ON THE MONOTONICITY OF THE PERIOD FUNCTION OF REVERSIBLE CENTERS*

Kuilin Wu

Abstract In this paper we study the period function of centers for a class of reversible systems and give a criteria to determine the monotonicity of the period functions.

Keywords Period functions, critical periods, reversible centers.

MSC(2000) 34C05, 34A34, 34C14.

1. Introduction and main results

It is well known that the center is surrounded by a continuous set of periodic orbits, which is called the period annulus of the center and denoted by $\mathcal{P}$. The period function is the period corresponding to the periodic orbits in $\mathcal{P}$. The center is called an isochronous center if the period function is a constant (see [1]). The critical periods are the zeros of the derivative of period function. It can be shown that the number of the critical periods does not depend on the particular parametrization used (see [6, 7]). A system with a center is said to be reversible if its orbits are symmetric with respect to a straight line passing through the center. In this paper we study the monotonicity of period functions of a class of reversible centers. Consider the reversible systems:

$$\begin{cases} \dot{x} & = & -U(x)y, \\ \dot{y} & = & f(x,y). \end{cases} \tag{1.1}$$

Suppose that (1.1) has a first integral of the form

$$H(x,y) = F(x)y^2 + G(x).$$

Then

$$f(x,y) = \frac{U(x)(y^2 F'(x) + G'(x))}{2F(x)}.$$

Denote by $(x_I, \ x_S)$ is the projection of the period annulus $\mathcal{P}$ to the $x-$axis, i.e.

$$(x_I, \ x_S) = \{x \in R \mid \exists y \in R, \text{ such that } (x,y) \in \mathcal{P}\}.$$

Assume that $F(x)$, $G(x)$, $U(x)$ are analytic functions on $(x_I, \ x_S)$. It is easy to verify that the origin is a center and $M(x) = \frac{2F(x)}{U(x)}$ is an integral factor of system (1.1). Without loss of generality, we always assume that

    $(H1)$: The origin is a nondegenerate centers of (1.1), and $U(0) > 0$.

   Email address: wukuilin@126.com(K.Wu)
   Department of Mathematics, GuiZhou University, Guiyang, 550025, China

$(H2): \mathcal{P}$ is the period annulus of the origin and $H(0,0) = 0, H(x,y) > 0$ for all $(x,y) \in \mathcal{P}\setminus\{(0,0)\}$.

Denote the range of H on $\mathcal{P}$ by $(0, h_0)$, where $\{H = h_0\}$ is corresponding to the boundary of the period annulus $\mathcal{P}$ with $h_0 \leq \infty$. From (H1), it is easy to obtain that

$$F(0) \neq 0, \quad G'(0) = 0, \quad F(0)G''(0) > 0. \tag{1.2}$$

For every $h \in (0, h_0)$, let us denote the periodic orbit of $\mathcal{P}$ corresponding to $\{H = h\}$ by γ_h and denote by $T(h)$ its period. Moreover, we define

$$[x_0(h), x_1(h)] = \{x \in R \mid \exists\ y \in R, \text{ such that } (x,y) \in \gamma_h\}.$$

Recall that an analytic diffeomorphism σ is said to be an involution if $\sigma \circ \sigma = Id$ and $\sigma \neq Id$. Since $xG'(x) > 0$ for all $x \in (x_I, x_S) \setminus \{0\}$ (See Lemma 2.1), there exists an analytic involution $\sigma(x)$ on (x_I, x_S) such that

$$G(x) = G(\sigma(x)) \quad \text{ for all } x \in (x_I, x_S).$$

In fact, we may take $\sigma(x) = g^{-1}(-g(x))$, where $g(x) = Sgn(x)\sqrt{G(x)}$.

In this paper, we obtain the following results for the reversible system (1.1).

Theorem 1.1. *Assume that the origin is a nondegenerate center of system (1.1) and that the function $T(h)$ is the period function of the periodic orbit γ_h. We take*

$$\mu_1(x) = \frac{(F(x)G(x))'U(x)G'(x) - 2F(x)G(x)(U(x)G'(x))'}{\sqrt{F(x)}((U(x)G'(x)))^2}$$

and define

$$\mu_{k+1}(x) = \mu_k(x) + \frac{2}{2k-1}\left(\frac{\mu_k(x)G(x)}{G'(x)}\right)'$$

and

$$S_\sigma(\mu_k)(x) = \frac{\mu_k(x)}{G'(x)} - \frac{\mu_k(\sigma(x))}{G'(\sigma(x))}.$$

If $S_\sigma(\mu_k)(x) > 0$ (or < 0) for $x \in (0, x_S)$, then the period function of system (1.1) is monotone.

Recall that the behavior of the period function plays an important role in the study of Abelian integrals (see [2, 8] for instance). Moreover it is also important in the study of other dynamical problems (see [3, 4]). Over the years the problem for the period function have been extensively studied. F. Mañsas and J. Villadelprat [11] studied the period functions of centers of Hamiltonian potential systems and gave a criteria to bound the number of critical periods. Chicone [5] conjectured that the period function of the quadratic reversible centers have at most two critical periods. To illustrate the applicability of Theorem 1.1, we study the monotonicity of period functions of some quadratic reversible centers(see Section 3 below). In the literature there are a lot of papers dealing with the period functions of the quadratic centers satisfied some Picard-Fuchs differential equations (see [9, 10, 13, 15, 16, 18, 19] and references therein).

The paper is organized in the following way. In Section 2 we give the proof of Theorem 1.1 by using some results in [17]. In Section 3 we study the period functions of two quadratic reversible systems by applying Theorem 1.1.

2. The proof of main results

In what follows we give the following lemma in [17].

Lemma 2.1. *[17] Under the assumptions $(H1)$ and $(H2)$, the following statements hold:*
(a) $F(x) > 0$, $U(x) > 0$, for all $x \in (x_I, x_S)$.
(b) $G(x) > 0$, $xG'(x) > 0$, for all $x \in (x_I, x_S) \setminus \{0\}$. In addition,

$$G(0) = G'(0) = 0, \ G''(0) > 0.$$

(c) $G(x) \to h_0$ as $x \searrow x_I$ or $x \nearrow x_S$.
(d) The period of the periodic orbit γ_h is given by

$$T(h) = 2 \int_{x_0(h)}^{x_1(h)} \frac{\sqrt{F}}{U\sqrt{h-G}} dx, \tag{2.1}$$

where $G(x_0(h)) = G(x_1(h)) = h$.

In order to character the period function $T(h)$ of the periodic orbit γ_h we study the auxiliary function $g(x) = Sgn(x)\sqrt{G(x)}$.

Lemma 2.2. *The following statements hold:*
(a) The function $g(x)$ is analytic on (x_I, x_S), $g(0) = 0$ and $g'(x) > 0$ for all $x \in (x_I, x_S)$. The inverse function $g^{-1}(x)$ is well defined and analytic on $(-\sqrt{h_0}, \sqrt{h_0})$.
(b) If $h \in (0, h_0)$, then the period of the periodic orbit γ_h is given by

$$T(h) = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{F(g^{-1}(\sqrt{h}\sin\theta))}}{U(g^{-1}(\sqrt{h}\sin\theta))g'(g^{-1}(\sqrt{h}\sin\theta))} d\theta. \tag{2.2}$$

Moreover,

$$T'(h) = \frac{1}{\sqrt{h}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\, \alpha(g^{-1}(\sqrt{h}\sin\theta))(g^{-1})'(\tau)|_{\tau=\sqrt{h}\sin\theta} d\theta, \tag{2.3}$$

where

$$\alpha(x) = \left(\frac{\sqrt{f(x)}}{U(x)g'(x)} \right)'.$$

(c) $\mu_k(x)$ are analytic functions on (x_I, x_S).

Proof. It is clear that $g(x)$ is well defined and analytic on (x_I, x_S) and that $g(0) = 0$ by (b) in Lemma 2.1. Since

$$g'(x) = \frac{sgn(x)G'(x)}{2\sqrt{G(x)}},$$

it follows from (b) in Lemma 2.1 that $g'(x) > 0$ for all $x \in (x_I, x_S) \setminus \{0\}$. On the other hand, we have

$$g'(0) = \frac{\sqrt{2G''(0)}}{2} > 0.$$

Therefore $g^{-1}(x)$ is well defined and analytic on $(-\sqrt{h_0}, \sqrt{h_0})$ and $g'(x) > 0$ for all $x \in (x_I, x_S)$.

Now turn to prove the statement (b). We make the variable $z = g(x)$ in the expression of $T(h)$ given by (2.1). Noting that for all $h \in (0, h_0)$, it holds

$$g(x_0(h)) = -\sqrt{h} \quad \text{and} \quad g(x_1(h)) = \sqrt{h}.$$

Since $x_0(h) < 0 < x_1(h)$, we obtain

$$T(h) = 2 \int_{-\sqrt{h}}^{\sqrt{h}} \frac{\sqrt{F(g^{-1}(z))}}{U(g^{-1}(z))g'(g^{-1}(z))\sqrt{h - z^2}} dz. \tag{2.4}$$

Making the variables $z = \sqrt{h} \sin \theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we get (2.2). Direct derivation with respect to h on (2.1), we have (2.3).

Since $\alpha(x)$ and $\frac{G(x)}{G'(x)}$ are analytic functions on (x_I, x_S), we have that $\mu_k(x)$ is an analytic function on (x_I, x_S). The proof is finished. □

Lemma 2.3. *Let γ_h be an oval inside the level curve $F(x)y^2 + G(x) = h$ and let $\frac{A(x)}{G'(x)}$ be any analytic function which is regular at $x = 0$. Then, for any $k \in \mathbb{N}$*

$$\oint_{\gamma_h} A(x) \left(\sqrt{F(x)} y \right)^{2k-3} dx = \frac{2}{2k-1} \oint_{\gamma_h} \left(\frac{A(x)}{G'(x)} \right)' \left(\sqrt{F(x)} y \right)^{2k-1} dx.$$

Proof. If $(x, y) \in \gamma_h \subset F(x)y^2 + G(x) = h$, then

$$\frac{dy}{dx} = -\frac{F'(x)y^2 + G'(x)}{2F(x)y}.$$

Moreover,

$$2d \left(\beta(x) y^{2k-1} \right)$$

$$= \left(2\beta'(x) y^{2k-1} - (2k-1)\beta(x) y^{2k-2} \frac{F'(x)y^2 + G'(x)}{F(x)y} \right) dx$$

$$= \left(2\beta'(x) - (2k-1)\beta(x) \frac{F'(x)}{F(x)} \right) y^{2k-1} dx - \left((2k-1)\beta(x) \frac{G'(x)}{F(x)} \right) y^{2k-3} dx.$$

Taking $A(x)(\sqrt{F(x)})^{2k-3} = \beta(x) \frac{G'(x)}{F(x)}$ in the above equality, then

$$\oint_{\gamma_h} d \left(\beta(x) y^{2k-1} \right) = 0.$$

This completes the proof of the result. □

The next result gives explicit expression of the derivative of the period function.

Lemma 2.4. *Suppose that the function $T(h)$ is the period function of the periodic orbit γ_h of system (1.1). We take*

$$\mu_1(x) = \frac{(F(x)G(x))'U(x)G'(x) - 2F(x)G(x)(U(x)G'(x))'}{\sqrt{F(x)}((U(x)G'(x)))^2},$$

and define

$$\mu_{k+1}(x) = \mu_k(x) + \frac{2}{2k-1} \left(\frac{\mu_k(x)G(x)}{G'(x)} \right).$$

Then, for any $k \in \mathbb{N}$, the following equalities hold:

$$T'(h) = \frac{1}{h^k} \oint_{\gamma_h} \mu_k(x)(F(x))^{\frac{2k-3}{2}} y^{2k-3} dx.$$

Proof. We prove the result by induction on k. Making the variable $g^{-1}(\sqrt{h}\sin\theta) = x$, it follows from (2.3) that

$$
\begin{aligned}
T'(h) &= \frac{2}{h} \int_{x_0(h)}^{x_1(h)} \frac{(F(x)G(x))'U(x)G'(x) - 2F(x)G(x)(U(x)G'(x))'}{\sqrt{F(x)}(U(x)G'(x))^2 \sqrt{h - G(x)}} dx \\
&= \frac{1}{h} \oint_{\gamma_h} \frac{(F(x)G(x))'U(x)G'(x) - 2F(x)G(x)(U(x)G'(x))'}{\sqrt{F(x)}((U(x)G'(x)))^2} \frac{dx}{\sqrt{F(x)}y} \\
&= \frac{1}{h} \oint_{\gamma_h} \mu_1(x)(F(x))^{\frac{2k-1}{2}} y^{2k-1} dx.
\end{aligned}
$$

This proves the case $k = 1$ in the statement. Suppose that the equality holds for $k = n$, it yields that

$$
\begin{aligned}
T'(h) &= \frac{1}{h^k} \oint_{\gamma_h} \mu_k(x)(F(x))^{\frac{2k-3}{2}} y^{2k-3} dx \\
&= \frac{1}{h^{k+1}} \oint_{\gamma_h} \mu_k(x)(F(x)y^2 + G(x))(F(x))^{\frac{2k-3}{2}} y^{2k-3} dx \\
&= \frac{1}{h^{k+1}} \oint_{\gamma_h} \left(\mu_k(x) + \frac{2}{2k-1}\left(\frac{\mu_k(x)G(x)}{G'(x)}\right)'\right) (F(x))^{\frac{2k-1}{2}} y^{2k-1} dx \\
&= \frac{1}{h^{k+1}} \oint_{\gamma_h} \mu_{k+1}(x)(F(x))^{\frac{2k-1}{2}} y^{2k-1} dx,
\end{aligned}
$$

where in the above equality we apply Lemma 2.3. This shows that the equality holds for $k = n + 1$. Therefore the proof is completed. $\qquad\square$

In the following, we prove the main result of the paper.

Proof of Theorem 1.1. By the Lemma 6, it follows that

$$
\begin{aligned}
T'(h) &= \frac{1}{h^k} \oint_{\gamma_h} \mu_k(x)(F(x))^{\frac{2k-3}{2}} y^{2k-3} dx \\
&= \frac{2}{h^k} \int_{x_0(h)}^{x_1(h)} \mu_k(x) \left(\sqrt{h - G(x)}\right)^{2k-3} dx.
\end{aligned}
$$

Taking the variables $x = g^{-1}(\tau)$ in the second integral above, we have that

$$
\begin{aligned}
T'(h) &= \frac{2}{h^k} \int_{-\sqrt{h}}^{\sqrt{h}} \frac{\mu_k(g^{-1}(\tau))}{g'(g^{-1}(\tau))} \left(\sqrt{h - \tau^2}\right)^{2k-3} d\tau \\
&= \frac{2}{h^k} \int_{0}^{\sqrt{h}} \frac{\mu_k(g^{-1}(\tau))}{g'(g^{-1}(\tau))} \left(\sqrt{h - \tau^2}\right)^{2k-3} d\tau \\
&\quad + \frac{2}{h^k} \int_{-\sqrt{h}}^{0} \frac{\mu_k(g^{-1}(\tau))}{g'(g^{-1}(\tau))} \left(\sqrt{h - \tau^2}\right)^{2k-3} d\tau \\
&= \frac{2}{h^k} \int_{0}^{\sqrt{h}} \left(\frac{\mu_k(g^{-1}(\tau))}{g'(g^{-1}(\tau))} + \frac{\mu_k(g^{-1}(-\tau))}{g'(g^{-1}(-\tau))}\right) \left(\sqrt{h - \tau^2}\right)^{2k-3} d\tau
\end{aligned}
$$

Define

$$\delta(\tau) = \frac{\mu_k(g^{-1}(\tau))}{g'(g^{-1}(\tau))} + \frac{\mu_k(g^{-1}(-\tau))}{g'(g^{-1}(-\tau))}.$$

Then $\delta(\tau) > 0$ (or < 0) for $\tau \in (0, \sqrt{h})$ imply $T'(h) > 0$ (or < 0) for $h \in (0, h_0)$. Taking the variable $g^{-1}(\tau) = x$, then $\delta(\tau) = \dfrac{\mu_k(x)}{g'(x)} + \dfrac{\mu_k(\sigma(x))}{g'(\sigma(x))}$. Note that $g^2(x) = g^2(\sigma(x)) = G(x) = G(\sigma(x))$ for $x \in (x_I, x_S)$. Therefore, $S_\sigma(\mu_k)(x) > 0$ (or $< o$) for $x \in (0, x_S)$, which implies that the period function of system (1.1) is monotone. This completes the proof of the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.1. Note that $S_\sigma(\mu_k)(x)$ is σ-odd(i.e. $S_\sigma(\mu_k)(x)= -S_\sigma(\mu_k)(\sigma(x))$), then the number of the zeros $S_\sigma(\mu_k)(x)$ on $(0, x_S)$ is equal to the number of the zeros $S_\sigma(\mu_k)(x)$ on $(x_I, 0)$. If $G(x)$ is an even function on (x_I, x_S), then $\sigma(x) = -x$, and $S_\sigma(\mu_k)(x) = \dfrac{2\mu_k(x)}{g'(x)}$. In addition, If $S_\sigma(\mu_k)(x)$ is monotone on (x_I, x_S), then the period function is a monotone function.

3. Applications

In the following examples, we shall apply Theorem 1 to prove the period functions of two reversible Lotka-Volterra systems Q_3^{LV} is monotone.

Example 3.1.

$$\begin{cases} \dot{x} &= -y(1 + \frac{4}{3}x), \\ \dot{y} &= x + \frac{4}{3}x^2 - \frac{4}{3}y^2. \end{cases} \tag{3.1}$$

The first integral of (3.1) is $H(x,y) = F(x)y^2 + G(x)$, where

$$G(x) = \frac{-108x - 144x^2 + 9(3 + 4x)^2 \ln(1 + \frac{4}{3}x)}{8(3 + 4x)^2} \quad \text{and } F(x) = \frac{9}{(3 + 4x)^2}.$$

System (2.4) has a center at $(0,0)$ and a nilpotent singularities at $(-\frac{3}{4}, 0)$, respectively. Moreover $(x_I, x_S) = (-\frac{3}{4}, \infty)$. Straightforward computation, we have that

$$\frac{\mu(x)}{G'(x)} = \frac{12x(1 + 2x) - (3 + 4x)^2 \ln(\frac{3+4x}{3})}{16x^3}.$$

We claim that $\frac{\mu(x)}{G'(x)}$ is a monotone increasing function on $(-\frac{3}{4}, \infty)$. In order to prove the claim, we compute the derivative of $\frac{\mu(x)}{G'(x)}$ and have that

$$\left(\frac{\mu(x)}{G'(x)}\right)' = \frac{-4x(9 + 10x) + (3 + 4x)(9 + 4x) \ln(\frac{3+4x}{3})}{16x^4}.$$

On the other hand, we take

$$a(x) = \ln(\frac{3 + 4x}{3}) - \frac{4x(9 + 10x)}{(3 + 4x)(9 + 4x)}.$$

Then

$$a'(x) = \frac{256x^3}{((3 + 4x)(9 + 4x))^2}.$$

It is clear that $a'(x) < 0$ on $(-\frac{3}{4}, 0)$ and $a'(x) > 0$ on $(0, \infty)$, respective. Since $a(x) \geq a(0) = 0$ on $(-\frac{3}{4}, \infty)$, $\left(\frac{\mu(x)}{G'(x)}\right)' \geq 0$ on $(-\frac{3}{4}, \infty)$, one gets that $\frac{\mu(x)}{G'(x)}$ is a monotone increasing function on $(-\frac{3}{4}, \infty)$. Consequently, $S_\sigma(\mu_k)(x) > 0$ for all $x \in (0, +\infty)$. By applying Theorem 1.1, we have that the period function of the center is (globally) monotone increasing.

Example 3.2.

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x + 2x^2 - 2y^2. \end{cases} \tag{3.2}$$

The first integral of (3.2) is $H(x, y) = F(x)y^2 + G(x)$, where

$$G(x) = \frac{1}{4} - \frac{(1/2 + x)^2}{e^{4x}} \quad \text{and } F(x) = \frac{1}{e^{4x}}.$$

System (3.2) has a center at $(0, 0)$ and a saddle at $(-\frac{1}{2}, 0)$, respectively. Moreover $(x_I, 0) = (-\frac{1}{2}, 0)$. Straightforward computation, we have that

$$\frac{\mu(x)}{G'(x)} = \frac{(1 + 2x)^3 + e^{4x}(-1 - 2x + 4x^2)}{8e^{-2x}x^3(1 + 2x)^3}.$$

We claim that $\frac{\mu(x)}{G'(x)}$ is a monotone function on $(-\frac{1}{2}, \infty)$. In order to prove the claim, we compute the derivative of $\frac{\mu(x)}{G'(x)}$ and have that

$$\left(\frac{\mu(x)}{G'(x)}\right)' = \frac{e^{2x}((-3 + 2x)(1 + 2x)^4 + e^{4x}(3 + 10x - 8x^2 - 32x^3 + 48x^4))}{8x^4(1 + 2x)^4}.$$

By Sturm's Theorem, we have that $3 + 10x - 8x^2 - 32x^3 + 48x^4 > 0$ on $(-\frac{1}{2}, \infty)$. It is clear that $\left(\frac{\mu(x)}{G'(x)}\right)' > 0$ on $[\frac{3}{2}, \infty)$.

On the other hand, for all $x \in (-\frac{1}{2}, \frac{3}{2})$, we take

$$b(x) = 4x - \ln(\frac{(3 - 2x)(1 + 2x)^4}{3 + 10x - 8x^2 - 32x^3 + 48x^4}).$$

Then

$$b'(x) = \frac{32x^3(-4 + 11x - 46x^2 + 24x^3)}{(-3 + 2x)(1 + 2x)(3 + 10x - 8x^2 - 32x^3 + 48x^4)}.$$

By Sturm's Theorem, we have that $-4 + 11x - 46x^2 + 24x^3 < 0$ on $(-\frac{1}{2}, \frac{3}{2})$. Since $b'(x) < 0$ on $(-\frac{1}{2}, 0)$ and $b'(x) > 0$ on $(0, \frac{3}{2})$, respective, then $b(x) \geq b(0) = 0$ on $(-\frac{1}{2}, \frac{3}{2})$. Hence $\left(\frac{\mu(x)}{G'(x)}\right)' \geq 0$ on $(-\frac{1}{2}, \infty)$, and $\frac{\mu(x)}{G'(x)}$ is a monotone increasing function on $(-\frac{1}{2}, \infty)$. Consequently, $S_\sigma(\mu_k)(x) > 0$ for all $x \in (0, +\infty)$. By applying Theorem 1.1, we have that the period function of the center is (globally) monotone.

References

[1] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qual. Theory Dyn. Syst., 1 (1999), 1-70.

[2] F. Chen, C. Li, J. Llibre and Z. Zhang, *A unified proof on the weak Hilbert 16th problem for n = 2,* J. Differential Equations, 221 (2006), 309-342.

[3] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields,* J. Differential Equations, 69 (1987), 310-321.

[4] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields,* Trans. Amer. Math. Soc., 312 (1989), 433-486.

[5] C. Chicone, *Review in MathSciNet,* ref. 94hL58072.

[6] A. Gasull, C. Liu and J. Yang, *On the number of critical periods for planar polynomial systems of arbitrary degree*, J. Differential Equations, 249 (2010), 684-692.

[7] M. Grau and J. Villadelprat, *Bifurcation of critical periods from Pleshkan's isochrones*, J. London Math. Soc., 81 (2010), 142-160.

[8] I. D. Iliev, *Perturbations of quadratic centers,* Bull. Sci. Math., 122 (1998), 107-161.

[9] H. Liang and Y. Zhao, *On the period function of reversible quadratic centers with their orbits inside quartics,* Nonlinear Anal., 71 (2009), 5655-5671.

[10] F. Mañosas and J. Villadelprat, *The bifurcation set of the period function of the dehomogenized Loud's centers is bounded,* Proc. Amer. Math. Soc., 136 (2008), 1631-642.

[11] F. Mañosas and J. Villadelprat, *Criteria to bound the number of critical periods,* J. Differential Equations, 246 (2009), 2415-2433.

[12] P. Mardešić, C. Rousseau, and B. Toni, *Linearization of isochronous centers,* J. Differential Equations, 121 (1995), 67-108.

[13] P. Mardešić, D. Marín and J. Villadelprat, *The period function of reversible quadratic center,* J. Differential Equations, 224 (2006), 120-171.

[14] M. Sabatini, *On the period function of Liénard system,* J. Differential Equations, 152 (1999), 467-487.

[15] J. Villadelprat, *On the reversible quadratic centers with monotonic period function,* Proc. Amer. Math. Soc., 135 (2007), 2555-2565.

[16] J. Villadelprat, *The period function of the generalized Lotka−Volterra centers,* J. Math. Anal. Appl., 341 (2007), 834-854.

[17] K. Wu and Y. Zhao, *Isochronicity for a class of reversible systems*, J. Math. Anal. Appl., 365 (2010), 300-307.

[18] Y. Zhao, *The monotonicity of period function for codimension four quadratic system Q_4,* J. Differential Equations, 185 (2002), 370-387.

[19] Y. Zhao, *On the monotonicity of the period function of a quadratic system,* Discrete Contin. Dyn. Syst., 13 (2005), 795-810.