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APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING A CLASS OF NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract In this paper, the nonlinear Volterra-Fredholm integro-differential equations are solved by using the homotopy analysis method (HAM). The approximation solution of this equation is calculated in the form of a series which its components are computed easily. The existence and uniqueness of the solution and the convergence of the proposed method are proved. A numerical example is studied to demonstrate the accuracy of the presented method.

Keywords Volterra and Fredholm integral equations, integro-differential equations, Homotopy analysis method (HAM).

MSC(2000) 35A16, 34B13, 34C15.

1. Introduction

Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations for example [2, 3, 5, 8, 9, 10, 11, 21]. The homotopy analysis method (HAM) is based on homotopy, a fundamental concept in topology and differential geometry [18]. The HAM has successfully been applied to many situations [1, 2, 6, 7, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In [4] we had studied the high-order nonlinear Volterra-Fredholm integro-differential equation by using HAM of the form

$$\sum_{j=0}^{k} p_j(x)y^{(j)}(x) = f(x) + \mu_1 \int_a^x k_1(x,t)\ G_1\big(y^{(p)}(t)\big)\ dt + \mu_2 \int_a^b k_2(x,t)\ G_2\big(y^{(m)}(t)\big)\ dt,\ 0 \le p, m \le k.$$

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In this study, we develop HAM to solve the high-order nonlinear Volterra-Fredholm integro-differential equations as follows:

$$\begin{aligned}\sum_{j=0}^{k} p_j(x)y^{(j)}(x) = f(x) + \mu_1 \int_a^x k_1(x,t)\ G_1\big(y^{(p)}(t)\big)\ dt \\ + \mu_2 \int_a^b k_2(x,t)\ G_2\big(y^{(m)}(t)\big)\ dt,\ 0 \le p, m \le k,\end{aligned} \tag{1.1}$$

with initial conditions

$$y^{(r)}(a) = b_r, \quad r = 0, 1, 2, \ldots, k-1, \tag{1.2}$$

where a, b, μ_1, μ_2, b_r are constant values, $f(x), k_1(x,t), k_2(x,t), G_1(y(t)), G_2(y(t))$ and $p_j(x)$, $j = 0, 1, \ldots, k$ are functions that have suitable derivatives on an interval $a \le t \le x \le b$ and $p_k(x) \ne 0$.

The paper is organized as follows. In section 2, the HAM is briefly presented. In section 3, this method is presented for solving (2.1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved. Finally, a numerical example and computational complexity of the proposed algorithm are shown in section 4.

2. Preliminaries

Consider

$$N[y] = 0,$$

where N is a nonlinear operator, $y(x)$ is an unknown function and x is an independent variable. Let $y_0(x)$ denote an initial guess of the exact solution $y(x)$, $h \ne 0$ an auxiliary parameter, $H(x) \ne 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L\big[\phi(x;q) - y_0(x)\big] - qhH(x)N\big[\phi(x;q)\big] = \hat{H}\big[\phi(x;q); y_0(x), H(x), h, q\big]. \tag{2.1}$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.

Enforcing the homotopy (2.1) to be zero, i.e.,

$$\hat{H}\big[\phi(x;q); y_0(x), H(x), h, q\big] = 0, \tag{2.2}$$

we have the so-called zero-order deformation equation

$$(1-q)L\big[\phi(x;q) - y_0(x)\big] = qhH(x)N\big[\phi(x;q)\big]. \tag{2.3}$$

When $q = 0$, the zero-order deformation (2.3) becomes

$$\phi(x;0) = y_0(x), \tag{2.4}$$

and when $q = 1$, since $h \ne 0$ and $H(x) \ne 0$, the (2.3) is equivalent to

$$\phi(x;1) = y(x). \tag{2.5}$$

Thus, according to (2.4) and (2.5), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such a kind of continuous variation is called deformation in homotopy [12].

Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m, \tag{2.6}$$

where

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m}\,|_{q=0}\ .$$

Let the initial guess $y_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series (2.6) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$y(x) = \phi(x;1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{2.7}$$

From (2.6), we can write (2.3) as follows

$$L\Big[\sum_{m=1}^{\infty} y_m(x)\ q^m\Big] - qL\Big[\sum_{m=1}^{\infty} y_m(x) q^m\Big] = qhH(x)N\big[\phi(x,q)\big]. \tag{2.8}$$

By differentiating (2.8) m times with respect to q, we obtain

$$m!\ L\big[y_m(x) - y_{m-1}(x)\big] = hH(x)m\frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\,|_{q=0}\ .$$

Therefore,

$$\begin{aligned} &L\big[y_m(x) - \chi_m y_{m-1}(x)\big] = hH(x)\Re_m\big(y_{m-1}(x)\big), \\ &y_m(0) = 0, \end{aligned} \tag{2.9}$$

where,

$$\Re_m\big(y_{m-1}(x)\big) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{2.10}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases}$$

Note that the high-order deformation (2.9) is governing the linear operator L, and the term $\Re_m(y_{m-1}(x))$ can be expressed simply by (2.10) for any nonlinear operator N.

3. Description of the method

To obtain the approximation solution of (1.2) based on the HAM, let

$$\begin{aligned} N[y] &= y(x) - L^{-1}\Big(\frac{f(x)}{p_k(x)}\Big) - \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ &- \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y^{(p)}(t)\big)\ dt\Big) + L^{-1}\Big(\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}\ y^{(j)}(x)\Big) \\ &- \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y^{(m)}(t)\big)\ dt\Big) = 0, \end{aligned}$$

where L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) = \int_a^x \int_a^x \dots \int_a^x (.)dxdx\dots dx\ , \quad (k\ times).$$

We obtain the term $\sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$ from the initial conditions. We have

$$\begin{aligned} &\Re_m\big(y_{m-1}(x)\big) \\ &= y_{m-1}(x) - \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y_{m-1}^{(p)}(t)\big)\ dt\Big) \\ &- \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y_{m-1}^{(m)}(t)\big)\ dt\Big) + L^{-1}\Big(\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}\ y_{m-1}^{(j)}(x)\Big) \\ &- (1-\chi_m)\Big(L^{-1}\big(\frac{f(x)}{p_k(x)}\big) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r\Big), \qquad m \geq 1. \end{aligned} \tag{3.1}$$

Substituting (3.1) into (2.9),

$$\begin{aligned} &L\big[y_m(x) - \chi_m y_{m-1}(x)\big] \\ &= hH(x)\Big[y_{m-1}(x) - \mu_1 L^{-1}\big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y_{m-1}^{(p)}(t)\big)\ dt\big) \\ &- \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y_{m-1}^{(m)}(t)\big)\ dt\Big) + L^{-1}\Big(\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}\ y_{m-1}^{(j)}(x)\Big) \\ &- (1-\chi_m)\Big(L^{-1}\big(\frac{f(x)}{p_k(x)}\big) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r\Big)\Big]. \end{aligned} \tag{3.2}$$

We take an initial guess $y_0(x) = F(x) = L^{-1}\big(\frac{f(x)}{p_k(x)}\big) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$, an auxiliary nonlinear operator $Ly = y$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x) = 1$. This is substituted into (3.2) to give the recurrence relation

$$y_0(x) = L^{-1}\big(\frac{f(x)}{p_k(x)}\big) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r,$$

$$y_m(x) = \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y_{m-1}^{(p)}(t)\big)\ dt\Big) - L^{-1}\Big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\ y_{m-1}^{(j)}(x)\Big) \quad (3.3)$$

$$+\,\mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y_{m-1}^{(m)}(t)\big)\ dt\Big),\quad m\geq 1.$$

Relation (3.3) will enable us to determine the components $y_m(x)$ recursively for $m \geq 0$.

Since $p_k(x)\neq 0$,we can write (1.2) in the form

$$y(x) = L^{-1}\big(\frac{f(x)}{p_k(x)}\big) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r + \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y^{(p)}(t)\big)\ dt\Big)$$

$$+\,\mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y^{(m)}(t)\ dt\big)\Big) - L^{-1}\Big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\ y^{(j)}(x)\Big). \quad (3.4)$$

The following relations has been mentioned in [21]:

$$L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y^{(p)}(t)\big)\ dt\Big) = \frac{1}{k!}\int_a^x (x-t)^k\frac{k_1(x,t)}{p_k(x)}\ G_1\big(y^{(p)}(t)\big)\ dt, \quad (3.5)$$

$$\sum_{j=0}^{k-1} L^{-1}\big(\frac{p_j(x)}{p_k(x)}\big)\ y^{(j)}(t) = \sum_{j=0}^{k-1}\frac{1}{k!}\int_a^x (x-t)^k\frac{p_j(x)}{p_k(x)}\ y^{(j)}(t)\ dt. \quad (3.6)$$

By substituting (3.5) and (3.6) in (3.4), we obtain

$$\begin{aligned} y(x) = F(x) &+ \mu_2\int_a^b L^{-1}\big(\frac{k_2(x,t)}{p_k(x)}\big)\ G_2\big(y^{(m)}(t)\big)\ dt \\ &+ \frac{\mu_1}{k!}\int_a^x (x-t)^k\frac{k_1(x,t)}{p_k(x)}\ G_1\big(y^{(o)}(t)\big)\ dt \\ &- \sum_{j=0}^{k-1}\frac{1}{k!}\int_a^x (x-t)^k\frac{p_j(x)}{p_k(x)}\ y^{(j)}(t)\ dt. \end{aligned} \quad (3.7)$$

In (3.7), we assume that $F(x)$ is bounded for all t in $C=[a,b]$ and

$$\left|\frac{\mu_1 k_1(x,t)(x-t)^k}{k!\ p_k(x)}\right| \leq M^{'},\quad \left|\mu_2 L^{-1}\big(\frac{k_2(x,t)}{p_k(x)}\big)\right| \leq M^{''},$$

$$\left|\frac{(x-t)^k p_j(x)}{p_k(x)\ k!}\right| \leq M_j,\quad j=0,1,...,k-1\quad ,\forall a\leq t\leq x\leq b.$$

Also, we suppose the nonlinear terms $G_1\big(y^{(p)}(t)\big), G_2\big(y^{(m)}(t)\big)$ and $D^j(y(t)) = \frac{d^j}{dt^j}y(t)$ are Lipschitz continuous with

$$|G_1(y^{(p)}) - G_1(y^{*(p)})| \leq L^{'}|y^{(p)} - y^{*(p)}|,$$

$$|G_2(y^{(p)}) - G_2(y^{*(p)})| \leq L^{''}|y^{(m)} - y^{*(m)}|,$$

$$|D^j(y) - D^j(y^*)| \leq L_j|y-y^*|\ \text{ for } j=0,1,...,k-1.$$

If we set,

$$\alpha = (L^{'}M^{'} + L^{''}M^{''} + kLM)(b-a),$$
$$M = max|M_j|,\ L = max|L_j|,\quad j = 0,1,...,k-1,$$

then the following theorems can be proved by using the above assumptions.

Theorem 3.1. *The nonlinear Volterra-Fredholm integro-differential equation in* (1.2) *has a unique solution whenever* $0 < \alpha < 1$.

Proof. Let y and y^* be two different solutions of (3.7), then

$$\begin{aligned}
|y-y^*| =& |\int_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{p_k(x)k!}\big[G_1(y)-G_1(y^*)\big]\ dt \\
&+ \int_a^b \mu_2 L^{-1}(\frac{k_2(x,t)}{p_k(x)})\big[G_2(y)-G_2(y^*)\big]dt \\
&- \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^k p_j(x)}{p_k(x)\ k!}\big[D^j(y)-D^j(y^*)\big]dt| \\
\le& \int_a^x |\frac{\mu_1 k_1(x,t)(x-t)^k}{p_k(x)k!}|\ |G_1(y)-G_1(y^*)|dt \\
&+ \int_a^b |\mu_2 L^{-1}(\frac{k_2(x,t)}{p_k(x)})|\ |G_2(y)-G_2(y^*)|dt \\
&+ \sum_{j=0}^{k-1}\int_a^x |\frac{(x-t)^k\ p_j(x)}{p_k(x)k!}|\ |D^j(y)-D^j(y^*)|dt \\
\le& (b-a)(L^{'}M^{'} + L^{''}M^{''} + kLM)|y-y^*|,
\end{aligned}$$

from which we get $(1-\alpha)|y-y^*| \le 0$. Since $0 < \alpha < 1$, so $|y-y^*| = 0$, therefore $y = y^*$ and this completes the proof. □

Theorem 3.2. *If the series solution* $y(x) = \sum_{m=0}^{\infty} y_m(x)$ *obtained from* (3.3) *is convergent then it converges to the exact solution of the problem* (1.2).

Proof. We assume:

$$\begin{aligned}
&y^{(j)}(x) = \textstyle\sum_{m=0}^{\infty} y_m^{(j)}(x), \\
&S_1\big(y^{(p)}(x)\big) = \textstyle\sum_{m=0}^{\infty} G_1\big(y_m^{(p)}(x)\big), \\
&S_2\big(y^{(m)}(x)\big) = \textstyle\sum_{m=0}^{\infty} G_2\big(y_m^{(m)}(x)\big),
\end{aligned} \tag{3.8}$$

where,

$$\lim_{m\to\infty} y_m(x) = 0.$$

We can write

$$\sum_{m=1}^{n}\big[y_m(x) - \chi_m y_{m-1}(x)\big] = y_1 + (y_2-y_1) + ... + (y_n - y_{n-1}) = y_n(x). \tag{3.9}$$

Hence, from (3.9),

$$\lim_{n\to\infty} y_n(x) = 0. \tag{3.10}$$

So using (3.10) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L\big[y_m(x) - \chi_m y_{m-1}(x)\big] = L\Big[\sum_{m=1}^{\infty}\big[y_m(x) - \chi_m y_{m-1}(x)\big]\Big] = 0.$$

Therefore, from (2.9) we can obtain that

$$\sum_{m=1}^{\infty} L\big[y_m(x) - \chi_m y_{m-1}(x)\big] = hH(x)\sum_{m=1}^{\infty}\Re_m\big(y_{m-1}(x)\big) = 0.$$

Since $h \neq 0$ and $H(x) \neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_m\big(y_{m-1}(x)\big) = 0. \tag{3.11}$$

By applying the relations (3.1) and (3.8),

$$\begin{aligned}
&\sum_{m=1}^{\infty}\Re_m\big(y_{m-1}(x)\big)\\
&= \sum_{m=1}^{\infty}\Bigg[y_{m-1} - \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ G_1\big(y_{m-1}^{(p)}(t)\big)dt\Big)\\
&\quad - \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ G_2\big(y_{m-1}^{(m)}(t)\big)dt\Big) + L^{-1}\Big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\ y_{m-1}^{(j)}(x)\Big)\\
&\quad - (1-\chi_m)F(x)\Bigg]\\
&= y(x) - F(x) - \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\big[\sum_{m=1}^{\infty} G_1(y_{m-1}^{(p)}(t))\big]dt\Big)\\
&\quad - \mu_2 L^{-1}\Bigg(\int_a^b \frac{k_2(x,t)}{p_k(x)}\Big[\sum_{m=1}^{\infty} G_2\big(y_{m-1}^{(m)}(t)\big)\Big]dt\Bigg) + L^{-1}\Big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\big[\sum_{m=1}^{\infty} y_{m-1}^{(j)}(x)\big]\Big)\\
&= y(x) - F(x) - \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ S_1\big(y^{(p)}(t)\big)dt\Big)\\
&\quad - \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)}\ S_2\big(y^{(m)}(t)\big)dt\Big) + L^{-1}\big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\ y^{(j)}(x)\big).
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned}
y(x) = F(x) &+ \mu_1 L^{-1}\Big(\int_a^x \frac{k_1(x,t)}{p_k(x)}\ S_1\big(y^{(p)}(t)\big)dt\Big)\\
&+ \mu_2 L^{-1}\Big(\int_a^b \frac{k_2(x,t)}{p_k(x)} S_2\big(y^{(m)}(t)\big)dt\Big) - L^{-1}\big(\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}\ y^{(j)}(x)\big),
\end{aligned}$$

therefore, $y(x)$ must be the exact solution of (1.2). $\square$

4. Numerical examples

In this section, we compute a numerical example which is solved by the HAM . The programs have been provided with mathematica 6 according to the following algorithms where ε is a given positive value.

Algorithm:

Step 1. $n \leftarrow 0$,
Step 2. Calculate the recursive relation using (3.3),
Step 3. If $| y_{n+1} - y_n | < \varepsilon$ then go to step 4 else $n \leftarrow n+1$ and go to step 2,
Step 4. Print $y(x) = \sum_{i=0}^{n} y_i$ as the approximate of the exact solution.

Lemma 4.1. *The computational complexity of the algorithm is $O(k^2 + n)$.*

Proof. The number of computations including division, production, sum and subtraction and without considering the number of computations in the term

$$\sum_{j=0}^{k-1} L^{-1}\big(\frac{p_j(x)}{p_k(x)} y_n^{(j)}(x)\big), \quad n \geq 1.$$

In step 2,
$y_0 : \frac{k^2}{2} + \frac{9}{2}k + 2$,
$y_1 : 7$,
$\vdots$
$y_{n+1} : 7,\ n \geq 0$.

In step 4, the total number of the computations is equal to

$$y_0 + \sum_{i=1}^{n} y_i = 7n + \frac{k^2}{2} + \frac{9}{2}k + 2 = O(k^2 + n). \quad \square$$

Example 4.1. Let us now study the nonlinear integro-differential equation

$$u''(x)+xu'(x) = e^x(2+x^2+3x)-(0.5892858)x+\int_0^x [u(t)]^2 dt+\int_0^{0.5} xt(1+u(t))^2 dt,$$

with initial conditions $u(0) = 0$, $u'(0) = 1$. The exact solution is $u(x) = xe^x$. $\epsilon = 10^{-3}$ and $\alpha = 0.350099$.

Table 1 Numerical results for Example 4.1

x	App.Sol ($n = 3$)	Errors
0.05	0.0528836	0.00071584
0.1	0.1118610	0.00127473
0.12	0.1372840	0.00146522
0.15	0.1775160	0.00163066
0.2	0.2505650	0.00196136
0.25	0.3318580	0.00217893
0.3	0.4223900	0.00235548
0.35	0.5233110	0.00268905
0.4	0.6359360	0.00297645
0.45	0.7617640	0.00314973
0.5	0.9024930	0.00347981

Table 1 shows that, the approximation solution of the nonlinear Fredholm-Volterra integral equation is convergent with 3 iterations by using the HAM.

5. Conclusion

Homotopy analysis method has been known as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations and so on. The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are rapidly convergent to the exact solution. In this work, the HAM has been successfully employed to obtain the approximate or analytical solution of the nonlinear Volterra-Fredholm integro-differential equations.

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