p(t)\|^2 + \frac{1}{2\varepsilon_k}\|\mu(t)\|^2_*,$$

Hence

$$\sup_{\mu\in M}\|\sqrt{\varepsilon_k}p^h_{k\sigma}[\mu]\|_S \le c,$$
$$\sup_{\mu\in M}\|p^h_{k\sigma}[\mu]\|_{P_0} \le c,$$
$$\sup_{\mu\in M}\|g_\sigma[h]p^h_{k\sigma}[\mu]\|_{2,Q} \le c.$$

Then we obtain the inequality

$$\sup_{\mu\in M}\|(g_\sigma[h])^2 p^h_{k\sigma}[\mu]\|_{q',Q} \le \|g_\sigma[h]\|_{2q/\rho,Q}\sup_{\mu\in M}\|g_\sigma[h]p^h_{k\sigma}[\mu]\|_{2,Q} \le c.$$

By formula (4.5)

$$\sup_{\mu\in M}\|(p^h_{k\sigma}[\mu])'\|_{L_2(0,T;Z)}$$
$$\le c[\varepsilon_k\|p^h_{k\sigma}[\mu]\|_S + \|\Pi_t p^h_{k\sigma}[\mu]\|_S + \sup_{\mu\in M}\|(g_\sigma[h])^2 p^h_{k\sigma}[\mu]\|_{q',Q} + \|\mu\|_{2,Q}] \le c.$$

Extracting subsequences, we get the convergences $p^h_{k\sigma}[\mu] \to s$ *-weakly in P_* and $(g_\sigma[h])^2 p^h_{k\sigma}[\mu] \to r$ weakly in $L_{q'}(Q)$ uniformly with respect to $\mu \in M$ because of the obtained estimates. By Theorem 5.1 (see [11], Chapter 1) the embeddings of the spaces Y_2 and P_* in $L_2(Q)$ are compact. Extracting subsequences, we have $y'_k[u+\sigma h] \to y'_k[u]$ and $p^h_{k\sigma}[\mu] \to s$ strongly in $L_2(Q)$ and a.e. in Q. Then $(g_\sigma[h])^2 p^h_{k\sigma}[\mu] \to (\rho+1)|y'_k[u]|^\rho s$ a.e. in Q. By Lemma 1.3 (see [11], Chapter 1) $(g_\sigma[h])^2 p^h_{k\sigma}[\mu] \to (\rho+1)|y'_k[u]|^\rho s$ weakly in $L_{q'}(Q)$ uniformly with respect to $\mu \in M$. Passing to the limit in (4.2) for $p = p^h_{k\sigma}[\mu]$, we get $s = p_k[\mu]$. This completes the proof of Lemma 4.1. □

Prove the extended differentiability of the solution of the regularized equation with respect to control. Consider the spaces $R_1 = \{p|\, p(0) = 0,\, p' \in R_0\}$ and $S_1 = \{p|\, p(0) = 0,\, p' \in S\}$ with norm $\|p\|_{S_1} = \|p'\|_S$.

Lemma 4.2. *The map $y_k : V \to Y_1$ is $(V, R_1; V, S_1)$-extended differentiable at the arbitrary point $u \in V$, besides its extended derivative $D_k(u)$ is defined by equality*

$$\int\limits_Q \mu (D_k[u]h)' dQ = \int\limits_Q hp_k[\mu] dQ \;\forall h \in V, \mu \in R_0'. \tag{4.6}$$

Proof. Equality (4.5) definite an operator $D_k[u] : V \to R_1$ in really. Definite $\lambda = p^h_{k\sigma}[\mu]$ in (3.5). We get

$$\int\limits_Q \mu\eta'_\sigma[h] dQ = \int\limits_Q hp^h_{k\sigma}[\mu] dQ \;\forall h \in V, \mu \in R'_\sigma.$$

Then

$$\|\eta'_\sigma[h] - (D_k[u]h)'\|_S = \sup_{\mu\in M} |\int\limits_Q (\eta_\sigma[h] - D_k[u]h)\mu dQ|$$

$$= \sup_{\mu\in M} |\int\limits_Q h(p^h_{k\sigma}[\mu] - p_k[\mu]) dQ|.$$

So $(y_k[u + \sigma h] - y_k[u])/\sigma \to D_k[u]h$ in S_1 for all $h \in V$ as $\sigma \to 0$. Thus the considered map is in really extended differentiable. □

Note that we cannot any possibilities to prove the extended differentiability of the control-state mapping for equation (2.1) because the solution of the corresponding analogue of equation (4.3) does not have the estimate in the space S_1.

Now we can prove the differentiability of the regularized functional.

Lemma 4.3. *Suppose the function F satisfies the assumptions of Theorem 3.1, besides the function $F(x, t; \cdot, \cdot)$ has the partial derivatives F'_1 and F'_2 , that are Caratheodory functions on $Q \times R^2$ with inequality $|F'_i(x, t, ; \xi_1, \xi_2)| \le a'_i(x, t) + b'_i(|\xi_1| + |\xi_2|)$ for all $(x, t) \in Q$ and $\xi_1, \xi_2 \in R$, where $a'_i \in L_2(Q)$, $b'_i > 0$, $i = 1, 2$. Then the functional I_k has Gateaux derivative $I'_k(u_k) = p_k + F'_{2k}$ at the point u_k, where p_k is a solution of the equation*

$$-\varepsilon_k \Delta p_k - p'_k - \Delta\Pi_t p_k + (\rho + 1)|y'_k|^\rho p_k = F'_{1k} \tag{4.7}$$

with homogeneous boundary conditions, besides

$$y_k = y_k[u_k],\; F'_{ik}(x, t) = F'_i(x, t, ; y_k(x, t), u_k(x, t)).$$

Proof. From Krasnosel'skiy Theorem follows that the operator $\Phi : L_2(Q)^2 \to L_1(Q)$ definite by equality $\Phi(\varphi, \psi)(x, t) = F(x, t; \varphi(x, t)\psi(x, t))$, is Frechet differentiable; besides its partial derivatives are determined by

$$[\Phi'_i(\varphi, \psi)h](x, t) = F'_i(x, t; \varphi(x, t)\psi(x, t)),\; i = 1, 2.$$

Define operator $\Psi : V \to L_1(Q)$ by equality $\Psi v = \Phi(y[v], v)$. Using Implicit Function Theorem (see [7], p.637), we obtain Gateaux differentiability of the map $y_k[\cdot] : L_2(Q) \to L_2(Q)$. Then the operator Ψ is Gateaux differentiable at the point

u_k because of the Composite Function Theorem, besides its derivative is determined by equality

$$\Psi'(u_k)h = \Phi_1'(y_k,u_k)D_k(u_k)h + \Phi_2'(y_k,u_k)h.$$

Then we can find Gateaux derivative of the functional I_k by

$$\langle I_k(u_k),h\rangle = \int\limits_Q \Psi'(u_k)hdQ = \int\limits_Q [\Phi_1'(y_k,u_k)D_k(u_k)h + \Phi_2'(y_k,u_k)h]dQ \ \forall h\in V.$$

Define $u = u_k$ and $\mu = \Phi_1'(y_k,u_k)$ in (4.5) and (4.6). Then we obtain (4.7). The previous equality is transformed to

$$\langle I_k(u_k),h\rangle = \int\limits_Q (p_k + F_{2k}')hdQ \ \forall h\in V.$$

This completes the proof of Lemma 4.3. □

We could prove the differentiability of the functional at an arbitrary point. But we need find its derivative only at the point u_k. By Lemma 4.2 the space S characterizes not only the function but its space derivatives too. Then we can prove the differentiability of the regularized functional whenever the function F depends from space derivatives of the state function too. However the strong convergence $y_k[v]\to y[v]$ was proved only for the function but not for its derivatives because of Theorem 3.1. So we cannot prove the convergence of the regularization method if the functional depends from derivatives of the state function.

Replacing the value of the derivative of the regularized functional in variational inequality (4.1), we get

Theorem 4.1. *Under the conditions of Lemma 4.3 the solution of Problem P_k satisfies the variational inequality*

$$\int\limits_Q (p_k + F_{2k}')(v-u_k)dQ \geq 0 \ \forall v\in U. \tag{4.8}$$

Thus the necessary optimality conditions of the regularized problem include equation (3.1) for the control u_k, the boundary problem for adjoint equation (4.7) and variational inequality (4.8). This system can be solved with using of the standard iterative methods [2]. The corresponding control can be chosen as an approximate solution of the initial optimization problem for large enough number k.

Remark 4.2. The regularization methods for optimization control problems for nonlinear hyperbolic equations was used by Tiba [19]. But he has an integral nonsmooth nonlinear term and uses smooth regularization. We have non-integral smooth nonlinear term. Our general difficulty is an absence of sufficient a priory estimates for the adjoint equation. So we use other form of regularization. We add high order term with small parameter to the equation for obtaining additional a priory estimates. Hence we consider other equation and other regularization method.

Analogical results could be obtained for other optimization problems with strong nonlinearity. The corresponding regularization should be so strong because an analogue of equation (4.4) requires additional a priory estimates for proof the extended differentiability of the control-state mapping for regularized equation. But this regularization should be so weak because of the necessity to prove the convergence of the regularization method.

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