ALGEBRAIC ASPECTS OF INTEGRABILITY FOR POLYNOMIAL DIFFERENTIAL SYSTEMS*

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Abstract In this article we summarize the results on algebraic aspects of integrability for polynomial differential systems and its application, which include the Darboux, elementary and Liouvelle integrability. Darboux theory of integrability was found by Darboux in 1878, and it becomes extremely useful in study of the center focus problem, of bifurcation, of limit cycle problem and of global dynamics. The importance of Darboux theory of integrability is also presented by the Singer's theorem for planar polynomial differential system. That is, if a polynomial system is Liouville integrable, then it is Darboux integrable, i.e. the system has a Darboux first integral or a Darboux integrating factor.

Keywords Darboux integrability, elementary integrability, Liouville integrability, invariant algebraic curves.

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1. Introduction

The integrability theory of differential systems is classic, and it becomes a very active subject. Because it is extremely useful in the study of dynamics of differential system, for instance, the center focus problem, the bifurcation, the limit cycle problem and the global dynamics.

Integrability has different definition in different fields. Here we mainly summarize some results related to algebraic aspects of polynomial differential systems. The algebraic theory of integrability involves the real and complex analysis, algebraic geometry and the field extension and so on. For further information on this subject, we refer readers to Daboux [20, 21], Jouanolou [24], Schlomiuk [42],Carnicer [2], Chavarriga *al al* [3], Llibre [26], Dumortier and Llibre *et al* [22], Christopher *et al* [19], Llibre and Zhang [32].

2. Darboux theory of integrability

Darboux theory of integrability was founded by Darboux in 1878. He established an essential relation between integrability and invariant algebraic curves or surfaces of polynomial differential systems (see Darboux [20] and Poincaré [39]).

Consider polynomial differential systems

$$\dot{x} = P(x), \qquad x \in \mathbb{K}^n, \tag{2.1}$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $P(x) = (P_1(x), \ldots, P_n(x))$ are vector valued polynomial functions, $\max\{\deg P_1, \ldots, \deg P_n\} =: m \in \mathbb{N}$ is called *the degree of polynomial differential systems* (2.1). In this paper we will always use m to denote the degree of system (2.1). We also use

$$\mathcal{X}_P = P_1(x)\frac{\partial}{\partial x_1} + \ldots + P_n(x)\frac{\partial}{\partial x_n}$$

to represent the vector field associated to system (2.1).

Let $\mathbb{K}[x]$ be the polynomial ring in x in the field $\mathbb{K}$. $f(x) \in \mathbb{C}[x]$ is a *Darboux polynomial* of system (2.1), if there exists a $k(x) \in \mathbb{C}[x]$ such that

$$\mathcal{X}_P(f) = kf. \tag{2.2}$$

We call $k(x)$ *cofactor*. Clearly, if $f(x)$ is a Darboux polynomial of (2.1), then the set $\{x \in \mathbb{K}^n | \, f(x) = 0\}$ is an invariant set of (2.1). For simplifying notations we use $f = 0$ to denote $\{x \in \mathbb{K}^n | \, f(x) = 0\}$.

We remark that even a vector field $\mathcal{X}_P$ is real, its Darboux polynomial and cofactor can be complex.

If f is a Darboux polynomial of (2.1), we call $f = 0$ an *invariant algebraic curve*($n = 2$), or *invariant algebraic surface*($n = 3$), or *invariant algebraic hypersurface*($n > 3$) of system (2.1).

Next we introduce Darboux first integral, integrating factor, Jacobi multiplier and their properties. The first one is the relation between the Darboux polynomials and their irreducible factors.

Proposition 2.1. *Assume that $f \in \mathbb{C}[x]$, and $f = f_1^{m_1} \ldots f_r^{m_r}$ are its irreducible decomposition in $\mathbb{C}[x]$. Then the following statements hold.*

   a) *f is a Darboux polynomial of (2.1) if and only if each f_j is a Darboux polynomial of (2.1), $j = 1, \ldots, r$.*

   b) *if $k(x), k_1(x), \ldots, k_r(x)$ are respectively the cofactors of $f(x), f_1(x), \ldots, f_r(x)$, then*

$$k(x) = m_1 k_1(x) + \ldots, m_r k_r(x).$$

Proof. Its proof is direct, the readers can check it easily. □

From the above results, in what follows our Darboux polynomials are irreducible if we do not specify it.

Proposition 2.2. *For $f \in \mathbb{C}[x]$ irreducible, the following statements hold.*

   a) *If $f = 0$ is invariant for (2.1), i.e. $\mathcal{X}_P(f)|_{f=0} = 0$, then there exists $k(x) \in \mathbb{K}[x]$ such that (2.2) hold.*

   b) *If $k(x)$ is a cofactor of $f(x)$, then $\deg k \leq m - 1$.*

Proof. *b*) It can be directly checked from the definition (2.2).

a) Its proof follows from Fulton [23, p18, Corrolary 1]. It can also be obtained from the next result, see for example Olver [37, Proposition 2.10]. □

Proposition 2.3. *Let $M \subset \mathbb{R}^n$ be an m dimensional smooth manifold, $F = (f_1, \ldots, f_l) : M \to \mathbb{R}^l$ be smooth maps. If F has maximal rank on $V(F) := \{x \in M | F(x) = 0\}$, i.e. for $\forall x \in V(F)$, the rank of Jacobian matrix of F at x is equal to $\min\{m, l\}$, then any smooth function $g : M \to \mathbb{R}$ vanishes on $V(F)$ if and only if there exist smooth functions $q_1, \ldots, q_l$ such that*

$$g(x) = q_1(x)f_1(x) + \ldots + q_l(x)f_l(x), \qquad x \in M.$$

In fact, since f is irreducible on $\mathbb{R}^n$, the derivative of f has only finitely many zeros on $V(f)$, so it is of full rank. So there exists a smooth function $k(x)$ on $\mathbb{R}^n$ such that

$$\mathcal{X}_P(f)(x) = k(x)f(x), \qquad x \in \mathbb{R}^n.$$

Moreover, we get from the above equality that $k(x)$ is a polynomial.

Suppose that $f, g \in \mathbb{C}[x]$, if there exists $L(x) \in \mathbb{C}_{m-1}[x]$ with $\mathbb{C}_{m-1}[x]$ the set of polynomials with coefficients in $\mathbb{C}$ of degree no more than $m - 1$, such that

$$\mathcal{X}_P(\exp(g/f)) = L \exp(g/f),$$

we call $\exp(g/f)$ *exponential factor* of differential system (2.1), L *cofactor* of $\exp(g/f)$. Without loss of generality, in what follows when we say exponential factor $\exp(g/f)$, we always mean that g, f are relative coprime, i.e. $(g, f) = 1$.

The next result presents a relation between exponential factor and invariant algebraic hypersurface.

Proposition 2.4. *The function $\exp(g/f)$ are exponential factor of polynomial differential system (2.1) if and only if f is a Darboux polynomial of (2.1), and*

$$\mathcal{X}_P(g) = gL + fk,$$

where L and k are respectively the cofactors of $\exp(g/f)$ and f.

Proof. *Necessity.* By the definition of cofactor we have

$$\frac{\mathcal{X}_P(g)f - g\mathcal{X}_P(f)}{f^2} = L,$$

i.e.

$$\mathcal{X}_P(g)f - Lf^2 = g\mathcal{X}_P(f).$$

Since $(g, f) = 1$, there is a $k \in \mathbb{C}[x]$ such that $\mathcal{X}_P(f) = kf$. So we have

$$\mathcal{X}_P(g) = Lf + gk.$$

Sufficiency. From the proof of necessity we have

$$\mathcal{X}_P\left(\exp\left(\frac{g}{f}\right)\right) = \exp\left(\frac{g}{f}\right)\frac{\mathcal{X}_P(g)f - g\mathcal{X}_P(f)}{f^2} = L \exp\left(\frac{g}{f}\right).$$

□

For $g, f, f_1, \ldots, f_r \in \mathbb{C}[x]$, $s_1, \ldots, s_r \in \mathbb{C}$, the function of the form

$$H := \exp\left(\frac{g}{f}\right) f_1^{s_1} \ldots f_r^{s_r},$$

is called *Darboux function*. A Darboux function is usually multi–valued.

If the first integral (or integrating factor or Jacobi multiplier) of a polynomial differential system (2.1) is a Darboux function, then it will be called *Darboux first* (or *Darboux integrating factor* or *Jacobi multiplier of Darboux type*). If a first integral is a polynomial function (or a rational function), it is called *polynomial first integral* (or *rational first integral*).

Proposition 2.5. *Suppose that polynomial differential system* (2.1) *has p irreducible Darboux polynomials* $f_1, \ldots, f_p$ *with the corresponding cofactors* $k_1, \ldots, k_p$, *and q relative different exponential factors* $E_1, \ldots, E_q$ *with the corresponding cofactors* $L_1, \ldots, L_q$. *For* $s_1, \ldots, s_p, r_1, \ldots, r_q \in \mathbb{C}$, *the following statements hold.*

a) $H = f_1^{s_1} \ldots f_p^{s_p} E_1^{r_1} \ldots E_q^{r_q}$ *is a Darboux first integral of differential system* (2.1) *if and only if* $s_1 k_1 + \ldots + s_p k_p + r_1 L_1 + \ldots + r_q L_q = 0$.

b) $M = f_1^{s_1} \ldots f_p^{s_p} E_1^{r_1} \ldots E_q^{r_q}$ *is a Jacobian multiplier or integrating factor of differential system* (2.1) *if and only if* $s_1 k_1 + \ldots + s_p k_p + r_1 L_1 + \ldots + r_q L_q = -\mathrm{div} P$.

Proof. The proof of $a)$ follows from the fact that

$$\mathcal{X}_P(H) = \sum_{i=1}^{p} \left(\prod_{i=1}^{p} s_i f_i^{s_i-1} \mathcal{X}_P(f_i)\right) \prod_{j=1}^{q} E_j^{r_j} + \prod_{i=1}^{p} f_i^{s_i} \sum_{j=1}^{q} \left(\prod_{j=1}^{q} r_j E_j^{r_j-1} \mathcal{X}_P(E_j)\right)$$

$$= \left(\sum_{i=1}^{p} s_i k_i + \sum_{j=1}^{q} r_j L_j\right) H.$$

The proof of $b)$ can be obtained from the fact that M is a Jacobi multiplier if and only if $\mathcal{X}_P(M) = -M \mathrm{div}(f)$, and $a)$. $\qquad\square$

We remark that if a real polynomial differential system $\mathcal{X}_P$ has a complex Darboux first integral, it will have a real Darboux first integral. This can be got from the facts that the product of two first integrals is also a first integral and that if f is a Darboux polynomial of $\mathcal{X}_P$, its conjugacy $\overline{f}$ is also a Darboux polynomial of $\mathcal{X}_P$. In addition, for $\forall k \in \mathbb{C}$ we have

$$f^k \overline{f}^{\overline{k}} = \left((\mathrm{Re}\, f)^2 + (\mathrm{Im}\, f)^2\right)^{\mathrm{Re}\, k} \exp\left(-2\,\mathrm{Im}\, k \, \arctan \frac{\mathrm{Im} f}{\mathrm{Re} f}\right),$$

where Re and Im are respectively the real and imaginary parts. This last equality can be proved with the help of

$$\arctan z = \ln\left(\left(\frac{1 - \mathbf{i}\, z}{1 + \mathbf{i}\, z}\right)^{\mathbf{i}/2}\right), \qquad z \in \mathbb{C},$$

where $\mathbf{i} = \sqrt{-1}$.

Theorem 2.1. (Darboux-Jouanolou *Theorem*) *Suppose that polynomial differential system* (2.1) *has p irreducible Darboux polynomials. Set*

$$N = \left(\begin{array}{c} m + n - 1 \\ n \end{array}\right).$$

a) *If $p \geq N + 1$, system (2.1) has a Darboux first integral.*

b) *System (2.1) has a rational first integral if and only if $p \geq N + n$.*

Statement a) of Theorem 2.1 was established by Darboux [20, 21] in 1878. b) was obtained by Jouanolou [24] in 1979 using the sophisticated tools of algebraic geometry. In 2000, Christopher and Llibre provided an elementary proof of the Jouanolou's result in the two dimensional case. An elementary proof for higher dimensional case was got by Llibre and Zhang until 2010. The next are the proof of Llibre and Zhang [34].

We say that functions $H_1, \ldots, H_m$ are k–*functionally independent* on $\mathcal{D}_1 \subset \mathbb{K}^n$, if among $H_1, \ldots, H_m$ there are k functionally independent elements on D_1, whereas any $k + 1$ elements are not functionally independent on any positive Lebesgue measure subset of $\mathcal{D}_1$.

Lemma 2.1. *Suppose that $H_1, \ldots, H_m$ are k $(< m)$ functionally independent first integrals of polynomial vector fields $\mathcal{X}_P$, and are analytic on the full Lebesgue measure subset of $\mathbb{K}^n$. Without loss of generality, we assume that $H_1, \ldots, H_k$ are functionally independent.*

(a) *For $\forall s \in \{k+1, \ldots, m\}$, there exist analytic functions $C_{s1}(x), \ldots, C_{sk}(x)$ on the full Lebesgue measure subset of $\mathbb{C}^n$ such that*

$$\partial_x H_s(x) = C_{s1}(x)\partial_x H_1(x) + \ldots + C_{sk}(x)\partial_x H_k(x). \qquad (2.3)$$

(b) *For $\forall s \in \{k+1, \ldots, m\}$, $j \in \{1, \ldots, k\}$, if the function $C_{sj}(x)$ is not a constant, then it is first integral of the vector field $\mathcal{X}_P$.*

Proof. Let $\mathcal{D}_1$ be a full Lebesgue measure subset of $\mathbb{K}^n$, such that $H_1, \ldots, H_m$ are k functionally independent on $\mathcal{D}_1$.

By the assumption there exists a full Lebesgue measure subset $\mathcal{D}_2 \subset \mathcal{D}_1$ such that for $\forall x \in \mathcal{D}_2$, $\forall s \in \{k+1, \ldots, m\}$, $\partial_x H_1(x), \ldots, \partial_x H_k(x)$ are linearly independent on $\mathbb{C}^n$, whereas $\partial_x H_s(x)$ and $\partial_x H_1(x), \ldots, \partial_x H_k(x)$ are linearly dependent on $\mathbb{C}^n$. So there exist $C_{s1}(x), \ldots, C_{sk}(x)$ such that the equality (2.3) hold on $\mathcal{D}_2$. By the Cramer rule, these functions $C_{s1}(x), \ldots, C_{sk}(x)$ defined on $\mathcal{D}_2$ can be expressed as functions of $\partial_x H_1, \ldots, \partial_x H_k, \partial_x H_s$, and so they are analytic on $\mathcal{D}_2$. This proves statement (a).

The next proof on (b) is processed on $\mathcal{D}_2$. For $\forall i, j \in \{1, \ldots, n\}$, we get from (2.3) that

$$\begin{aligned}
\frac{\partial H_s}{\partial x_i} &= C_{s1}(x)\frac{\partial H_1}{\partial x_i} + \ldots + C_{sk}(x)\frac{\partial H_k}{\partial x_i}, \\
\frac{\partial H_s}{\partial x_j} &= C_{s1}(x)\frac{\partial H_1}{\partial x_j} + \ldots + C_{sk}(x)\frac{\partial H_k}{\partial x_j}.
\end{aligned}$$

Differentiating these last two equalities respectively with respect to x_j and x_i, and subtracting the resulting expressions, we get

$$\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} + \ldots + \frac{\partial C_{sk}}{\partial x_i}\frac{\partial H_k}{\partial x_j} - \frac{\partial C_{sk}}{\partial x_j}\frac{\partial H_k}{\partial x_i} = 0. \qquad (2.4)$$

Since $k \leq n - 1$, we distinguish two cases. First we assume that $k = n - 1$. From

(2.4) we have

$$\sum_{1\le i<j\le n} \left(\left(\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} + \ldots + \frac{\partial C_{sk}}{\partial x_i}\frac{\partial H_k}{\partial x_j} - \frac{\partial C_{sk}}{\partial x_j}\frac{\partial H_k}{\partial x_i} \right) \cdot \sum_{\sigma(k_1,k_2\ldots,k_{n-2})} (-1)^{\tau(ijk_1k_2\ldots,k_{n-2})} \frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}}\cdots\frac{\partial H_{n-1}}{\partial x_{k_{n-2}}} \right) = 0,$$

where σ is the permutation of $\{1,\ldots,n\}\setminus\{i,j\}$, and the second summation takes over all these permutations, τ is the minimal times permutating a give order to $\{1,\ldots,n\}$. In fact, the last equation can be written in

$$\begin{vmatrix} \frac{\partial C_{s1}}{\partial x_1} & \frac{\partial C_{s1}}{\partial x_2} & \cdots & \frac{\partial C_{s1}}{\partial x_n} \\ \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \cdots & \frac{\partial H_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_{n-1}}{\partial x_1} & \frac{\partial H_{n-1}}{\partial x_2} & \cdots & \frac{\partial H_{n-1}}{\partial x_n} \end{vmatrix} = 0. \tag{2.5}$$

It can be got from the next two equalities:

$$\sum_{1\le i<j\le n} \left(\frac{\partial C_{s1}}{\partial x_i}\frac{\partial H_1}{\partial x_j} - \frac{\partial C_{s1}}{\partial x_j}\frac{\partial H_1}{\partial x_i} \right)$$
$$\cdot \sum_{\sigma(k_1,k_2\ldots,k_{n-2})} (-1)^{\tau(ijk_1k_2\ldots,k_{n-2})} \frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}}\cdots\frac{\partial H_{n-1}}{\partial x_{k_{n-2}}}$$
$$= \begin{vmatrix} \frac{\partial C_{s1}}{\partial x_1} & \frac{\partial C_{s1}}{\partial x_2} & \cdots & \frac{\partial C_{s1}}{\partial x_n} \\ \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \cdots & \frac{\partial H_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_{n-1}}{\partial x_1} & \frac{\partial H_{n-1}}{\partial x_2} & \cdots & \frac{\partial H_{n-1}}{\partial x_n} \end{vmatrix},$$

and for $l = 2,\ldots,k$

$$\sum_{1\le i<j\le n} \left(\frac{\partial C_{sl}}{\partial x_i}\frac{\partial H_l}{\partial x_j} - \frac{\partial C_{sl}}{\partial x_j}\frac{\partial H_l}{\partial x_i} \right)$$
$$\cdot \sum_{\sigma(k_1,k_2\ldots,k_{n-2})} (-1)^{\tau(ijk_1k_2\ldots,k_{n-2})} \frac{\partial H_2}{\partial x_{k_1}}\frac{\partial H_3}{\partial x_{k_2}}\cdots\frac{\partial H_{n-1}}{\partial x_{k_{n-2}}}$$
$$= \begin{vmatrix} \frac{\partial C_{sl}}{\partial x_1} & \frac{\partial C_{sl}}{\partial x_2} & \cdots & \frac{\partial C_{sl}}{\partial x_n} \\ \frac{\partial H_l}{\partial x_1} & \frac{\partial H_l}{\partial x_2} & \cdots & \frac{\partial H_l}{\partial x_n} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \cdots & \frac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_{n-1}}{\partial x_1} & \frac{\partial H_{n-1}}{\partial x_2} & \cdots & \frac{\partial H_{n-1}}{\partial x_n} \end{vmatrix} = 0.$$

For $\forall\, x \in \mathcal{D}_2$, denote by $\mathcal{P}_{n-1}(x)$ the $n-1$ dimensional vector space generated by $\{\partial_x H_1(x), \ldots, \partial_x H_{n-1}(x)\}$. Then we get from (2.5) that the vector $\partial_x C_{s1}(x)$ belongs to $\mathcal{P}_{n-1}(x)$. By the very definition of first integral, for $\forall\, x \in \mathcal{D}_2$

$$\frac{\partial H_j(x)}{\partial x_1} P_1(x) + \ldots + \frac{\partial H_j(x)}{\partial x_n} P_n(x) = 0, \qquad \text{for } j = 1, \ldots, n-1.$$

This means that $\mathcal{X}_P(x) = (P_1(x), \ldots, P_n(x))$ is orthogonal to the $n-1$ dimensional vector space $\mathcal{P}_{n-1}(x)$ on $\mathcal{D}_2$. Hence we have

$$\frac{\partial C_{s1}(x)}{\partial x_1} P_1(x) + \ldots + \frac{\partial C_{s1}(x)}{\partial x_n} P_n(x) = 0, \qquad \text{for all } x \in \mathcal{D}_2.$$

This proves that $C_{s1}(x)$ (if not a constant) is a first integral of the vector field $\mathcal{X}_P$ on $\mathcal{D}_2$.

Similarly we can prove that functions $C_{sj}(x)$ (if not constants), $j = 2, \ldots, k$, are first integrals of $\mathcal{X}_P$. This proves statement (b) in the case $k = n-1$.

Now we assume that $k < n-1$. Similar to the proof of the case $k = n-1$, suppose that $H_1, \ldots, H_m$ are k functionally independent on $\mathcal{D}_2$. For arbitrary $i_1, \ldots, i_{k+1}$ satisfying $1 \le i_1 < i_2 < \ldots < i_{k+1} \le n$ and $\forall\, x \in \mathcal{D}_2$, we have

$$\begin{vmatrix} \dfrac{\partial C_{s1}}{\partial x_{i_1}} & \dfrac{\partial C_{s1}}{\partial x_{i_2}} & \cdots & \dfrac{\partial C_{s1}}{\partial x_{i_{k+1}}} \\[2mm] \dfrac{\partial H_1}{\partial x_{i_1}} & \dfrac{\partial H_1}{\partial x_{i_2}} & \cdots & \dfrac{\partial H_1}{\partial x_{i_{k+1}}} \\[2mm] \vdots & \vdots & \ddots & \vdots \\[2mm] \dfrac{\partial H_k}{\partial x_{i_1}} & \dfrac{\partial H_k}{\partial x_{i_2}} & \cdots & \dfrac{\partial H_k}{\partial x_{i_{k+1}}} \end{vmatrix} = 0.$$

That is, for $\forall\, x \in \mathcal{D}_2$, $\partial_x C_{s1}(x)$ belonging to the k dimensional vector space generated by $\{\partial_x H_1(x), \ldots, \partial_x H_k(x)\}$, denoted by $\mathcal{P}_k(x)$.

On the other hand, since $H_1(x), \ldots, H_k(x)$ are the first integrals of $\mathcal{X}_P$, so for $\forall\, x \in \mathcal{D}_2$ the vector fields $\mathcal{X}_P(x)$ and $\mathcal{P}_k(x)$ are orthogonal. This means that $\mathcal{X}_P(x)$ and $\partial_x C_{s1}(x)$ are orthogonal. Hence we prove that $C_{s1}(x)$ are first integrals of the vector fields $\mathcal{X}_P$ on $\mathcal{D}_2$. Similarly we can prove that $C_{s2}, \ldots, C_{sk}$ are first integrals of the vector field $\mathcal{X}_P$. This proves statement (b). $\qquad \square$

Proof of Theorem 2.1. Suppose that $f_1(x), \ldots, f_p(x)$ are the Darboux polynomials of $\mathcal{X}_P$, $k_1(x), \ldots, k_p(x)$ are the corresponding cofactors.

$a)$ Since each polynomial in $\mathbb{C}[x]$ is uniquely determined by its coefficients, and so $\mathbb{C}_{m-1}[x]$ is an N dimensional vector space. By the assumption $p \ge N + 1$ we get that the cofactors $k_1(x), \ldots, k_p(x)$ are linearly dependent on $\mathbb{C}_{m-1}[x]$. So there exist $s_1, \ldots, s_p \in \mathbb{C}$ such that

$$s_1 k_1(x) + \ldots + s_p k_p(x) = 0.$$

It follows from Proposition 2.5 that $H = f_1^{s_1} \ldots f_p^{s_p}$ is a first integral of the vector field $\mathcal{X}_P$. This proves (a).

For proving (b), we only need to prove the sufficiency, because the necessity is obvious.

Let $\mathcal{V}$ be the vector space generated by $\{k_1(x), \ldots, k_{N+n}(x)\}$. Set $\rho := \dim \mathcal{V}$, then $\rho \leq N$. For simplifying the notation we assume that $p = N$ and $k_1(x), \ldots, k_N(x) \in \mathcal{V}$ are linearly independent. The case $p < N$ can be proved similarly, and so the details are omitted.

For $\forall s \in \{1, \ldots, n\}$, there exist $(\sigma_{s1}, \ldots, \sigma_{sN}, 1) \in \mathbb{C}^{N+1}$ such that

$$\sigma_{s1} k_1(x) + \ldots + \sigma_{sN} k_N(x) + k_{N+s}(x) = 0. \tag{2.6}$$

Since $k_i = \mathcal{X}_P(f_i)/f_i$, equation (2.6) can be written as

$$\mathcal{X}_P \left(\log \left(f_1^{\sigma_{s1}} \ldots f_N^{\sigma_{sN}} f_{N+s} \right) \right) = 0.$$

This means that

$$H_s = \log \left(f_1^{\sigma_{s1}} \ldots f_N^{\sigma_{sN}} f_{N+s} \right), \qquad s = 1, \ldots, n,$$

are analytic first integrals of the vector field $\mathcal{X}_P$ on some full Legesgue measure subset of $\mathbb{C}^n$.

Clearly these n first integrals $H_1, \ldots, H_n$ are functionally dependent on some positive Lebesgue measure subset of $\mathcal{D}_3$. Otherwise these first integrals are functionally independent on some subset $\mathcal{D}_4$ of $\mathcal{D}_3$, then

$$\frac{\partial H_i(x)}{\partial x_1} P_1(x) + \ldots + \frac{\partial H_i(x)}{\partial x_n} P_n(x) = 0, \qquad i = 1, \ldots, n, \quad x \in \mathcal{D}_4.$$

From their functionally independence, this n dimensional homogenous algebraic equations can have only trivial solution $P_i(x) = 0$, $i = 1, \ldots, n$ on $\mathcal{D}_4$. And so the polynomial vector field $\mathcal{X}_P \equiv 0$, $x \in \mathbb{C}^n$, a contradiction.

Define

$$r(x) := \operatorname{rank}\{\nabla H_1(x), \ldots, \nabla H_n(x)\}, \qquad d = \max\{r(x) : x \in \mathcal{D}_3\}.$$

There exists an open subset $\mathcal{O}$ of $\mathcal{D}_3$ such that $d = r(x)$, $x \in \mathcal{O}$, and $d < n$. Without loss of generality we assume that $\{\partial_x H_1(x), \ldots, \partial_x H_d(x)\}$ has rank d on $\mathcal{O}$. Hence by Lemma 2.1(a), for $\forall x \in \mathcal{O}$ there exist $C_{k1}(x), \ldots, C_{kd}(x)$ such that

$$\partial_x H_k(x) = C_{k1}(x)\partial_x H_1(x) + \ldots + C_{kd}(x)\partial_x H_d(x), \quad k = d+1, \ldots, n. \tag{2.7}$$

By Lemma 2.1(b), the function $C_{kj}(x)$ (if not a constant), $j \in \{d+1, \ldots, n\}$, is a first integral of the vector field $\mathcal{X}_P$ on $\mathcal{O}$.

Next we prove that $C_{kj}(x)$ are rational first integrals. From the construction of $H_1, \ldots, H_n$, each $\partial_x H_i$, $i = 1, \ldots, n$, is a vector valued rational function. Since $\{\partial_x H_1(x), \ldots, \partial_x H_d(x)\}$ are linearly independent on $\mathcal{O}$, linear algebraic system (2.7) has a unique solution $(C_{k1}(x), \ldots, C_{kd}(x))$, $k = d+1, \ldots, n$, on $\mathcal{O}$. Obviously each $C_{kj}(x)$, $j \in \{1, \ldots, d\}$, is rational, and satisfies

$$\frac{\partial C_{kj}(x)}{\partial x_1} P_1(x) + \ldots + \frac{\partial C_{kj}(x)}{\partial x_n} P_n(x) = 0 \quad x \in \mathcal{O}.$$

Since $\mathcal{O}$ is an open subset of $\mathbb{C}^n$, and so $C_{kj}(x)$ is a rational function. Hence $C_{kj}(x)$ satisfies the above equation on a full Lebesgue measure subset of $\mathbb{C}^n$ (except where C_{kj} has no definition). This proves that if function $C_{kj}(x)$ is not a constant, it should be a rational first integral of $\mathcal{X}_P$.

Last we prove that there are C_{kj} not a constant.If all functions $C_{k1}, \ldots, C_{kd}$ are constant, we get from the linear algebraic equations (2.7) that

$$H_k(x) = C_{k1}H_1(x) + \ldots + C_{kd}H_d(x) + \log C_k,$$

where C_k are constant. This forces that for $k \in \{d+1, \ldots, n\}$

$$f_1^{\sigma_{k1}} \ldots f_N^{\sigma_{kN}} f_{N+k} = C_k \left(f_1^{\sigma_{11}} \ldots f_N^{\sigma_{1N}} f_{N+1}\right)^{C_{k1}} \ldots \left(f_1^{\sigma_{d1}} \ldots f_N^{\sigma_{dN}} f_{N+d}\right)^{C_{kd}}.$$

This is in contradiction with the facts that the polynomials $f_1, \ldots, f_{N+d}$ are irreducible and are pairwise different. Hence there exist some $j_0 \in \{1, \ldots, d\}$ such that $C_{k_0j_0}(x)$ is a non constant function. This proves Theorem 2.1. $\qquad\square$

Darboux–Jouanolou theorem was generalized from different aspects. Christopher [8, 11, 12] in 1994 introduced the exponential factor, Chavarriga, Llibre and Sotomayor [6] in 1997 bringed independent singularities into the generalization of Darboux–Jouanolou theorem. This theory had be extended to surfaces, see example [31].

Pereira [38], and Christopher, Lliber and Pereira [19] further generalized the Darboux–Jouanolou theorem to take into account the multiplicity of invariant algebraic curve by using the Extactic curves. Llibre and Zhang [32–35] improved the Darboux theory of integrability to higher dimensional systems taking into account not only the algebraic multiplicity of invariant algebraic hypersurface but also that of infinity using the Poincaré compactification.

Darboux theory of integrability depends on the number of invariant algebraic curves or surfaces. In [28–30] Llibre and Zhang provided an effective theory to find invariant algebraic surfaces of Lorenz system, Rikitake system and Rössler system. Related to the dynamics of these systems having an invariant algebraic surfaces Zhang *et al* [1, 7] characterized their dynamics on the invariant algebraic surfaces.

On the degree of invariant algebraic curves, there are also lots of works, see e.g. [2, 4, 13, 26, 27]. Related to the classification of invariant algebraic curves of degree 4 for quadratic differential systems, there is an excellent work [5] by Chavarriga, Llibre and Sorolla.

For the inverse problem on Darboux theory of integrability, Christopher, Llibre, Pantazi and Walcher did a series works [14–17]. They deal with multiple invariant algebraic curves, Darboux integrating factor and Darboux theory of integrability. Related to the inverse problem, see also Christopher, Llibre, Pantaziand Zhang [18].

3. Liouville and elementary integrability

Liouville and elementary first integrals are respectively first integrals and are Liouville and elementary functions, which will be defined later on. Prelle and Singer [40] in 1983 presented an reduction of elementary first integrals. Singer [43] in 1992 proved the equivalence between the existence of Liouville first integral and Darboux integrating factor of planar polynomial differential systems. So in two dimensional case Liouville integrability is equivalent to the Darboux integrability. For introducing the results of Prelle and Singer, we need the knowledge on field extension.

3.1. Elementary on differential field extension

Let K be a ring. Its summation and product are denoted by $\oplus$ and $\otimes$, the zero and unit element of K under the summation and product are respectively 0_K and 1_K.

The ring K has characteristic 0 if for arbitrary $r \in \mathbb{N}$,

$$\underbrace{1_K + \ldots + 1_K}_{r} = 0_K,$$

does not hold. If there exists a positive integer r such that the above equality holds, such minimal positive integer is called *characteristic of K*, denoted by char(K). For example, $\mathbb{R}$ and $\mathbb{Z}$ are respectively field and ring of characteristic 0. Whereas $\mathbb{Z}/(r\mathbb{Z})$ is a ring of characteristic r.

A *derivative* on a ring K is an operator $\delta : K \to K$ satisfying

$$\delta(x + y) = \delta x + \delta y, \quad \delta(xy) = (\delta x)y + x(\delta y), \qquad \forall\, x, y \in K.$$

A *differential field* (K, Δ) consists of the field K and the set Δ of commutative derivatives defined on K. In this paper all fields have characteristic 0.

An *extension of differential field* of a differential field (K, Δ) is a differential field (L, Δ'), which satisfies $K \subset L$ and for $\forall\, \delta' \in \Delta'$ we have $\delta'|_K \in \Delta$.

Since the relation between the derivative of differentia field (K, Δ) and its field extension (L, Δ'), we also use Δ to represent Δ'. For simplifying notations we also use L/K to denote that (L, Δ) is a field extension of (K, Δ).

For differential field extension L/K,

- $\alpha \in L$ is called

  - *an algebraic element of K*, if there exists a polynomial with coefficients in K such that $F(\alpha) = 0$.
  - *transcendental element of K*, if α is not an algebraic element over K.

- A subset $S \subset L$ is *algebraic independent over K*, if for $\forall\, s_1, \ldots, s_r \in S$, there does not exist polynomial $P(Z_1, \ldots, Z_r)$ with coefficients in K such that $P(s_1, \ldots, s_r) \equiv 0$.

- The maximal cardinality of sets consisting of algebraic independent elements of L over K is called *transcendental degree of L/K*.

- If each element of L is algebraic over K, we call *L/K an algebraic extension of field K*.

Given a field K,

- A *separating field* of a polynomial $p(x)$ over K is a minimal field extension of K such that $p(x)$ can be decomposed into product of linear factors over this field extension, i.e. $p(x) = \prod(x - a_i)$, $a_i \in L$, L/K is the minimal field extension such that this decomposition can happen.

- We say that an algebraic field extension L/K of K is *normal*, if L is a separating field of polynomials in $K[x]$.

- The *normal closure* of an algebraic field extension L/K is a field extension $\overline{L}$ of L such that $\overline{L}/K$ is normal, and $\overline{L}$ is the minimal field extension satisfying this property.

- *Field automorphism* over field K is a bijective map $\varphi : K \to K$ which keeps the algebraic properties of K. Keeping algebraic properties means that $\varphi(0_K) = 0_K$, $\varphi(1_K) = 1_K$, $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$.

- The set of all field automorphisms over field K fixing elements of a subfield $K' \subset K$ forms a group under the composition of maps, it is called *Galois group*.
- The *order of a group* is the number of elements of a group G, denoted by $|G|$.

Give a field extension L/K,

- L can be considered as a *vector space over K*: the elements of L are treated as vectors, and elements of K are treated as scalars, and the summation of vectors is that of elements of field and the product of elements of L and K is the that of elements of field L.
- The dimension of this vector space is called *degree of this differential field extension*, denoted by $[L : K]$.
- If $[L : K] \in \mathbb{N}$, we call L/K *finite field extesion*.
- Let $S \subset L$,
  - $K(S)$ denotes the minimal subfield of L including K and S.
  - If S contains only one element, we call $K(S)$ *the minimal field extension of K*.

A differential field extension is *elementary*, if this differential field extension can be written in differential field extensions of tower form

$$K = K_0 \subset K_1 \subset \ldots \subset K_r = L,$$

such that each extension satisfies

(a) K_{i+1} is a finite algebraic extension of K_i, or

(b) $K_{i+1} = K_i(t)$, where t satisfies : for each $\delta \in \Delta$, there exists a $x \in K_i$ such that $\frac{\delta t}{t} = x$, or

(c) $K_{i+1} = K_i(t)$, where t satisfies: for each $\delta \in \Delta$, there exists a $x \in K_i$ such that $\delta t = \frac{\delta x}{x}$.

We note that an elementary extension of a field is finitely many times to add algebraic elements to the original field, the minimal extension of exponential and logarithm of the element in the original field. The K_i in the tower is called *tower element*, $i = 0, 1, \ldots, r$.

A differential field extension is *Liouville*, if this differential field extension can be written in tower form such that

(a) K_{i+1} is a finite algebraic extension of K_i, or

(b) $K_{i+1} = K_i(t)$, where t satisfies: for each $\delta \in \Delta$, $\dfrac{\delta t}{t} \in K_i$, or

(c) $K_{i+1} = K_i(t)$, where t satisfies: for $\delta \in \Delta$, $\delta t \in K_i$.

We note that the Liouville extension of a field is the minimal field extension consisting of elementary extension and finitely many times integrating of original fields.

Is the integrating of an elementary function still an elementary one? Liouville proved the following result, see e.g [36, 41]. Let (K, Δ) be a differential field. Set

$$\mathrm{Con}(K, \Delta) = \{k \in K | \ \delta k = 0, \ \forall \delta \in \Delta\}.$$

We also use $\mathrm{Con}(K)$ to denote $\mathrm{Con}(K, \Delta)$ if there is no confusion.

Theorem 3.1. (Liouville) *Let* (L, δ) *be an elementary field extension of* (K, δ). *Assume that* $\mathrm{Con}(K) = \mathrm{Con}(L)$. *If* $x \in K$, $y \in L$ *satisfy* $\delta y = x$, *then there exist* $c_1, \ldots, c_m \in \mathrm{Con}(K)$, *and* $w_0, w_1, \ldots, w_m \in K$ *such that*

$$x = \delta w_0 + \sum_{j=1}^{m} c_i \frac{\delta w_i}{w_i}.$$

3.2. Integrability theorem of Prelle and Singer

The results introduced in this section were obtained by Prelle and Singer [40] in 1983, and Singer [43] in 1992.

Theorem 3.2. (Prelle *and* Singer *theorem*) *Assume that* (L, Δ) *is a field extension of* (K, Δ), *and* $\mathrm{Con}(L) = \mathrm{Con}(K)$. *Choose* $\delta_1, \ldots, \delta_n \in \Delta$, $y_1, \ldots, y_n \in K$, *and set* $\mathcal{X} = y_1 \delta_1 + \ldots + y_n \delta_n$. *If* $\mathrm{Con}(L, \Delta)$ *is a proper subset of* $\mathrm{Con}(L, D)$, *then there exist* $c_1, \ldots, c_m \in \mathrm{Con}(K, \Delta)$, *and the algebraic elements* $w_0, w_1, \ldots, w_m$ *over* K, *such that*

$$\mathcal{X}w_0 + \sum_{i=1}^{m} c_i \frac{\mathcal{X}w_i}{w_i} = 0, \qquad \delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0, \ \forall \delta \in \Delta.$$

From Theorem 3.2 of Prelle and Singer, we can obtain the next result.

Corollary 3.1. *If a planar polynomial differentia system has an elementary first integral, then it must have an integrating factor of the form*

$$f_1^{n_1} \ldots f_p^{n_p}, \qquad f_i \in \mathbb{C}[x, y], \ \ n_i \in \mathbb{Z},$$

where f_i *are Darboux polynomials,* $i = 1, \ldots, 0$.

In 1992 Singer [43, Theorem 1] characterized properties of planar polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{3.1}$$

when some orbits are contained in vanishing set of a Liouville function, where $P, Q \in \mathbb{C}[x, y]$.

Theorem 3.3. (Singer theorem) *Assume that polynomial differential system* (3.1) *has an analytic solution* $(x, y) = (\varphi(t), \psi(t))$ *defined on some open subset* V *of* $\mathbb{C}$. *If there is a* Liouville *function* $F(x, y)$, *which is analytic on some open subset containing* $\mathcal{S} := \{(\varphi(t), \psi(t)) | \ t \in V\}$, *and* $F(x, y)|_{\mathcal{S}} = 0$, *then either* $\mathcal{S}$ *is an algebraic solution, or system* (3.1) *has an integrating factor*

$$R(x, y) := \exp\left(\int_{(x_0, y_0)}^{(x, y)} U(x, y)dx + V(x, y)dy \right),$$

where $U, V \in \mathbb{C}(x, y)$ *satisfy* $\partial_y U = \partial_x V$.

This theorem indicates that invariant algebraic curves play an important role in the study of dynamics of planar polynomial differential systems. For planar polynomial differential systems, differential $(K, \Delta) = (\mathbb{C}(x, y), \{\partial_x, \partial_y\})$.

An important corollary of this last theorem characterize the equivalence of Liouville integrability and Darboux integrabilty [43, Corollary] for planar polynomial differential systems.

Theorem 3.4. (Singer theorem) *A polynomial differential system* (3.1) *has a Liouville first integral if and only if it has* Darboux *integrating factor.*

Proof. This proof follows from Christopher [10]. For differential field $(K, \Delta) = (\mathbb{C}(x,y), \{\partial_x, \partial_y\})$, $\delta \in \Delta$ implies $\delta = \partial_x$, or $\delta = \partial_y$.
Sufficiecy. Assume that differential system (3.1) has a Darboux integrating factor R. Then by the definition of Darboux, we have $\dfrac{\delta R}{R} \in \mathbb{C}(x,y)$, i.e. R satisfies the condition (b) in the tower elements of Liouville extension. So RP, RQ belong to some tower element. Hence we get from the condition (c) of the tower elements in the Liouville extension that

$$H(x,y) = \int_{(x_0,y_0)}^{(x,y)} RPdy - RQdx,$$

is a Liouville function, and consequently is a Liouville first integral of system (3.1).
Necessity. Denote by $\mathcal{X}$ the vector field associated to system (3.1). Without loss of generality we assume $(P, Q) = 1$.

We separate the proof in two parts. First we prove that

Lemma 3.1. *If system* (3.1) *has a Liouville first integral, then it has an integrating factor*

$$R = \exp\left(\int U dx + V dy\right), \qquad U, V \in \mathbb{C}(x,y), \quad \partial_y U = \partial_x V.$$

Proof. In order for proving this lemma, we only need to prove that there exist $U, V \in \mathbb{C}(x,y)$ such that

$$\partial_y U = \partial_x V, \quad PU + QV = \partial_x P + \partial_y Q, \tag{3.2}$$

We now prove Lemma 3.1. By the assumption there exists an element $H(x,y)$ belonging to some Liouville extension L of the field $\mathbb{C}(x,y)$, such that

$$\mathcal{X}(H) \equiv 0, \qquad P\partial_x H + Q\partial_y H \equiv 0.$$

Since $(P, Q) = 1$, there exists $h \in L$ such that

$$h\partial_x H = Q, \quad h\partial_y H = -P. \tag{3.3}$$

Set

$$A = \frac{\partial_x h}{h}, \quad B = \frac{\partial_y h}{h}.$$

Then $A, B \in L$, and

$$\partial_y A = \partial_x B, \quad PA + QB = \partial_x P + \partial_y Q, \tag{3.4}$$

where the second equality was obtained by differentiating the first equation of (3.3) with respect to y, and the second equation with respect to x, and then subtracting these two resulting equations.

The functional equation (3.4) indicates that

$$R(x,y) := \exp\left(\int_{(x_0,y_0)}^{(x,y)} -A(x,y)dx - B(x,y)dy\right),$$

is an integrating factor of system (3.1). From the construction of A, B, they belong to the same tower element of the Liouville field extension. Next we prove that if $A, B \in K_{i+1}$, then we can find $C, D \in K_i$ such that instead of A, B by C, DR, (3.4) still hold. And so by induction we can find $U, V \in \mathbb{C}(x, y)$ satisfying (3.4).

By the tower form of the Liouville extension, we distinguish three cases (a), (b) and (c). In (b) and (c), we assume without loss of generality that t is a transcendental element. Otherwise $K_i(t)$ is an algebraic extension of K_i, and so it belongs to (a).
(a) Assume that K_{i+1} is a finite algebraic extension of K_i. Let $\overline{K}_{i+1}$ be the normal closure of K_{i+1}, and $\mathcal{G}$ is the group formed by the automorphisms over $\overline{K}_{i+1}$ keeping K_i unchanged. Then from Emil Artin's result (see e.g. [25, Theorem 1.1] and its proof) we get that the group $\mathcal{G}$ is finite order, denoted by $N = |\mathcal{G}|$, and $N = [\overline{K}_{i+1} : K_i]$.

Since the element of $\mathcal{G}$ keep elements of K_i unchanged and keep the algebraic structure of K_{i+1}, and $P, Q \in \mathbb{C}(x, y) \subset K_i$, it follows from (3.4) that

$$\sum_{\varphi \in \mathcal{G}} \varphi(PA + QB) = \sum_{\varphi \in \mathcal{G}} \varphi\left(\partial_x P + \partial_y Q\right)$$

$$\implies P \sum_{\varphi \in \mathcal{G}} \varphi(A) + Q \sum_{\varphi \in \mathcal{G}} \varphi(B) = \sum_{\varphi \in \mathcal{G}} \left(\partial_x \varphi(P) + \partial_y \varphi(Q)\right) = N\left(\partial_x P + \partial_y Q\right),$$

$$\sigma(\partial_y A) = \sigma(\partial_x B) \implies \partial_y \sigma(A) = \partial_x \sigma(B), \quad \forall \sigma \in \mathcal{G}.$$

Since A, B are algebraic functions of K_i, there exist minimal polynomials $f(z)$, $g(z)$ with coefficients in K_i such that $f(A) = 0$, $g(B) = 0$. Now $\varphi \in \mathcal{G}$ keeps elements of K_i, we have $f(\varphi(A)) = 0$, $g(\varphi(B)) = 0$. That is

$$\varphi(A), \ \varphi(B) \in K_{i+1}, \qquad \forall \, \varphi \in \mathcal{G}.$$

Set

$$C := \frac{1}{N} \sum_{\varphi \in G} \varphi(A), \qquad D := \frac{1}{N} \sum_{\varphi \in G} \varphi(B).$$

Using C, D to replace A, B, we also have (3.4). Since C and D are the averages of all elements of the Galois group G acting on A and B, we have

$$\varphi(C) = C, \qquad \varphi(D) = D, \qquad \forall \, \varphi \in \mathcal{G}.$$

This proves that $C, D \in K_i$.
(b) Assume that $K_{i+1} = K_i(t)$ with t a transcendental element over K_i, and $\delta t / t \in K_i$ and $\delta \in \{\partial_x, \partial_y\}$. Since $A, B \in K_i(t)$, we assume without loss of generality that $A = a(t), B = b(t) \in K_i(t)$, they are rational functions in t with coefficients in K_i. Expanding $a(t), b(t)$ in Laurent series in the transcendental element t

$$a(t) = C + \sum_{i \neq 0} a_i t^i, \quad b(t) = D + \sum_{i \neq 0} b_i t^i.$$

Then the coefficients of this series all belong to K_i. Direct calculations show that

$$\partial_y A = \partial_y a(t) = \partial_y C + \sum_{i \neq 0} (\partial_y a_i + i a_i p) t^i,$$

$$\partial_x B = \partial_x b(t) = \partial_x D + \sum_{i \neq 0} (\partial_x b_i + i b_i q) t^i,$$

where $p, q \in K_i$ satisfies $p = \partial_y t/t$, $q = \partial_x t/t$. Substituting the expansions of $A = a(t)$, $B = b(t)$ and the partial derivatives in to (3.4), equating the coefficients of t^0 gives

$$\partial_y C = \partial_x D, \quad PC + QD = \partial_x P + \partial_y Q.$$

(c) Assume that $K_{i+1} = K_i(t)$ with t transcendental over K_i, and $\delta t \in K_i$ and $\delta \in \{\partial_x, \partial_y\}$. Set $A = a(t)$, $B = b(t) \in K_i(t)$, and expand $a(t), b(t)$ in Laurent series in the transcendental $1/t$

$$a(t) = \sum_{i=-\infty}^{r} a_i t^i, \quad b(t) = \sum_{i=-\infty}^{r} b_i t^i,$$

where $a_i, b_i \in K_i$. Direct calculations show that

$$\partial_y A = \partial_y a(t) = \sum_{i=-\infty}^{r} (\partial_y a_{i-1} + i a_i p) t^{i-1} + \partial_y a_r t^r,$$

$$\partial_x B = \partial_x b(t) = \sum_{i=-\infty}^{r} (\partial_x b_{i-1} + i b_i q) t^{i-1} + \partial_x b_r t^r,$$

where $p, q \in K_i$ satisfy $p = \partial_y t$, $q = \partial_x t$. Substituting the expansions of $A = a(t)$, $B = b(t)$ and the above partial derivatives into (3.4), equating the coefficients of t^r gives

$$\text{when} \quad r \neq 0, \quad \partial_y a_r = \partial_x b_r, \quad Pa_r + Qb_r = 0; \quad \text{or}$$
$$\text{when} \quad r = 0, \quad \partial_y a_0 = \partial_x b_0, \quad Pa_0 + Qb_0 = \partial_x P + \partial_y Q.$$

If the latter holds, choose $C = a_0$, $D = b_0$, we can finish the proof.

If the former holds, since $Pa_r + Qb_r = 0$, $(P, Q) = 1$, there $h \in K_i$ such that

$$P = -b_r h, \qquad Q = a_r h.$$

So we from $\partial_y a_r = \partial_x b_r$ that

$$\partial_x P + \partial_y Q = -b_r \partial_x h + a_r \partial_y h = P \frac{\partial_x h}{h} + Q \frac{\partial_y h}{h}.$$

Choose

$$C = \frac{\partial_x h}{h}, \qquad D = \frac{\partial_y h}{h}.$$

Then we have $C, D \in K_i$, and $\partial_y C = \partial_x D$.

By induction, there exist $U, V \in K_0 = \mathbb{C}(x, y)$ such that (3.2) holds. Lemma 3.1 follows. $\qquad \square$

According to the plan of the proof, we next prove

Lemma 3.2. *If polynomial differential system* (3.1) *has integrating factor*

$$R = \exp\left(\int U \, dx + V \, dy \right), \qquad \text{where } U, V \in \mathbb{C}(x, y), \text{ and } \partial_y U = \partial_x V.$$

Then it has a Darboux integrating factor

$$\exp\left(\frac{g}{f} \right) \prod_i f_i^{l_i},$$

where $g, f, f_i \in \mathbb{C}[x, y]$, $l_i \in \mathbb{C}$.

Proof. We now prove Lemma 3.2. Since $U, V \in \mathbb{C}(x, y)$, we treat their numerators and denominators as polynomial in x with coefficients in $\mathbb{C}(y)$. Let K be the minimal normal algebraic field extension of $\mathbb{C}(y)$ such that it is the separating field of the numerators and denominators of U, V. Then U, V can be expanded over K as

$$
\begin{aligned}
U &= \sum_{i=1}^{r} \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{(x - \beta_i)^j} + \sum_{i=0}^{p} \xi_i x^i, \\
V &= \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{\gamma_{ij}}{(x - \beta_i)^j} + \sum_{i=0}^{q} \eta_i x^i,
\end{aligned}
$$

where $\alpha_{ij}, \gamma_{ij}, \beta_i, \xi_i, \eta_i \in K$, and they can be partially zero. Since

$$
\begin{aligned}
\partial_y U &= \sum_{i=1}^{r} \sum_{j=1}^{n_i} \left(\frac{\alpha'_{ij}}{(x - \beta_i)^j} - \frac{j \alpha_{ij} \beta'_i}{(x - \beta_i)^{j+1}} \right) + \sum_{i=0}^{p} \xi'_i x^i, \\
\partial_x V &= \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{-j \gamma_{ij}}{(x - \beta_i)^{j+1}} + \sum_{i=0}^{q} (i + 1) \eta_{i+1} x^i,
\end{aligned}
$$

where '$\prime$' denotes the derivative with respect to y. Comparing the coefficients of $\partial_y U = \partial_x V$ we get

$$
\xi'_i = (i + 1) \eta_{i+1}, \qquad \alpha'_{i, j+1} - j \beta'_i \alpha_{ij} + j \gamma_{ij} = 0. \tag{3.5}
$$

Set $j = 0$ in the second equality, we get $\alpha_{i1} \in \mathbb{C}$.

Set

$$
\Phi(x, y) = \sum_i \alpha_{i1} \log(x - \beta_i) + \sum_{i,j} \frac{-1}{j - 1} \frac{\alpha_{ij}}{(x - \beta_i)^{j-1}} + \sum_i \frac{\gamma_i x^{i+1}}{i + 1} + \int \eta_0 \, dy,
$$

where the last integrating represents any primitive function of η_0. Direct calculations show that $\partial_x \Phi = U$. Since

$$
\partial_y \Phi(x, y) = \sum_i \alpha_{i1} \frac{-\beta'_i}{x - \beta_i} + \sum_{i,j} \frac{-1}{j - 1} \left(\frac{\alpha'_{ij}}{(x - \beta_i)^{j-1}} - \frac{(j - 1) \alpha_{ij}}{(x - \beta_i)^j} \right) + \sum_i \frac{\gamma'_i x^{i+1}}{i + 1} + \eta_0,
$$

by the equality (3.5) we get $\partial_y \Phi = V$. Hence $\Phi(x, y) = \int U \, dx + V \, dy$.

Denote by $\mathcal{G}$ the automorphism group keeping $\mathbb{C}(y)$ over K. Then by the selection of K, $\mathcal{G}$ is a finite group. Denote by $N = |\mathcal{G}|$ the order of $\mathcal{G}$. Set

$$
\Psi = \frac{1}{N} \sum_{\sigma \in \mathcal{G}} \sigma(\Phi).
$$

Since $\sigma \in \mathcal{G}$ keeps the algebraic structure of K, we get from the property of automorphisms over algebraic field extension that

$$
\sigma(\alpha_{i1} \log(x - \beta_i)) = \alpha_{i1} \log(x - \sigma(\beta_i)), \quad \sigma\left(\int \eta_0 \, dy \right) = \int \sigma(\eta_0) \, dy,
$$

where integrating equality is in the sense that they maybe have a constant difference. Since $\sigma \in \mathcal{G}$ keeps $\mathbb{C}(x, y)$, we have

$$
\begin{aligned}
\sigma(\partial_x \Phi) = \sigma(U) &\implies \partial_x \sigma(\Phi) = \sigma(U) = U, \\
\sigma(\partial_y \Phi) = \sigma(V) &\implies \partial_y \sigma(\Phi) = \sigma(V) = V.
\end{aligned}
$$

It follows that

$$\partial_x \Psi = U, \qquad \partial_y \Psi = V.$$

Furthermore, we get from the expression of Φ and the definition of Ψ that

$$\Psi(x,y) = \sum_i l_i \log R_i(x,y) + R(x,y) + \int S(y)dy,$$

where R_i, $R \in \mathbb{C}(x,y)$, $S \in \mathbb{C}(y)$. Note that $R_i \in \mathbb{C}(x,y)$. It can be obtained from

$$R_i = \prod_{\varphi \in \mathcal{G}} (x - \varphi(\beta_i)),$$

and that R_i is unchanged under the action of each $\varphi \in \mathcal{G}$. Since $S(y)$ has partial fractional expansions in $\mathbb{C}$, so we have

$$\int S(y)dy = \sum_j k_j \log(S_j(y)) + S_0(y),$$

where $S_j \in \mathbb{C}[y]$, $j \neq 0$, $S_0 \in \mathbb{C}(y)$.

Taking exponential of Ψ, we get the Darboux integrating factor

$$\exp(\Psi) = \exp(R(x,y) + S_0(y)) \prod_i R_i^{l_i}(x,y) \prod_j S_j^{k_j}(y).$$

This proves Lemma 3.2.   □

Summarizing Lemma 3.1 and 3.2, we complete the proof of Theorem 3.4.   □

For differential systems having an Darboux integrating factor but its first integral not Darboux, Zoladek posed the definition of Darboux–Schwatz–Christoffel first integral and Darboux–Hyperelliptic first integrals. These first integrals can be distinguished by holonomy group. Note that Darboux–Hyperelliptic first integrals are elementary, whereas Darboux–Schwartz–Christoffel first integrals are not elementary, see e.g. Christopher [9].

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