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NUMERICAL ANALYSIS FOR A LOCALLY DAMPED WAVE EQUATION*

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Abstract We consider a semi-discrete finite element formulation with artificial viscosity for the numerical approximation of a problem that models the damped vibrations of a string with fixed ends. The damping coefficient depends on the spatial variable and is effective only in a sub-interval of the domain. For this scheme, the energy of semi-discrete solutions decays exponentially and uniformly with respect to the mesh parameter to zero. We also introduce an implicit in time discretization. Error estimates for the semi-discrete and fully discrete schemes in the energy norm are provided and numerical experiments performed.

Keywords Damped wave equation, artificial viscosity, finite element method, error estimate, numerical simulations.

MSC(2000) 35L05, 65M15, 65M60.

1. Introduction

In recent years, the study of mathematical models related to flexible structures subject to vibrations has been significantly driven by a growing number of issues of practical interest. Among these models, we highlight those related to structural engineering that require modern mechanisms of active control to stabilize inherently unstable structures or have a very weak natural damping.

In this work, we are interested in studying a model that describes the transverse vibrations of a string of finite length L fixed in its extremities and subject to an axial force $a(x)$. The position $y(x,t)$ of a point x in the string, at instant t, satisfies

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) + a(x)y_t(x,t) = 0, & 0 < x < L, \quad t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & 0 < x < L, \end{cases} \tag{1.1}$$

where $a \in L^\infty(0,L)$ is a non-negative function such that $a(x) > a_0 > 0$, $\forall x \in (b,c)$, $0 \le b < c \le L$. The energy associated with the model is given by

$$E(t) = \frac{1}{2}\int_0^L (y_t^2 + y_x^2)\, dx$$

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and it can be shown through a direct calculation that

$$\frac{d}{dt}E(t) = -\int_0^L a(x)y_t^2 dx$$

and, therefore, $E(t)$ is decreasing. Hence, we can say that (1.1) has a dissipative nature, and the term $a(x)y_t$ acts as a mechanism of control (dissipation) and is strong enough to stabilize the energy associated with (1.1). Decay estimates of the energy of the wave equation with a localized nonlinear damping term in a bounded domain were studied by Tébou [8].

Semi-discrete in space finite difference schemes for a more general one-dimensional hyperbolic equation with a constant damping coefficient a were considered by Gao & Chi [3]. After discretization in space the authors give the exact solution in time for the semi-discrete problem. Then the fully discrete solution is found using Padé approximations. Convergence results were not discussed.

A priori error estimates for Galerkin approximations to the solution of hyperbolic equations were obtained by Dupont [2] and Wheeler [10].

Finite element methods for a strongly damped wave equation were studied by Larsson etc. [4].

A finite difference approximation with artificial viscosity, semi-discrete in space, for the damped wave equation was considered by Tébou & Zuazua in [7]. Exponential decay, uniform with respect to the mesh parameter, of the semi-discrete energy is proved and convergence of the scheme was obtained but no error bounds were provided. See also Münch & Pazoto [6] where a more general domain was considered. It is the purpose of this work to restate their scheme as a spatial finite element method and to obtain error estimates for the approximation of (1.1). Moreover, a fully discrete implicit scheme is proposed and analysed. The main idea is to introduce an auxiliary problem with artificial viscosity that is a critical part of the definition of the numerical method and to split the error in two parts.

An outline of the contents of this paper is as follows. In section 2 we introduce an auxiliary problem and provide existence, uniqueness and regularity results. In section 3 a semi-discrete Galerkin approximation to the solution of (1.1) is analysed and in section 4 a fully discrete scheme is considered. We use piecewise linear functions in space and the backward Euler method in time. We show that the discrete energy decays and derive some stability and error estimates. Finally, the results of numerical experiments illustrating the theoretical results are presented in section 5.

Throughout the paper we indicate by $(\ .\ )$ and $|\ .\ |$ the inner product and the norm in $L^2(0,L)$, respectively, and the letter C denotes positive constants that may depend on data.

2. An auxiliary problem

The numerical approximation proposed by Tébou [7] is based on the auxiliary problem that includes artificial viscosity

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + a(x)u_t(x,t) - \epsilon^2 u_{txx}(x,t) = 0, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = y_0(x), \quad u_t(x,0) = y_1(x), \end{cases} \tag{2.1}$$

where ϵ was chosen equal to the spatial mesh size parameter. For completeness, we prove the existence, uniqueness and regularity of a solution to (2.1) using the Faedo-Galerkin method (see Lions [5] and Teman [9]).

Theorem 2.1. *Assume that* $y_0,\ y_1 \in H_0^1(0,L) \cap H^2(0,L)$. *Then problem* (2.1) *admits a unique solution* u *satisfying the following conditions:*

i. $u \in L^\infty(0,T;H_0^1(0,L))$.

ii. $u_t \in L^\infty(0,T;H_0^1(0,L))$.

iii. $u_{tt} \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$.

iv. $(u_x + \epsilon^2 u_{tx})_x \in L^\infty(0,T;L^2(0,L))$.

v. $u_{tt} - (u_x + \epsilon^2 u_{tx})_x + a(x)u_t = 0 \quad a.e.\ in \quad Q = [0,L] \times [0,T]$.

vi. $u(0) = y_0 \quad and \quad u_t(0) = y_1$.

Proof. For convenience we introduce the new variable $\tilde{u}(x,t) = u(x,t) - y_0(x) - ty_1(x)$ satisfying $\tilde{u}(x,0) = \tilde{u}_t(x,0) = 0$. In order to establish the result, we represent by V_m the subspace spanned by $\{\phi_1, \phi_2, ..., \phi_m\}$, where $\{\phi_\nu\}_{\nu=1}^\infty$ is an orthogonal basis to $H_0^1(0,L)$. We write $\tilde{u}^m(x,t) = \sum_{\nu=1}^m d_{\nu m}(t)\phi_\nu(x)$ and consider the approximation problem

$$(\tilde{u}_{tt}^m, v) + (\tilde{u}_x^m + y_{0x} + ty_{1x}, v_x) + (a(x)(\tilde{u}_t^m + y_1), v) + \epsilon^2(\tilde{u}_{tx}^m + y_{1x}, v_x) = 0, \quad (2.2)$$

with $\tilde{u}^m(0) = \tilde{u}_t^m(0) = 0$ and $v \in V_m$. This system has a local solution $\tilde{u}^m = \tilde{u}^m(x,t)$ in the interval $(0,T_m)$ by Carathéodory Theorem. To extend the local solution to the interval $(0,T)$ independent of m and to take the limit in (2.2), *a priori* estimates are needed.

Taking $v = 2\tilde{u}_t^m$ we obtain

$$\begin{aligned} & \frac{d}{dt}|\tilde{u}_t^m(t)|^2 + \frac{d}{dt}|\tilde{u}_x^m(t)|^2 + 2\Big(a(x)\tilde{u}_t^m, \tilde{u}_t^m\Big) + 2\epsilon^2|\tilde{u}_{tx}^m(t)|^2 \\ = \quad & -2(y_{0x} + ty_{1x}, \tilde{u}_{tx}^m) - 2(a(x)y_1, \tilde{u}_t^m) - 2\epsilon^2(y_{1x}, \tilde{u}_{tx}^m) \end{aligned}$$

and integration from 0 to $t < T_m$ yields

$$\begin{aligned} & |\tilde{u}_t^m(t)|^2 + |\tilde{u}_x^m(t)|^2 + 2\int_0^t \Big(a(x)\tilde{u}_t^m(s), \tilde{u}_t^m(s)\Big)\, ds + 2\epsilon^2 \int_0^t |\tilde{u}_{tx}^m(s)|^2\, ds \\ \leq \quad & C\int_0^t \left(|y_{0xx} + sy_{1xx}|^2 + |a(x)y_1|^2 + \epsilon^2|y_{1xx}|^2 + |\tilde{u}_t^m(s)|^2\right)\, ds. \end{aligned}$$

Since a is non-negative we can use the Gronwall inequality and the assumptions on the initial data to deduce that

$$\tilde{u}^m \text{ is bounded in } L^\infty(0,T;H_0^1(0,L)),$$

$$\tilde{u}_t^m \text{ is bounded in } L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$$

and the solution $\tilde{u}^m$ can be extended to the whole interval $[0,T]$. Also, there exists a sub-sequence, still denoted by $(\tilde{u}^m)$, such that

$$\begin{aligned} & \tilde{u}^m \longrightarrow \tilde{u} \text{ weak-star in } L^\infty(0,T;H_0^1(0,L)), \\ & \tilde{u}_t^m \longrightarrow \tilde{u}_t \text{ weak-star in } L^\infty(0,T;L^2(0,L)), \\ & \tilde{u}_t^m \longrightarrow \tilde{u}_t \text{ weakly in } L^2(0,T;H_0^1(0,L)). \end{aligned} \quad (2.3)$$

This result together with the regularity of the initial data implies that

$$u \in L^\infty(0,T;H_0^1(0,L)), \quad u_t \in L^\infty(0,T;L^2(0,L)), \quad u_t \in L^2(0,T;H_0^1(0,L)).$$

Differentiating equation (2.2) with respect to t and introducing $w^m = \tilde{u}_t^m$, we have

$$(w_{tt}^m, v) + (w_x^m + y_{1x}, v_x) + (a(x)w_t^m, v) + \epsilon^2(w_{tx}^m, v_x) = 0. \tag{2.4}$$

Taking $v = 2w_t^m$ in (2.4) we obtain, after integration by parts,

$$\begin{aligned} & \frac{d}{dt}|w_t^m(t)|^2 + \frac{d}{dt}|w_x^m(t)|^2 + 2(a(x)w_t^m, w_t^m) + 2\epsilon^2|w_{tx}^m|^2 \\ = \quad & 2(y_{1xx}, w_t^m) \le C + |w_t^m(t)|^2. \end{aligned}$$

Integrating from 0 to t and keeping in mind that a in non-negative yields

$$\begin{aligned} & |w_t^m(t)|^2 + |w_x^m(t)|^2 + 2\epsilon^2 \int_0^t |w_{tx}^m(s)|^2 \, ds \\ \le \quad & C + |w_t^m(0)|^2 + |w_x^m(0)|^2 + \int_0^t |w_t^m(s)|^2 \, ds. \end{aligned} \tag{2.5}$$

Since we have that $w^m(0) = \tilde{u}_t^m(0) = 0$, it remains to bound $|w_t^m(0)|^2$.

Letting $t = 0$ in (2.2) and $v = \tilde{u}_{tt}^m(0)$ we find

$$\big(\tilde{u}_{tt}^m(0), \tilde{u}_{tt}^m(0)\big) - \big(y_{0xx}, \tilde{u}_{tt}^m(0)\big) + \big(a(x)y_1, \tilde{u}_{tt}^m(0)\big) - \epsilon^2(y_{1xx}, \tilde{u}_{tt}^m(0)) = 0$$

and applying the Cauchy-Schwarz inequality we get

$$|\tilde{u}_{tt}^m(0)|^2 \le |y_{0xx} - a(x)y_1 + \epsilon^2 y_{1xx}| \, |\tilde{u}_{tt}^m(0)|.$$

Then the regularity of the initial data implies that

$$|w_t^m(0)| \le C$$

and from (2.5) and Gronwall's inequality we conclude that

$$\begin{aligned} & w^m = \tilde{u}_t^m \text{ is bounded in } L^\infty(0,T;H_0^1(0,L)), \\ & w_t^m = \tilde{u}_{tt}^m \text{ is bounded in } L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L)). \end{aligned}$$

As a consequence, we can select a sub-sequence, denoted by $(w^m = \tilde{u}_t^m)$, such that

$$\begin{aligned} & w^m = \tilde{u}_t^m \longrightarrow \tilde{u}_t \text{ weak-star in } L^\infty(0,T;H_0^1(0,L)), \\ & w_t^m = \tilde{u}_{tt}^m \longrightarrow \tilde{u}_{tt} \text{ weak-star in } L^\infty(0,T;L^2(0,L)), \\ & w_t^m = \tilde{u}_{tt}^m \longrightarrow \tilde{u}_{tt} \text{ weakly in } L^2(0,T;H_0^1(0,L)). \end{aligned} \tag{2.6}$$

Using the convergence results obtained in (2.3) and (2.6), we can pass to the limit in (2.2) to obtain, after reversing the change of variables,

$$(u_{tt}, v) + (u_x, v_x) + (a(x)u_t, v) + \epsilon^2(u_{tx}, v_x) = 0, \quad \forall v \in H_0^1(0,L)$$

with

$$\begin{aligned} & u_t = \tilde{u}_t + y_1 \in L^\infty(0,T;H_0^1(0,L)), \\ & u_{tt} = \tilde{u}_{tt} \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L)). \end{aligned}$$

It follows that

$$(u_x + \epsilon^2 u_{tx})_x = u_{tt} + a(x)u_t \in L^\infty(0,T;L^2(0,L)),$$

and

$$u_{tt} - (u_x + \epsilon^2 u_{tx})_x + a(x)u_t = 0 \quad \text{a.e. in} \quad Q.$$

For the proof of uniqueness let u_1 and u_2 be two solutions e let $w = u_1 - u_2$. Then,

$$|w_t(t)|^2 + |w_x(t)|^2 \leq 0$$

and therefore $w = 0$. □

The following result arises from the proof of the previous theorem.

Theorem 2.2. *Assume that* $y_0,\ y_1 \in H_0^1(0,L) \cap H^2(0,L)$*. Then problem* (1.1) *admits a unique solution* y *satisfying the following conditions:*

i. $y \in L^\infty(0,T;H_0^1(0,L) \cap H^2(0,L))$.

ii. $y_t \in L^\infty(0,T;H_0^1(0,L))$.

iii. $y_{tt} \in L^\infty(0,T;L^2(0,L))$.

iv. $y_{tt} - y_{xx} + a(x)y_t = 0 \quad$ *a.e. in* $\quad Q = [0,L] \times [0,T]$.

vi. $y(0) = y_0 \quad$ *and* $\quad y_t(0) = y_1$.

In the next theorem we provide an estimate of the difference between the true solution and the auxiliary solution in terms of the parameter ϵ.

Theorem 2.3. *If* $y_0,\ y_1 \in H_0^1(0,L) \cap H^2(0,L)$ *there exists a constant* C *such that, for* $0 \leq t \leq T$,

$$|y_t(t) - u_t(t)|^2 + |y_x(t) - u_x(t)|^2 \leq C\epsilon^2.$$

Proof. We have, $\forall v \in H_0^1(0,L)$,

$$(y_{tt} - u_{tt}, v) + (y_x - u_x, v_x) + (a(x)(y_t - u_t), v) - \epsilon^2(u_{tx}, v_x) = 0.$$

Thus, taking $v = y_t - u_t$ results in

$$\frac{1}{2}\frac{d}{dt}|y_t - u_t|^2 + \frac{1}{2}\frac{d}{dt}|y_x - u_x|^2 + (a(x)(y_t - u_t), y_t - u_t) \leq \epsilon^2|u_{tx}||y_{tx} - u_{tx}|$$

and integration with respect to t yields

$$|y_t(t) - u_t(t)|^2 + |y_x(t) - u_x(t)|^2 \leq C\epsilon^2,$$

where we used the regularity of y and u and the fact that a is non-negative.

3. Semi-discrete approximation

Let $0 = x_0 < x_1 < \ldots < x_{s+1} = L$ be a regular partition of the interval $(0,L)$ into sub-intervals $I_j = (x_{j-1}, x_j)$, $j = 1, \ldots, s+1$, of length $h = L/(s+1)$. We denote by $S_0^h \subset H_0^1(0,L)$ the space of continuous piecewise linear functions associated to this partition and by P_0^h the elliptic projection $P_0^h : H_0^1(0,L) \to S_0^h$, defined by $((P_0^h\eta)_x, \chi_x) = (\eta_x, \chi_x)\ \forall \chi \in S_0^h$, satisfying Ciarlet [1]

$$|\eta - P_0^h\eta| \leq Ch|\eta_x.| \tag{3.1}$$

Let $\epsilon = h$ as in Tébou [7]. The semi-discrete finite element method to (2.1) is to find $u^h(t) \in S_0^h$ for $0 < t \le T$ such that

$$(u_{tt}^h, W) + (u_x^h, W_x) + (a(x)u_t^h, W) + h^2(u_{tx}^h, W_x) = 0, \tag{3.2}$$

$\forall\ W \in S_0^h$, with $u^h(0) = P_0^h y_0$ and $u_t^h(0) = P_0^h y_1$.

We can show that the semi-discrete energy decays choosing $W = u_t^h$ as a test function. This gives

$$\frac{1}{2}\frac{d}{dt}\left(|u_t^h|^2 + |u_x^h|^2\right) + (a(x)u_t^h, u_t^h) + h^2|u_{tx}^h|^2 = 0$$

and keeping in mind that $a(x) > 0$ in (b, c) we conclude that

$$\frac{1}{2}\frac{d}{dt}\left(|u_t^h|^2 + |u_x^h|^2\right) \le 0.$$

Furthermore, integrating in time we get stability:

$$\begin{aligned}&|u_t^h(t)|^2 + |u_x^h(t)|^2 + 2\int_0^t (a(x)u_t^h(s), u_t^h(s))\ ds + 2h^2\int_0^t |u_{tx}^h(s)|^2\ ds\\ &= |u_t^h(0)|^2 + |u_x^h(0)|^2 \le C.\end{aligned}$$

3.1. Error estimate

In this section we derive error bounds for the piecewise linear approximation (3.2).

Theorem 3.1. *Under the assumptions of* Theorem 2.1 *the estimate*

$$|u_t(t) - u_t^h(t)|^2 + |u_x(t) - u_x^h(t)|^2 \le Ch^2$$

holds for $0 \le t \le T$.

Proof. Introducing $\hat{u} = u_t$, $\hat{u}^h = u_t^h$ we can re-write the continuous auxiliary and the semi-discrete problems as

$$(\hat{u}_t, w) + (u_x, w_x) + (a(x)\hat{u}, w) + h^2(\hat{u}_x, w_x) = 0, \tag{3.3}$$

$$(\hat{u}_x, v_x) = (u_{tx}, v_x), \tag{3.4}$$

where $w, v \in H_0^1(0, L)$ and

$$(\hat{u}_t^h, W) + (u_x^h, W_x) + (a(x)\hat{u}^h, W) + h^2(\hat{u}_x^h, W_x) = 0, \tag{3.5}$$

$$(\hat{u}_x^h, V_x) = (u_{tx}^h, V_x), \tag{3.6}$$

with $W, V \in S_0^h$. Then,

$$\begin{aligned}&(\hat{u}_t^h - P_0^h\hat{u}_t, W) + ((u^h - P_0^h u)_x, W_x) + (a(x)(\hat{u}^h - P_0^h\hat{u}, W)\\ &+h^2((\hat{u}^h - P_0^h\hat{u})_x, W_x)\\ =\ &-(P_0^h\hat{u}_t, W) - ((P_0^h u)_x, W_x) - (a(x)P_0^h\hat{u}, W) - h^2((P_0^h\hat{u})_x, W_x)\\ =\ &-(P_0^h\hat{u}_t, W) + (\hat{u}_t, W) + (a(x)\hat{u}, W) - (a(x)P_0^h\hat{u}, W),\end{aligned}$$

where we used the definition of P_0^h and (3.3) with $w = W$. Denoting $e^h \equiv u^h - P_0^h u$, $\hat{e}^h \equiv \hat{u}^h - P_0^h \hat{u}$ we can write

$$\begin{aligned} & (\hat{e}_t^h, W) + (e_x^h, W_x) + (a(x)(\hat{e}^h, W) + h^2(\hat{e}_x^h, W_x) \\ = \quad & (\hat{u}_t - P_0^h \hat{u}_t, W) + (a(x)(\hat{u} - P_0^h \hat{u}), W). \end{aligned}$$

From (3.6) and (3.4) we get

$$(\hat{e}_x^h, V_x) = ((u_t^h - P_0^h \hat{u})_x, V_x) = ((u^h - u)_{tx}, V_x).$$

Thus, taking $W = \hat{e}^h$ and $V = e^h$ we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\hat{e}^h|^2 + \frac{1}{2}\frac{d}{dt}|e_x^h|^2 + (a(x)(\hat{e}^h, \hat{e}^h) + h^2|\hat{e}_x^h|^2 \\ = \quad & (\hat{u}_t - P_0^h \hat{u}_t, \hat{e}^h) + (a(x)(\hat{u} - P_0^h \hat{u}, \hat{e}^h) \end{aligned}$$

and therefore,

$$\begin{aligned} & |\hat{e}^h(t)|^2 + |e_x^h(t)|^2 \\ \leq \quad & C\int_0^t (|\hat{u}_t(s) - P_0^h \hat{u}_t(s)|^2 + |a(x)(\hat{u}(s) - P_0^h \hat{u}(s)|^2 + |\hat{e}^h(s)|^2)\ ds. \end{aligned}$$

The standard splittings

$$u - u^h = u - P_0^h u + P_0^h u - u^h, \quad \hat{u} - \hat{u}^h = \hat{u} - P_0^h \hat{u} + P_0^h \hat{u} - \hat{u}^h$$

together with (3.1), the regularity of u, u_t, u_{tt} and the Gronwall inequality yield the result. □

Finally, we deduce from Theorems 2.3 and 3.1 the next result which gives a bound on the error between the solution of our main problem and the semi-discrete approximation.

Theorem 3.2. *Under the assumptions of* Theorem 2.2 *the estimate*

$$|y_t(t) - u_t^h(t)|^2 + |y_x(t) - u_x^h(t)|^2 \leq Ch^2,$$

holds for $0 \leq t \leq T$.

4. Fully discrete approximation

In this section we introduce and analyse a fully discrete finite element method to problem (2.1).

Given an integer $N > 0$, our numerical problem can be stated as find $U^n, n = 2, \dots, N$, such that, $\forall\ W \in S_0^h$,

$$\begin{aligned} & \frac{1}{(\Delta t)^2}(U^{n+1} - 2U^n + U^{n-1}, W) + (U_x^{n+1}, W_x) \\ & + \frac{1}{\Delta t}(a(x)(U^{n+1} - U^n), W) + \frac{h^2}{\Delta t}(U_x^{n+1} - U_x^n, W_x) = 0, \end{aligned} \tag{4.1}$$

where $\Delta t = T/N$ is the time step, $U^0 = P_0^h y_0$ and $U^1 = U^0 + \Delta t P_0^h y_1$.

Writing $U^n(x) = \sum_{i=1}^{s} d_i^n \eta_i(x)$ with $\{\eta_i\}_{i=1}^{s}$ the usual basis of S_0^h, we find that the method defined requires the linear system of algebraic equations

$$(M + [(\Delta t)^2 + \Delta t h^2]K + \Delta t D)\underline{d}^{n+1} = (2M + \Delta t D + \Delta t h^2 K)\underline{d}^n - M\underline{d}^{n-1},$$

where

$$M_{ij} = (\eta_i, \eta_j), \quad K_{ij} = (\eta_{ix}, \eta_{jx}), \quad D_{ij} = (a(x)\eta_i, \eta_j), \quad \{\underline{d}^n\}_i = d_i^n,$$

to be solved at each time step. Since the matrix $M + [(\Delta t)^2 + \Delta t h^2]K + \Delta t D$ is positive definite, this system has a unique solution.

Similarly to the continuous case, we prove now the decay of the energy associated to the discrete problem and obtain stability estimates.

Let

$$\hat{U}^{n+1} = \frac{U^{n+1} - U^n}{\Delta t}, \quad E^{n+1} = \frac{1}{2}(|\hat{U}^{n+1}|^2 + |U_x^{n+1}|^2), \quad n = 0, 1, \ldots, N-1.$$

Taking $W = \hat{U}^{n+1}$ in (4.1) it results

$$\begin{aligned}&\frac{1}{2\Delta t}(|\hat{U}^{n+1} - \hat{U}^n|^2 + |\hat{U}^{n+1}|^2 - |\hat{U}^n|^2) + \frac{1}{2\Delta t}(|U_x^{n+1} - U_x^n|^2 + |U_x^{n+1}|^2 - |U_x^n|^2)\\ &+(a(x)\hat{U}^{n+1}, \hat{U}^{n+1}) + h^2|\hat{U}_x^{n+1}|^2 = 0.\end{aligned}$$

Since a is positive in some interval, we have

$$\frac{1}{2\Delta t}(|\hat{U}^{n+1}|^2 - |\hat{U}^n|^2 + |U_x^{n+1}|^2 - |U_x^n|^2) \leq 0$$

and therefore,

$$\frac{E^{n+1} - E^n}{\Delta t} \leq 0.$$

Moreover,

$$\begin{aligned}&\sum_{i=1}^{n} |\hat{U}^{i+1} - \hat{U}^i|^2 + |\hat{U}^{n+1}|^2 + \sum_{i=1}^{n} |U_x^{i+1} - U_x^i|^2 + |U_x^{n+1}|^2\\ &+2\Delta t \sum_{i=1}^{n} (a(x)\hat{U}^{i+1}, \hat{U}^{i+1}) + 2h^2\Delta t \sum_{i=1}^{n} |\hat{U}_x^{i+1}|^2\\ =\; &\frac{1}{2}(|\hat{U}^1|^2 + |U_x^1|^2) \leq C.\end{aligned}$$

4.1. Error estimate

Let us rewrite (4.1) as

$$\frac{1}{\Delta t}(\hat{U}^{n+1} - \hat{U}^n, W) + (U_x^{n+1}, W_x) + (a(x)\hat{U}^{n+1}, W) + h^2(\hat{U}_x^{n+1}, W_x) = 0, \quad (4.2)$$

$$\frac{1}{\Delta t}(U_x^{n+1} - U_x^n, V_x) = (\hat{U}_x^{n+1}, V_x), \quad (4.3)$$

with $W, V \in S_0^h$ and $n = 1, \ldots, N-1$.

Theorem 4.1. *Suppose that $a(x) > a_0 > 0\ \forall x \in [0, L]$ and that the assumptions of* Theorem 2.1 *hold. Then, if $u_{ttt} \in L^2(0, T; L^2(I))$, we have*

$$|u_t(t_n) - \hat{U}^n|^2 + |u_x(t_n) - U_x^n|^2 + \leq C\left(h^2 + (\Delta t)^2 + (\Delta t)^4\right).$$

Proof. We use the standard decompositions

$$\begin{aligned} U^n - u(t_n) &= U^n - P_0^h u(t_n) + P_0^h u(t_n) - u(t_n), \\ \hat{U}^n - \hat{u}(t_n) &= \hat{U}^n - P_0^h \hat{u}(t_n) + P_0^h \hat{u}(t_n) - \hat{u}(t_n), \end{aligned}$$

and split the error in two parts.

Let $e^n = e(t_n) \equiv U^n - P_0^h u(t_n)$, $\hat{e}^n = \hat{e}(t_n) \equiv \hat{U}^n - P_0^h \hat{u}(t_n)$, $\rho^n = \rho(t_n) \equiv P_0^h u(t_n) - u(t_n)$, and $\hat{\rho}^n = \hat{\rho}(t_n) \equiv P_0^h \hat{u}(t_n) - \hat{u}(t_n)$.

From (4.2) we have, $\forall\ W \in S_0^h$,

$$\begin{aligned} & \frac{1}{\Delta t}(\hat{e}^{n+1} - \hat{e}^n, W) + (e_x^{n+1}, W_x) + (a(x)\hat{e}^{n+1}, W) + h^2(\hat{e}_x^{n+1}, W_x) \\ = \quad & -\frac{1}{\Delta t}(P_0^h \hat{u}(t_{n+1}) - P_0^h \hat{u}(t_n), W) - ((P_0^h u(t_{n+1}))_x, W_x) \\ & -(a(x)P_0^h \hat{u}(t_{n+1}), W) - h^2((P_0^h \hat{u}(t_{n+1}))_x, W_x) \end{aligned}$$

and the definition of P_0^h together with (3.3) gives

$$\begin{aligned} & \frac{1}{\Delta t}(\hat{e}^{n+1} - \hat{e}^n, W) + (e_x^{n+1}, W_x) + (a(x)\hat{e}^{n+1}, W) + h^2(\hat{e}_x^{n+1}, W_x) \\ = \quad & -\frac{1}{\Delta t}(P_0^h \hat{u}(t_{n+1}) - P_0^h \hat{u}(t_n), W) + (\hat{u}_t(t_{n+1}), W) \\ & +(a(x)\hat{u}(t_{n+1}), W) - (a(x)P_0^h \hat{u}(t_{n+1}), W). \end{aligned} \tag{4.4}$$

Now, equations (4.3) and (3.4) yield, $\forall\ V \in S_0^h$,

$$\begin{aligned} (\hat{e}_x^{n+1}, V_x) &= \frac{1}{\Delta t}(U_x^{n+1} - U_x^n, V_x) - ((P_0^h \hat{u}(t_{n+1}))_x, V_x) \\ &= \frac{1}{\Delta t}(U_x^{n+1} - U_x^n, V_x) - (u_{tx}(t_{n+1}), V_x). \end{aligned} \tag{4.5}$$

Choosing $W = \hat{e}^{n+1}$ in (4.4) and $V = e^{n+1}$ in (4.5) results in

$$\begin{aligned} & \frac{1}{\Delta t}(\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\Delta t}(U_x^{n+1} - U_x^n, e_x^{n+1}) - (u_{tx}(t_{n+1}), e_x^{n+1}) \\ & +(a(x)\hat{e}^{n+1}, \hat{e}^{n+1}) + h^2|\hat{e}_x^{n+1}|^2 \\ = \quad & \left(\hat{u}_t(t_{n+1}) - \frac{P_0^h \hat{u}(t_{n+1}) - P_0^h \hat{u}(t_n)}{\Delta t}, \hat{e}^{n+1}\right) \\ & +(a(x)(\hat{u}(t_{n+1}) - P_0^h \hat{u}(t_{n+1})), \hat{e}^{n+1}). \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{1}{2\Delta t}(|\hat{e}^{n+1} - \hat{e}^n|^2 + |\hat{e}^{n+1}|^2 - |\hat{e}^n|^2) + \frac{1}{2\Delta t}(|e_x^{n+1} - e_x^n|^2 + |e_x^{n+1}|^2 - |e_x^n|^2) \\ & +(a(x)\hat{e}^{n+1}, \hat{e}^{n+1}) + h^2|\hat{e}_x^{n+1}|^2 \\ = \quad & \left(u_{tx}(t_{n+1}) - \frac{u_x(t_{n+1}) - u_x(t_n)}{\Delta t}, e_x^{n+1}\right) \\ & +\left(\hat{u}_t(t_{n+1}) - \frac{P_0^h \hat{u}(t_{n+1}) - P_0^h \hat{u}(t_n)}{\Delta t}, \hat{e}^{n+1}\right) \\ & +(a(x)(\hat{u}(t_{n+1})) - P_0^h \hat{u}(t_{n+1}), \hat{e}^{n+1}) \end{aligned}$$

and using that $a(x) > a_0 > 0$, we find

$$\begin{aligned} & \frac{1}{\Delta t}(|\hat{e}^{n+1}-\hat{e}^n|^2+|\hat{e}^{n+1}|-|\hat{e}^n|^2)+\frac{1}{\Delta t}(|e_x^{n+1}-e_x^n|^2+|e_x^{n+1}|^2-|e_x^n|^2) \\ & +a_0|\hat{e}^{n+1}|^2+h^2|\hat{e}_x^{n+1}|^2 \\ \leq\ & |e_x^{n+1}|^2+a_0|\hat{e}^{n+1}|^2+C\Big(\Big|u_{tx}(t_{n+1})-\frac{u_x(t_{n+1})-u_x(t_n)}{\Delta t}\Big|^2\Big) \\ & +C\Big(\Big|\hat{u}_t(t_{n+1})-\frac{P_0^h\hat{u}(t_{n+1})-P_0^h\hat{u}(t_n)}{\Delta t}\Big|^2\Big) \\ & +C\Big(\big|a(x)(\hat{u}(t_{n+1})-P_0^h\hat{u}(t_{n+1}))\big|^2\Big). \end{aligned} \tag{4.6}$$

Let us estimate the last three terms. We have

$$\begin{aligned} I_1 &\equiv \left|u_{tx}(t_{n+1})-\frac{u_x(t_{n+1})-u_x(t_n))}{\Delta t}\right| = \left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)u_{ttx}(s)ds\right| \\ &\leq \frac{1}{\Delta t}\left((\Delta t)^3\int_{t_n}^{t_{n+1}}|u_{ttx}|^2dt\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} I_2 &\equiv \left|\hat{u}_t(t_{n+1})-\frac{P_0^h\hat{u}(t_{n+1})-P_0^h\hat{u}(t_n)}{\Delta t}\right| \\ &\leq \left|\hat{u}_t(t_{n+1})-\frac{\hat{u}(t_{n+1})-\hat{u}(t_n)}{\Delta t}\right|+\left|\frac{\hat{u}(t_{n+1})-\hat{u}(t_n)}{\Delta t}-\frac{P_0^h\hat{u}(t_{n+1})-P_0^h\hat{u}(t_n)}{\Delta t}\right| \\ &= \left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(s-t_n)\hat{u}_{tt}(s)ds\right|+\left|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\hat{\rho}_t(s)ds\right| \\ &\leq \frac{1}{\Delta t}\left((\Delta t)^3\int_{t_n}^{t_{n+1}}|\hat{u}_{tt}|^2\ dt\right)^{1/2}+\frac{1}{\Delta t}\left(\Delta t\int_{t_n}^{t_{n+1}}|\hat{\rho}_t|^2\ dt\right)^{1/2}. \end{aligned}$$

Finally,

$$I_3 \equiv \big|a(x)(\hat{u}(t_{n+1})-P_0^h\hat{u}(t_{n+1}))\big| = |a(x)\hat{\rho}^{n+1}|.$$

Collecting all estimates, noting (3.1) and summing (4.6) over n we obtain

$$\begin{aligned} & \sum_{i=1}^n|\hat{e}^{i+1}-\hat{e}^i|^2+|\hat{e}^{n+1}|^2+\sum_{i=1}^n|e_x^{i+1}-e_x^i|^2+|e_x^{n+1}|^2 \\ \leq\ & |\hat{e}^1|^2+|e_x^1|^2+\Delta t\sum_{i=1}^n|e_x^{i+1}|^2+C\Big((\Delta t)^2\int_{t_1}^{t_{n+1}}|u_{ttx}|^2dt+(\Delta t)^2\int_{t_1}^{t_{n+1}}|\hat{u}_{tt}|^2dt\Big) \\ & +C\Big(\int_{t_1}^{t_{n+1}}|\hat{\rho}_t|^2\ dt+\Delta t\sum_{i=1}^n|a(x)\hat{\rho}^{i+1}|^2\Big) \\ \leq\ & |\hat{e}^1|^2+|e_x^1|^2+\Delta t\sum_{i=1}^n|e_x^{i+1}|^2+C\Big((\Delta t)^2\int_{t_1}^{t_{n+1}}|u_{ttx}|^2dt+(\Delta t)^2\int_{t_1}^{t_{n+1}}|\hat{u}_{tt}|^2dt\Big) \\ & +C\Big(h^2\int_{t_1}^{t_{n+1}}|\hat{u}_{tx}|^2\ dt\Big)+\|a(x)\|_\infty^2h^2\Delta t\sum_{i=1}^n|\hat{u}(t_{i+1})_x|^2. \end{aligned} \tag{4.7}$$

Note now that, due to the definition of U^0, U^1 and P_0^h,

$$(\hat{e}_x^1, V_x) = \frac{1}{\Delta t}(U_x^1-U_x^0, V_x)-(P_0^h\hat{u}(t_1))_x, V_x) = (u_{tx}(0)-u_{tx}(t_1), V_x)$$

Table 1. Computed errors when $T = 1$.

h	$\lvert y_x(T) - U_x^N\rvert$	rate	$\lvert y_t(T) - \hat{U}^N\rvert$	rate
1/20	1.576×10^{-2}		6.145×10^{-3}	
1/40	6.890×10^{-3}	1.19	2.613×10^{-3}	1.23
1/80	3.214×10^{-3}	1.10	1.189×10^{-3}	1.14
1/160	1.554×10^{-3}	1.05	5.655×10^{-4}	1.07
1/320	7.585×10^{-4}	1.03	2.738×10^{-4}	1.05

and writing

$$u_{tx}(t_1) = u_{tx}(0) + u_{ttx}(\xi)\Delta t,$$

we get, using the Poincaré inequality,

$$|\hat{e}^1| \leq C|\hat{e}_x^1| \leq C\Delta t|u_{ttx}(\xi)|.$$

Further,

$$(e_x^1, V_x) = ((P_0^h u(0))_x + \Delta t(P_0^h u_t(0))_x - (P_0^h u(t_1))_x, V_x),$$

so that

$$|e_x^1| \leq \frac{(\Delta t)^2}{2}|u_{ttx}(\eta)|,$$

where we used the definition of P_0^h and the expansion

$$u_x(t_1) = u_x(0) + u_{tx}(0)\Delta t + \frac{(\Delta t)^2}{2}u_{ttx}(\eta).$$

Applying the discrete Gronwall inequality to (4.7) the proof is complete. □

The previous result together with Theorem 2.3 leads to

Theorem 4.2. *Under the assumptions of* Theorem 4.1 *we have*

$$|y_t(t_n) - \hat{U}^n|^2 + |y_x(t_n) - U_x^n|^2 + \leq C\left(h^2 + (\Delta t)^2 + (\Delta t)^4\right).$$

5. Numerical simulations

In our experiments we let $L = 1$ and numerical integration, namely trapezoidal rule, was used to compute the matrices M and D.

The accuracy of the method was examined numerically by comparing the solution obtained on coarse meshes with a known solution. We choose $a(x) = 2\pi$, $T = 1$, exact solution $u(x,t) = \sin(\pi x)\exp(-\pi t)$ and we evaluated the errors $|y_x(T) - U_x^N|$ and $|y_t(T) - \hat{U}^N|$. For the coarse meshes, $s + 1 = 20, 40, 80, 160, 320$ and $\Delta t = h$. Thus, if E_i denotes the L^2-error when $h = (5 \times 2^{i+1})^{-1}$, the rates of convergence are given by $\ln(E_i/E_{i+1})/\ln 2$. The computed errors and rates are shown in Table 1 where we observe that the errors decreased by a factor of approximately 2 when the mesh parameter h was halved which gives a convergence rate of approximately 1, as expected.

For completeness, the evolution in time of the string's position and the energy decay were observed for a non-constant function $a \geq 0$. We let $h = 1/200$, $\Delta t = 1 \times 10^{-5}$, $y_1(x) = 0$ and

$$a(x) = \begin{cases} 5, & \text{if } 0.7 < x < 0.8, \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 1 we display the results when the initial displacement is $y_0(x) = \sin(45\pi x)$. We see that the amplitude of oscillations are smaller when we introduce artificial viscosity. Next we choose a discontinuous initial condition

$$y_0(x) = \begin{cases} 2, & \text{if } 0.4 < x < 0.6, \\ 0, & \text{otherwise.} \end{cases}$$

Again, the artificial viscosity damps out faster the oscillations (see Figure 2). Finally in Figure 3 we observe the influence of artificial viscosity in the energy decay.

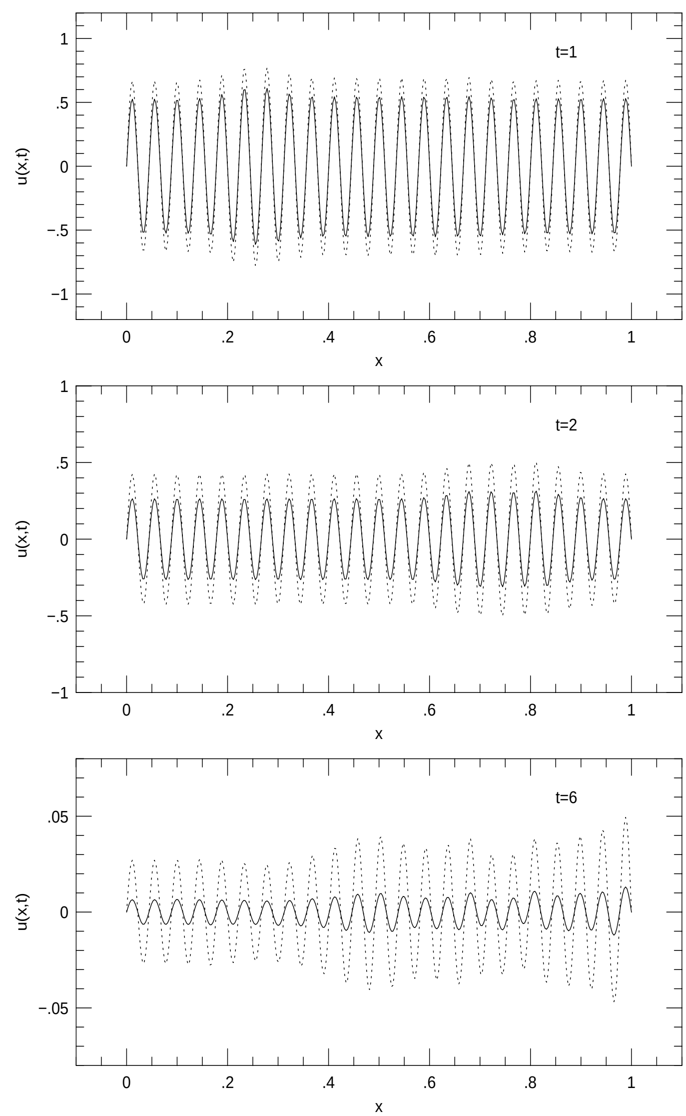

Figure 1. The time evolution of the string's position when $y_0(x) = \sin(45\pi x)$ with (solid line) and without (dot line) artificial viscosity.

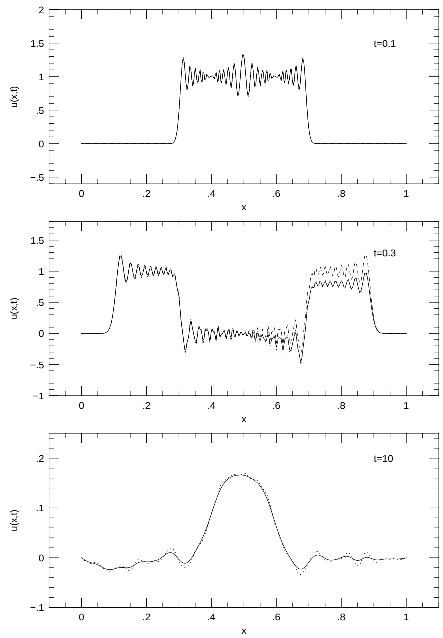

Figure 2. The time evolution of the string's position when $y_0(x)$ is a square wave with (solid line) and without (dot line) artificial viscosity. For comparison, when $t = 0.1$ and $t = 0.3$ the numerical solution of the wave equation is drawn (short dash).

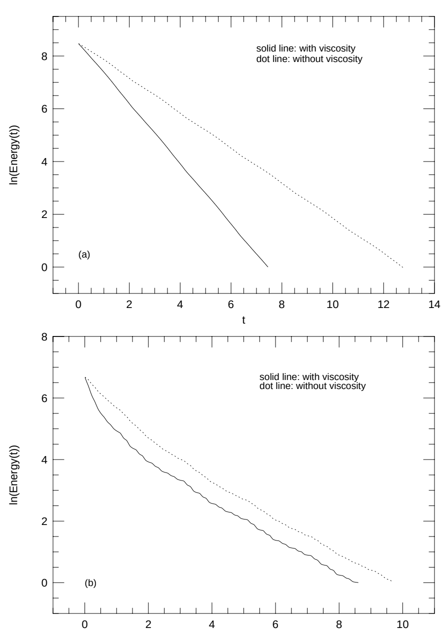

Figure 3. The time evolution of the energy when (a) $y_0(x) = \sin(45\pi x)$ and (b) $y_0(x)$ is a square wave.

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