Journal of Applied Analysis and Computation
Volume 2, Number 1, February 2012

Website:http://jaac-online.com/
pp. 103–109

EXISTENCE OF A POSITIVE SOLUTION FOR A FIRST-ORDER *P*-LAPLACIAN BVP WITH IMPULSIVE ON TIME SCALES*

Lijian Yin[a] and Zhiyu Zhang[b,†]

Abstract In this paper, we consider the existence of a positive solution for a first-order p-Laplacian BVP with impulsive on time scales:

$$\phi_p(y^{\Delta}(t)) = h(t)f(y(t)) \quad t \in [0, \mathrm{T}]_{\mathbb{T}} \backslash \{\tau\}, \quad \mathrm{Imp}(y(\tau)) = \mathrm{I}y(t), \quad y(0) = B_0(\mathrm{T}).$$

Using the fixed-point theory, we have established the problem in a Banach space with an appropriate operator. Our main contribution is to the combination of the first-order p-Laplacian BVP and the impulsive dynamic equation. We obtained some new results which have advanced recent developments on this type of problem.

Keywords Time scales, p-Laplacian operator, the fixed theory, cone.

MSC(2000) 34B18, 34B37.

1. Introduction

Recently years, there is much attention to the BVP with p-Laplacian operator on the time scales, some authors have obtained many important results, see [1, 4, 6, 7, 8, 9, 10, 11] and references therein. The theory of BVP on time scales provides powerful new tools for exploring connections between some traditionally separated fields. We refer to the books by M. Bohner and A.C.Perterson [2], M. Bohner and A.C.Perterson (Eds) [3] and references therein.

Moreover, impulsive differential equation on time scales has been attracted a large number of authors' interests. Specially, dealing with the problems in the mathematical models of real processes, impulse has led to several important applications, e.g., in the study of population dynamics, financial, optimal control, industrial robots, physical, and so on, refer to [5, 9, 11] and references therein.

In 2006, C.C.Tisdell [10] researched the existence of solutions to first-order periodic BVP:

$$\begin{cases} x' + b(t)x = g(t, x), \ t \in [0, 1], \\ x(0) = x(1). \end{cases} \tag{1.1}$$

[†]the corresponding author. Email address: zhangzhiyu008@yahoo.com.cn(Z. Zhang)
[a]Department of Mathematics, North University of China,Taiyuan, Shanxi, 030008, China
[b]Taiyuan Institute of Technology Taiyuan, Shanxi, 030008, China
*The authors were supported by Natural Science Foundation of Shanxi Province of China (2011011002-3).

In 2007, Zhen, Wang, Zhang and Zhou [4] discussed the double positive solutions of BVP for p-Laplacian impulsive functional dynamic questions on time scales:

$$\begin{cases} [\phi_p(y^{\Delta}(t))]^{\nabla} + a(t)f(y(t), y(\mu(t))) = 0, \ t \in [0,1]_{\mathbb{T}} \setminus \{\tau\}, \\ \mathrm{Imp}(y(\tau)) = \mathrm{I}y(\tau), \\ y_0(t) = \varphi(t), \ t \in [-r, 0], \\ y(0) = y^{\Delta}(1). \end{cases} \tag{1.2}$$

In 2011, Goodrich [7] studies the first-order p-Laplacian on time scales:

$$\begin{cases} \phi_p(y^{\Delta}(t)) = h(t)f(y^{\sigma}(t)), \ t \in (a,b), \\ y(a) = y(b), \\ y(a) = (y^{\Delta}(b))^m, \end{cases} \tag{1.3}$$

or

$$\begin{cases} \phi_p(y^{\Delta}(t)) = h(t)f(y^{\sigma}(t)), \ t \in (a,b), \\ y(a) = y(b), \\ y(a) = B_0(y^{\Delta}(b)). \end{cases} \tag{1.4}$$

Motivated by above articles, in this paper, we are concerned with the existence of a first-order p-Laplacian BVP with impulsive on time scales:

$$\begin{cases} \phi_p(y^{\Delta}(t)) = h(t)f(y(t)), \ t \in [0, \mathrm{T}]_{\mathbb{T}} \backslash \{\tau\}, \\ \mathrm{Imp}(y(\tau)) = \mathrm{I}y(t), \\ y(0) = B_0(\mathrm{T}), \end{cases} \tag{1.5}$$

Throughout this paper, it is assumed that:

(S1). $h : [0, \mathrm{T}] \to [0, +\infty]$ is continuous, and does not vanish identically on any closed subinterval of $[0, \mathrm{T}]_{\mathbb{T}}$;

(S2). $h : [0, \mathrm{T}] \to \mathrm{R}^+$ is left dense continuous(which is short for ld-continuous), where R^+ denotes the nonnegative real numbers;

(S3). $\mathrm{Imp}(y(t)) = y(\tau^+) - y(\tau^-) \geq 0$, $I : \mathrm{R}^+ \to \mathrm{R}^+$;

(S4). $B_0 : \mathrm{R} \to \mathrm{R}$ is continuous;

(S5). $\mathbb{T}$ is a closed nonempty subset of R and have the subspace topology inherited the Euclidean topology on R.

Appearing in above equations, the operator $\phi_p(\cdot)$ is one-dimensional p-Laplacian operator, which is defined by $\phi_p(z) = |z|^{p-2}z$, $p > 1$. It has two properties:

(1). $\phi_p(z_1 z_2) = \phi_p(z_1)\phi_p(z_2)$;

(2). $\phi_p^{-1}(z) = \phi_q(z)$;

where p and q are Hölder conjugate: $\frac{1}{p} + \frac{1}{q} = 1$. We should note that we adopt the standard notation:$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. The open and the half-open intervals are defined in an similar way. On the other hand, [a,b] will be used to denote intervals on the real line, i.e. [a,b]=$\{t \in \mathrm{R}; a \leq t \leq b\}$. This notational convention should help the reader to distinguish between ordinary and time scale intervals.

To place the problem (1.5) in the appropriate content, it often arises in abundant applied mathematical models and real processes. For example, it occurs in turbulence theory, some reaction-diffusion problems, plasma problem, non-Newtonian, astrophysics, nonlinear elasticity and flow through porous media; refer to [1, 6, 7, 8, 9, 11] and references therein. From a more mathematical perspective, BVP with p-Laplacian operator have been extensively studied over the past couple of decades, particularly in terms of the existence of one or more positive solutions; refer to [1, 4, 6, 7, 8, 9] and references therein.

In this paper, the main contribution is to combine the first-order p-Laplacian BVP with impulsive dynamic equation. Although some people have considered p-Laplacian BVP and impulsive dynamic equation; to the authors' best knowledge, the question of a positive solution for a first-order p-Laplacian BVP with impulsive on time scales has not been studied.

2. Preliminaries

For convenience, in this section of the first, we list the following well-known definitions.

Definition 2.1. For $t < \sup\mathbb{T}$ and $t > \inf\mathbb{T}$, define the forward jump operator and the backward jump operator, respectively;

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T}, \qquad \rho(s) = \sup\{\tau \in \mathbb{T} \mid \tau < s\} \in \mathbb{T}.$$

If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(s) < s$, s is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(s) < s$, s is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. The forward graininess is $\mu(t) = \sigma(t) - t$; the backward graininess is $\nu(t) = t - \rho(t)$.

Definition 2.2. For $f : \mathbb{T} \to \mathrm{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted by $f^{\Delta}(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ \forall s \in U.$$

For $f : \mathbb{T} \to R$ and $t \in \mathbb{T}_k$, the nable derivative of f at t, denoted by $f^{\nabla}(t)$, is the number (provided it exist) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset T$ of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|, \ \forall s \in U.$$

In this case $\mathbb{T} = \mathrm{R}$, $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$; when $\mathbb{T} = Z$, $f^{\Delta}(t) = f(t+1) - f(t)$ and $f^{\nabla}(t) = f(t) - f(t-1)$.

Definition 2.3. A function $f : \mathbb{T} \to \mathrm{R}$ is ld-continuous provided it is continuous at left dense point in $\mathbb{T}$ and its left-sided limit exist (finite) at left point $\mathbb{T}$.

Definition 2.4. If $F^{\Delta} = f(t)$, then we define the delta integral by $\int_a^t f(s)\Delta s = F(t) - F(a)$. If $\Phi^{\nabla}(t) = f(t)$, then we define the nable integra by $\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a)$.

In the next section, we will select an appropriate Banach space and a cone in the space, and provide argument conditions for the solution of problem (1.5). We first introduce two hypotheses that we shall make about $h(t), f(y), \mathrm{lmp}(f(y))$:

(F0). The continuous function $h(t)$ is not zero on $[\frac{\mathrm{T}}{4}, \frac{3\mathrm{T}}{4}]_{\mathbb{T}}$ and the continuous function $f(y)$ satisfies: $0 \leq K_1 \leq f(y) \leq K_2$, where K_1 and K_2 are positive numbers;

(F1). Impulsive dynamic equation $\mathrm{Imp}(f(y))$ satisfies: $0 \leq f(y) < \mathrm{R}_2$, Where $\mathrm{R}_2 > 0$.

We introduce a parameter γ, which is defined by

$$\gamma = \frac{\int_0^t \phi_q(h(s))(K_1)\Delta s}{\int_0^T \phi_q(h(s)(K_2)\Delta s}.$$

We have

$$0 \leq \gamma = \frac{\int_0^t \phi_q(h(s))(K_1)\Delta s}{\int_0^T \phi_q(h(s)(K_2)\Delta s} < \frac{\int_0^t \phi_q(h(s))(y(\mathrm{T}))\Delta s}{\int_0^T \phi_q(h(s)(y(\mathrm{T}))\Delta s} < 1. \tag{2.1}$$

Then we give an appropriate Banach spaces E and a cone P in E:

$$E = \{y : [0, \mathrm{T}]_{\mathbb{T}} \to \mathrm{R} | y \in C[0, \tau]_{\mathbb{T}}, y \in C[\tau, \mathrm{T}]_{\mathbb{T}}.y(\tau^+) \in \mathrm{R}\},$$

equipped with the norm:

$$\|y\| = \max\{\sup_{t\in[o,\tau]_{\mathbb{T}}}|y(\tau)|, \sup_{t\in[\tau,\mathrm{T}]_{\mathbb{T}}}|y(\tau)|\}.$$

The cone P is defined as following:

$$P = \{y \in E \mid y \geq 0, \min_{t\in[\frac{\mathrm{T}}{4}, \frac{3\mathrm{T}}{4}]_{\mathbb{T}}} (Ay)(t) \geq \ \gamma \parallel Ay \parallel,$$
$$y \in [0, \tau]_{\mathbb{T}}, y \in [\tau, \mathrm{T}]_{\mathbb{T}}, \mathrm{Imp}(y(\tau)) \geq 0\}.$$

Then we establish an operator as following:

$$(Ay)(t) = I(y(\tau))\chi_{(\tau,\mathrm{T}]_{\mathbb{T}}} + \int_0^t \phi_q(h(s)f(y(s)))\Delta s. \tag{2.2}$$

We note that y(t) is a solution of (1.5) if and only if

$$y(t) = I(y(\tau))\chi_{(\tau,\mathrm{T}]_{\mathbb{T}}} + \int_0^t \phi_q(h(s)f(y(s)))\Delta s,$$

where $\chi_{(\tau,\mathrm{T}]_{\mathbb{T}}}$ is the characteristic function.

Theorem 2.1. *Let (2.2) be defined as above and assume that (2.1) hold. Then $A : P \to P$.*

Theorem 2.2. *Let B be a Banach space and let $K \subseteq B$ that Ω_1 and Ω_2 are bounded open sets contained in B, such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Assume that L : is a completely continuous operator. If either*

(1). $\|Lu\| \leq \|u\|$ for $u \in K \cap \partial\overline{\Omega}_1$, and $\|Lu\| \leq \|u\|$ for $u \in K \cap \partial\overline{\Omega}_2$; or

(2). $\|Lu\| \leq \|u\|$ for $u \in K \cap \partial\overline{\Omega}_1$, and $\|Lu\| \leq \|u\|$ for $u \in K \cap \partial\overline{\Omega}_1$.

Then L has at least one fixed point in $K \bigcap(\overline{\Omega}_1 \setminus \Omega_2)$.

3. Main results

Lemma 3.1. *Assume that the function satisfies condition (F0), then condition (F2), which is defined by*

$$\lim_{y\to 0^+}\frac{f(y)}{\phi_p(y)}=+\infty \tag{3.1}$$

holds.

Proof. The proof can refer to C.S.Goodrich's proof. □

Theorem 3.1. *Suppose that conditions (F0)-(F1) and (S1)-(S5) hold, then (1.5) has at least one positive solution.*

Proof. Let h and f be given and satisfied condition (F0). Lemma 2.1 shows that $A : P \to P$, then, we show that it is a completely continuous operator. By the continuous of f, ϕ_q and I, one can easily derive that there is a constant $C > 0$ such that

$$|Ay(t)| \le |\mathrm{I}(y(\tau))| + \sup_{t\in[0,\mathrm{T}]_{\mathbb{T}}} |\int_0^{\mathrm{T}} \phi_q(h(s)f(y(s)))\Delta s| < C,$$

where $y \in \{y \in P | \|y\| \le \mathrm{R}_1\}$. We show that A maps bounded sets into bounded sets in P. Let $t_1, t_2 \in [0,\mathrm{T}]_{\mathbb{T}}$, $y \in \{y \in P | \|y\| \le \mathrm{R}_1\}$, then

$$|Ay(t_1) - Ay(t_2)| \le \phi_q(h(s)f(y(s)))|t_1 - t_2| < \phi_q(h(s))f(\mathrm{R}_1)|t_1 - t_2|.$$

Obviously, the right-hand side tends uniformly to zero when $|t_1 - t_2| \to 0$. By the arguments above and the Areola-Ascorem, we get that $A : P \to P$ is completely. Pick a number $c > 0$ such that

$$c\int_{[\frac{\mathrm{T}}{4},\frac{3\mathrm{T}}{4}]_{\mathbb{T}}} \gamma\phi_q(h(s))\Delta s \ge 1. \tag{3.2}$$

From lemma 3.1, there is a number $\mathrm{R}_1 > 0$ such that $0 < y < \mathrm{R}_1$, we obtained that

$$f(y) \ge \phi_p(c)\phi_p(y(s)) = \phi_p(cy). \tag{3.3}$$

Put $\Omega_1 = \{y \in P : \|y\| < P_1\}$ for $y \in P \bigcap \partial\Omega_1$, we find that

$$\begin{aligned}
\|Ay\| =&(Ay)(t)\\
=&\mathrm{Imp}(y(\tau))\chi_{[\tau,\mathrm{T}]_{\mathbb{T}}} + \int_0^{\mathrm{T}} \phi_q(h(s)f(y(s)))\Delta s\\
\ge& \int_0^{\mathrm{T}} \phi_q(h(s)f(y(s)))\Delta s\\
\ge& \int_0^{\mathrm{T}} \phi_q(h(s)\phi_p(c)\phi_p(y))\Delta s\\
=&c\int_0^{\mathrm{T}} \phi_q(h(s))y(s)\Delta s\\
\ge&\|y\|c\int_{[\frac{\mathrm{T}}{4},\frac{3\mathrm{T}}{4}]_{\mathbb{T}}} \gamma\phi_q(h(s))\Delta s \ge \|y\|,
\end{aligned} \tag{3.4}$$

where to get the first inequality we have used the fact that $\mathrm{Imp}(y(\tau))\chi_{(\tau,\mathrm{T}]_{\mathbb{T}}}(t) \geq 0$; the second inequality we have used (3.3); the last inequality we use the cone P. Thus, (3.4) implies that $\|Ay\| \geq \|y\|$ for $y \in P \bigcap \partial\Omega_1$.

From the hypothesis (F0), we know that $0 < K_1 \leq f(y) \leq K_2$, and $f(y)$ is bounded for $y > 0$, therefore, there is $\mathrm{R}_2^* > 0$ sufficiently large such that $f(y) \leq \phi_q(\mathrm{R}_2^*)$ for $y > 0$; from the hypothesis (F1), put $0 < \varepsilon < 1$, we get

$$\begin{aligned}
&\mathrm{Imp}(y(\tau))\chi_{(\tau,T]_{\mathbb{T}}} \leq (1-\varepsilon)\mathrm{R}_2,\\
&\mathrm{R}_2 = \max\{\frac{1}{\varepsilon}\mathrm{R}_1, \frac{\mathrm{R}_2^*}{\varepsilon}\int_0^{\mathrm{T}} \phi_q(h(s))\Delta s\},
\end{aligned}$$

Define the set Ω_2 by $\Omega_2 = \{y \in P : \|y\| < \mathrm{R}_2\}$, and $\mathrm{R}_2 > \mathrm{R}_1$, whenever $y \in P \cap \Omega_2$, we find that

$$\begin{aligned}
\|Ay\| =&(Ay)(\mathrm{T})\\
=&\mathrm{Imp}(y(\tau))\chi_{(\tau,\mathrm{T}]_{\mathbb{T}}} + \int_0^{\mathrm{T}} \phi_q(h(s)f(y(s)))\Delta s\\
\leq&(1-\varepsilon)\mathrm{R}_2 + \int_0^{\mathrm{T}} \phi_q(h(s)\phi_p(\mathrm{R}_2^*))\Delta s\\
\leq&(1-\varepsilon)\mathrm{R}_2 + \mathrm{R}_2^*\int_0^{\mathrm{T}} \phi_q(h(s))\Delta s\\
\leq&(1-\varepsilon)\mathrm{R}_2 + \varepsilon\mathrm{R}_2 = \mathrm{R}_2 = \|y\|.
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we conclude from lemma 2.2 that A has a fixed point in $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$. And this completes the proof. $\Box$

In this paper, we study the existence of a positive solution for a first-order p-Laplacian BVP with impulsive on time scales. Because we combine the first-order p-Lapulacian BVP with impulsive dynamic equation. Some assumptions are given, which are different from others, for example [1, 7, 8]. Significantly, our new results have advanced recent developments on this type of problem.

References

[1] D.R. Anderson, *Existence of solutions for a first-order p-Laplacian BVP on time scale*, Nonlinear Anal. TMA, 69 (2008), 4521-4525.

[2] M. Bohner and A.C.Peterson,*Dynamic equations on time scale:an introduction with applications* , Birkhäuser, Boston, 2001.

[3] M. Bonhner and A.C.Perterson(Eds), *Advances in dynamic equations on time scale* , Birkhäuser, Boston, 2003.

[4] H. Chen, H. Wang, Q. Zhang and T. Zhou, *Double positive solutions of boundary value problems for p-Laplacian impulsive functional dynamic equations on time scales*, Comput. Math. Appl., 53 (2007), 1473-1480.

[5] J. Chen, C.C.Tisdell and R. Yuan, *On the solvability of periodic value problem with impulse* , J. Math. Anal. Appl., 331 (2007), 902-912.

[6] B. Du, *Some new results on the existence of positive solutions for the one-dimensional p-Laplacian boundary value problems on time scales* , Nonlinear Anal, 69 (2008),385-392.

[7] C.S. Goodrich, *Existence of a positive solution to a first-oder p-Laplacian BVP on a time scale*, Nonlinear Anal, 74 (2011), 1926-1936.

[8] C.S. Goodrich, *The existence of a positive solution to a second-order delta-nabla p-Laplacian BVP on a time scale*, Appl. Math. Lett, 25 (2012), 157-162.

[9] M. He, *Double positive solutions of nth-order impulsive boundary value problems for p-Laplacian dynamic equations on time scales* , J. Comput. Appl. Math., 182 (2005), 304-315.

[10] C.C. Tisdell, *Existence of solutions to first-order periodic Boundary value problems* , J. Math. Anal. Appl., 323 (2006), 1325-1332.

[11] Zhang, Yang and Ge, *Positive solutions of nth-order impulsive boundary value problems with intergral boundary value conditions in Banach spaces* , Nonlinear Anal, 71 (2009), 5930-5945.