Journal of Applied Analysis and Computation Website:http://jaac-online.com/
Volume 2, Number 1, February 2012 pp. 29–56

LONG TIME BEHAVIOR OF AN ALLEN-CAHN TYPE EQUATION WITH A SINGULAR POTENTIAL AND DYNAMIC BOUNDARY CONDITIONS

Haydi Israel

Abstract The aim of this paper is to study the well-posedness and the long time behavior of solutions for an equation of Allen-Cahn type owing to proper approximations of the singular potential and a suitable definition of solutions. We also prove the existence of the finite dimensional global attractor as well as exponential attractors.

Keywords Allen-Cahn equation, dynamic boundary conditions, singular potentials, variational solutions, global attractor, exponential attractors.

MSC(2000) 35B40, 35B41, 35K55, 35J60, 80A22.

1. Introduction

In this article we are interested in the study of the following initial and boundary value problem, considered in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$:

$$\begin{cases} \partial_t \phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu, \ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + f(\phi) - \lambda\phi, \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, Δ_Γ is the Laplace-Beltrami operator on the boundary $\partial\Omega$, f and g are given nonlinear interaction functions and λ is some given positive constant. In particular, f is the derivative of a double-well potential whose wells correspond to the phases of the material. A thermodynamically relevant function f is the following logarithmic (singular) function:

$$f(s) = -2\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \ s \in (-1, 1), \ \kappa_0 > \kappa_1 > 0.$$

The boundary condition will be interpreted as an additional second-order parabolic equation on the boundary $\partial\Omega$.

Equation (1.1) may be viewed as a combination of the well-known Cahn-Hilliard equation

$$\partial_t u = -\Delta(\Delta u + f(u)), \quad u(0, x) = u_0(x),$$

Email adress: haydiisrael@hotmail.com(H. Israel)
Laboratoire de Mathématiques et Applications,Université de Poitiers, UMR CNRS 7348- SP2MI, Boulevard Marie et Pierre Curie-Téléport 2, BP30179 - 86962 Futuroscope Chasseneuil Cedex,France

and of the Allen-Cahn equation

$$\partial_t u = \Delta u + f(u), \quad u(0,x) = u_0(x).$$

This equation is associated with multiple microscopic mechanisms such as surface diffusion and absorption/desorption and was recently derived and studied in Karali & Katsoulakis [6], Katsoulakis & Vlachos [7], Israel [9], Hildebrand & Mikhailov [10].

This paper is organized as follows. In Section 2, we introduce regularized problems in which the singular nonlinearity is approximated by regular functions and we derive uniform a priori estimates on the corresponding solutions. In Section 3, we formulate the variational formulation of (2.1), we verify the existence and uniqueness of a solution and we study the further regularity of the solutions. In Section 4, we give sufficient conditions which ensure that solutions are separated from the singularities of f and that a variational solution coincides with a solution in the usual (distribution) sense. Finally, we study in Section 5 the asymptotic behavior of the system and we prove the existence of finite-dimensional (both global and exponential) attractors.

2. Approximations and uniform a priori estimates

We set $\tilde{f}(\phi) := f(\phi) - \lambda\phi$ and rewrite problem (1.1) in the form:

$$\begin{cases} \partial_t \phi = \Delta\mu - \mu = -(-\Delta + I)\mu = -A\mu, \ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{f}(\phi), \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - g(\psi) - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{2.1}$$

where the singular function f satisfies:

$$\begin{cases} f \in C^2((-1,1)), \\ f(0) = 0, \ \lim_{s\to\pm\infty} f(s) = \pm\infty, \\ f'(s) \geq 0, \ \lim_{s\to\pm\infty} f'(s) = +\infty, \\ f''(s) \operatorname{sgn} s \geq 0. \end{cases} \tag{2.2}$$

As a consequence, the following properties hold for $\tilde{f}$:

$$\tilde{f}'(s) \geq -\lambda \text{ and } -\tilde{c} \leq \tilde{F}(s) \leq \tilde{f}(s)s + \tilde{C}, \ \ \forall s \in (-1,1), \tag{2.3}$$

where $\tilde{F}(s) = \int_0^s \tilde{f}(r)dr$ and $\tilde{c}$, $\tilde{C}$ are strictly positive constants.

The nonlinear function $g \in C^2([-1,1])$ can be extended, without loss of generality, to the whole real line by writing:

$$g(s) = s + g_0(s), \quad \forall s \in \mathbb{R}, \text{ where } \|g_0\|_{C^2(\mathbb{R})} := C_0 < +\infty. \tag{2.4}$$

We set, for $r \geq 1$,

$$H^r(\Omega) \otimes H^r(\Gamma) := \{v \in H^r(\Omega), \ v|_\Gamma \in H^r(\Gamma)\},$$

which we endow with the norm:

$$\|v\|^2_{H^r(\Omega)\otimes H^r(\Gamma)} = \|v\|^2_{H^r(\Omega)} + \|v\|^2_{H^r(\Gamma)}.$$

Alternatively, the functions in $H^r(\Omega)\otimes H^r(\Gamma)$ can be viewed as pairs of functions $(v, v|_\Gamma)$.

We introduce a family of regular approximating functions: given any $N \in \mathbb{N}$, we set:

$$f_N(s) = \begin{cases} f(-1+1/N) + f'(-1+1/N)(s+1-1/N), & -1 < s < -1+1/N, \\ f(s), \quad |s| \le 1-1/N, \\ f(1-1/N) + f'(1-1/N)(s-1+1/N), & 1-1/N < s < -1. \end{cases} \tag{2.5}$$

Then, we denote F_N the primitive $F_N(s) = \int_0^s f_N(s)\mathrm{d}s$ and having set $\tilde{f}_N(s) = f_N(s) - \lambda s$, we define $\tilde{F}_N$ analogously, with $\tilde{f}_N$ instead of f_N. We recall the following properties (for more details, see Miranville & Zelik [12]), namely, there exist $\alpha > 0$, $c > 0$ and $C > 0$ such that:

$$\tilde{f}_N(s)s \ge \alpha/2|f_N(s)| - c, \tag{2.6}$$

and

$$1/2F_N(s) - C \le \tilde{F}_N(s) \le 2F_N(s) + C, \tag{2.7}$$

$\forall s \in \mathbb{R}$ and for $N \ge N_0(\lambda)$ large enough, where the constant C only depends on λ.

We then consider the approximate problems:

$$\begin{cases} \partial_t\phi = \Delta\mu - \mu = -(-\Delta+I)\mu = -A\mu, \ \partial_n\mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{f}_N(\phi), \ \phi_{|t=0} = \phi_0, \\ \partial_t\psi = \Delta_\Gamma\psi - g(\psi) - \partial_n\phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi. \end{cases} \tag{2.8}$$

It is convenient to rewrite problem (2.8) in an equivalent form by using the inverse of $A := (-\Delta + I)$ (endowed with Neumann boundary conditions). Applying A^{-1} to both side of (2.8), we obtain:

$$\begin{cases} A^{-1}\partial_t\phi - \Delta\phi + \tilde{f}_N(\phi) = 0, \ x \in \Omega, \ \phi_{|t=0} = \phi_0, \\ \partial_t\psi = \Delta_\Gamma\psi - g(\psi) - \partial_n\phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi. \end{cases} \tag{2.9}$$

We start with the usual energy equality.

Lemma 2.1. *Let the above assumptions hold and let ϕ be a sufficiently regular solution of* (2.9)*. Then, the following identities hold:*

$$\begin{aligned} &\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t \left(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + (\tilde{F}_N(\phi(s)),1)_\Omega\right)\mathrm{d}s \\ \le\ & C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)} + \int_0^t e^{-\nu(t-s)}\left(\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)}\right)\mathrm{d}s \\ &+ \int_0^t e^{-\nu(t-s)}\|f_N(\phi(s)\|_{L^1(\Omega)}\mathrm{d}s \le C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right)e^{-\nu t}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\int_t^{t+1}(\|\phi(s)\|^2_{H^1(\Omega)}+\|\psi(s)\|^2_{H^1(\Gamma)}+\alpha\|f_N(\phi(s)\|_{L^1(\Omega)})\mathrm{d}s\\ \le\ & C\left(1+(\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)})e^{-\nu t}\right),\end{aligned}\tag{2.12}$$

for some positive constants C and ν which are independent of t.

Proof. Multiplying (2.9) by ϕ and using the fact that:

$$\|\phi\|_{H^1(\Omega)}\le C(\|\nabla\phi\|_{L^2(\Omega)}+\|\psi\|_{H^1(\Gamma)}),\tag{2.13}$$

we find:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}\right)+\|\phi(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Gamma)}\\ &+(\tilde f_N(\phi(t)),\phi(t))_\Omega+(g_0(\psi(t)),\psi(t))_\Gamma\quad\le\quad 0.\end{aligned}\tag{2.14}$$

Using (2.3), the fact that g_0 is globally bounded and that $\|\psi(t)\|_{L^\infty(\Gamma)}\le 1$, we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}\right)+\ \|\phi(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Gamma)}\\ &+(\tilde F_N(\phi(t)),1)_\Omega\quad\le\quad C,\end{aligned}\tag{2.15}$$

for some positive constant C. Integrating (2.15) with respect to t, we deduce:

$$\begin{aligned}&\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}+\int_0^t\left(\|\phi(s)\|^2_{H^1(\Omega)}+\|\psi(s)\|^2_{H^1(\Gamma)}+(\tilde F_N(\phi(s)),1)_\Omega\right)\mathrm{d}s\\ \le& C\left(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}\right),\end{aligned}\tag{2.16}$$

for some constant C.

Now, using (2.6) and the fact that g_0 is globally bounded, (2.14) gives:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}\right)+\|\phi(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Gamma)}\\ &+\alpha/2\|f_N(\phi(t)\|_{L^1(\Omega)}\quad\le\quad C,\end{aligned}\tag{2.17}$$

for some positive constants α and C. Hence, for $\nu>0$ small enough, we obtain:

$$\begin{aligned}&\frac{d}{dt}(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)})+\nu\left(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)}\right)\\ &+\nu(\|\phi(t)\|^2_{H^1(\Omega)}+\|\psi(t)\|^2_{H^1(\Gamma)}+\|f_N(\phi(t)\|_{L^1(\Omega)})\quad\le\quad C.\end{aligned}\tag{2.18}$$

Applying Gronwall's lemma, we deduce estimate (2.11). Finally, integrating (2.17) with respect to t over $(t,t+1)$ and using (2.11), we obtain:

$$\begin{aligned}&\int_t^{t+1}(\|\phi(s)\|^2_{H^1(\Omega)}+\|\psi(s)\|^2_{H^1(\Gamma)}+\alpha\|f_N(\phi(s)\|_{L^1(\Omega)})\mathrm{d}s\\ \le\ & C(1+\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)})\\ \le\ & C\left(1+(\|\phi(t)\|^2_{H^{-1}(\Omega)}+\|\psi(t)\|^2_{L^2(\Gamma)})e^{-\nu t}\right).\end{aligned}\tag{2.19}$$

□

Lemma 2.2. *Let the assumptions of Lemma 2.1 hold. Then, the following identity holds:*

$$\begin{aligned}
&\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\
&+ \int_0^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq\ & C\left(1 + \|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(0)),1)_\Omega\right),
\end{aligned} \tag{2.20}$$

where the constant C is independent of t and of the initial data.

Proof. Multiplying the first equation of (2.9) by $\partial_t\phi$ and integrating over Ω, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \|\psi(t)\|^2_{L^2(\Gamma)}\right) + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} \\
&+ \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + (\tilde{f}_N(\phi(t)),\partial_t\phi(t))_\Omega + (g_0(\psi(t)),\partial_t\psi(t))_\Gamma \\
=&\frac{1}{2}\frac{d}{dt}(\|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} + \|\psi(t)\|^2_{L^2(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\
&+ 2(G_0(\psi(t)),1)_\Gamma) + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} \\
=&0,
\end{aligned} \tag{2.21}$$

where $G_0(t) = \int_0^t g_0(s)\mathrm{d}s$. Using (2.13), we find:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma\right) \\
&+ \|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)} = 0.
\end{aligned} \tag{2.22}$$

Integrating (2.22) with respect of t, taking into account that g_0 is globally bounded, we deduce (2.20). □

Lemma 2.3. *Let the assumptions of Lemma 2.1 hold, ϕ be a sufficiently regular solution of (2.9) and N be large enough. Then, for $t > 0$, the following smoothing property holds:*

$$\begin{aligned}
&t\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega\right) \\
&+ \int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq\ & C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right),
\end{aligned} \tag{2.23}$$

where the constant C is independent of N.

Proof. Multiplying (2.22) by t, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma)) \\
&+ t(\|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)}) \\
=&\frac{1}{2}(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma).
\end{aligned} \tag{2.24}$$

Integrating (2.24) with respect of t from 0 to t, we have:

$$
\begin{aligned}
& t(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega + 2(G_0(\psi(t)),1)_\Gamma \\
& + 2\int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq & \int_0^t (\|\phi(s)\|^2_{H^1(\Omega)} + \|\psi(s)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(s)),1)_\Omega + 2(G_0(\psi(s)),1)_\Gamma).
\end{aligned} \tag{2.25}
$$

Using (2.7), (2.10) and that g_0 is bounded globally, we deduce:

$$
\begin{aligned}
& t\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega\right) \\
& + \int_0^t s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq \ & C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right),
\end{aligned} \tag{2.26}
$$

and the proof is complete. □

Lemma 2.4. *Let the assumptions of Lemma 2.1 hold. Then, we have, for all $t \geq 1$ and N large enough, the following property:*

$$
\begin{aligned}
& \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \mathrm{d}s \\
& + \int_1^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq \ & C(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}).
\end{aligned} \tag{2.27}
$$

Moreover, for any $t > 0$ and N large enough, the following inequality holds:

$$
\begin{aligned}
& \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\
\leq \ & C\frac{t+1}{t}(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}).
\end{aligned} \tag{2.28}
$$

Proof. Integrating (2.22) with respect to t from 1 to t, we obtain:

$$
\begin{aligned}
& \|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(t)),1)_\Omega \\
& + \int_1^t (\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq \ & C\left(1 + \|\phi(1)\|^2_{H^1(\Omega)} + \|\psi(1)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(1)),1)_\Omega\right).
\end{aligned} \tag{2.29}
$$

From (2.23) and for $t = 1$, we find:

$$
\begin{aligned}
& \left(\|\phi(1)\|^2_{H^1(\Omega)} + \|\psi(1)\|^2_{H^1(\Gamma)} + 2(\tilde{F}_N(\phi(1)),1)_\Omega\right) \\
& + \int_0^1 s(\|\partial_t\phi(s)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\
\leq \ & C\left(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}\right).
\end{aligned} \tag{2.30}
$$

Estimates (2.29) and (2.30) allow us to find (2.27). Estimate (2.28) follows immediately from (2.23) and (2.27). □

We will now give some additional regularity results on $\partial_t\phi(t)$. To prove this, we differentiate (2.8) and set $(u(t), v(t), w(t)) := \partial_t(\phi(t), \mu(t), \psi(t))$. Then we have:

$$\begin{cases} \partial_t u = \Delta v - v = -(-\Delta + I)v = -Av, \ \partial_n v_{|\Gamma} = 0, \\ v = -\Delta u + \tilde{f}'_N(\phi)u, \\ \partial_t w = \Delta_\Gamma w - g'(\psi)w - \partial_n w, \ x \in \Gamma, \\ u_{|\Gamma} = w. \end{cases} \tag{2.31}$$

Lemma 2.5. *Let the assumptions of Lemma 2.1 hold. Then, the following estimate is valid for all $t > 0$:*

$$\begin{aligned} &\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|w(s)\|^2_{H^1(\Gamma)}) \mathrm{d}s \\ \leq\ & c(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)})e^{-\nu t} + c, \end{aligned} \tag{2.32}$$

for some positive constants c and ν independent of N. Moreover, for $t > 0$, we have the smoothing property:

$$\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \leq c\frac{t^2+1}{t^2}(1 + \|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}). \tag{2.33}$$

Proof. Multiplying the first equation of (2.31) by $A^{-1}u$, the second equation by u and the third one by w and taking the sum of the equations that we obtain, we have the following identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}\right) + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma w(t)\|^2_{L^2(\Gamma)} \\ &+ \|w(t)\|^2_{L^2(\Gamma)} + (\tilde{f}'_N(\phi(t))u(t), u(t))_\Omega + (g'_0(\psi(t))w(t), w(t))_\Gamma = 0. \end{aligned} \tag{2.34}$$

Using (2.3), (2.13) and that g'_0 is globally bounded, we find:

$$\begin{aligned} &\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}\right) + \|u(t)\|^2_{H^1(\Omega)} + \|w(t)\|^2_{H^1(\Gamma)} \\ \leq\ & c(\|u(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{2.35}$$

Using the interpolation inequality $\|u\|^2_{L^2(\Omega)} \leq c\|u\|_{H^1(\Omega)}\|u\|_{H^{-1}(\Omega)}$, we find:

$$\begin{aligned} &\frac{d}{dt}\left(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}\right) + \|u(t)\|^2_{H^1(\Omega)} + \|w(t)\|^2_{H^1(\Gamma)} \\ \leq\ & c(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{2.36}$$

Provided that $\nu > 0$ is small enough, the following estimate holds:

$$\begin{aligned} &\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}) + \nu(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}) \\ &+ \nu(\|u(t)\|^2_{H^1(\Omega)} + \|w(t)\|^2_{H^1(\Gamma)}) \\ \leq\ & c(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}). \end{aligned} \tag{2.37}$$

We consider the case $t \geq 1$. Applying Gronwall's inequality and using estimate

(2.27), we obtain:

$$\begin{aligned}&\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}\\ \le&(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}\\ &+c\int_1^t e^{-\nu(t-s)}(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \le&(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}\\ &+c\int_1^t(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \le&(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}\\ &+c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{2.38}$$

Integrating (2.37) over $(t,t+1),\ t\ge 1$, we find:

$$\begin{aligned}&\int_t^{t+1}(\|u(s)\|^2_{H^1(\Omega)}+\|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\\ \le&\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}+c\int_t^{t+1}(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \le&(\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)})e^{-\nu(t-1)}+c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}).\end{aligned}\tag{2.39}$$

We have:

$$\begin{aligned}&t^2(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)})\\ =&\ 2\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ &+\int_0^t s^2\partial_t(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s.\end{aligned}\tag{2.40}$$

Taking into account (2.26) and (2.36), we obtain for $t\in(0,1]$:

$$\begin{aligned}&t^2(\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)})\\ \le&\ 2\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ &+c\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \le&\ (2+c)\int_0^t s(\|u(s)\|^2_{H^{-1}(\Omega)}+\|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s\\ \le&\ c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}),\end{aligned}\tag{2.41}$$

for some positive constant c. For $t=1$, estimate (2.41) gives:

$$\|u(1)\|^2_{H^{-1}(\Omega)}+\|w(1)\|^2_{L^2(\Gamma)}\le c(1+\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)}),\tag{2.42}$$

then, replacing this estimate in (2.38) and (2.39), we deduce:

$$\begin{aligned}&\|u(t)\|^2_{H^{-1}(\Omega)}+\|w(t)\|^2_{L^2(\Gamma)}+\int_t^{t+1}(\|u(s)\|^2_{H^1(\Omega)}+\|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s\\ \le&\ c+c(\|\phi(0)\|^2_{H^{-1}(\Omega)}+\|\psi(0)\|^2_{L^2(\Gamma)})e^{-\nu t},\end{aligned}\tag{2.43}$$

for all $t \geq 1$. From estimates (2.41) and (2.43), we deduce:

$$\begin{aligned} & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \\ \leq \ & c(1+\frac{1}{t^2})(1+\|\phi(0)\|^2_{H^{-1}(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}),\ \forall t>0. \end{aligned} \tag{2.44}$$

To show estimate (2.32) for $t \in (0,1]$, we consider (2.37), in particular,

$$\frac{d}{dt}(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}) \leq c(\|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)}). \tag{2.45}$$

Applying Gronwall's inequality to (2.45), we find:

$$\begin{aligned} & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} \\ \leq \ & ce^{ct}(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)}) \\ \leq \ & c'e^{-\nu t}(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)}),\ t\in(0,1]. \end{aligned} \tag{2.46}$$

Integrating (2.37) over $(t,t+1)$, $t \in (0,1]$, using (2.43) and (2.46), we obtain:

$$\begin{aligned} & \int_t^{t+1} (\|u(s)\|^2_{H^1(\Omega)} + \|w(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leq \ & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + c\int_t^{t+1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leq \ & \|u(t)\|^2_{H^{-1}(\Omega)} + \|w(t)\|^2_{L^2(\Gamma)} + c\int_t^{1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ & + c\int_1^{t+1} (\|u(s)\|^2_{H^{-1}(\Omega)} + \|w(s)\|^2_{L^2(\Gamma)})\mathrm{d}s \\ \leq \ & c + c(\|u(0)\|^2_{H^{-1}(\Omega)} + \|w(0)\|^2_{L^2(\Gamma)})e^{-\nu t}. \end{aligned} \tag{2.47}$$

Hence, the proof is complete. □

Theorem 2.1. *Let the nonlinearities f and g satisfy (2.2) and (2.4) respectively and set $\Omega_\delta := \{x \in \Omega,\ d(x,\Gamma) > \delta\}$. Denote by $n = n(x)$ some smooth extension of the unit normal vector field at the boundary inside the domain Ω. Let also $D_\tau\phi := \nabla_x\phi - (\partial_n\phi)_n$ be the tangential part of the gradient $\nabla_x\phi$. Then, for every $\delta > 0$, the following estimate is valid:*

$$\begin{aligned} & \|\phi(t)\|^2_{C^\alpha(\Omega)} + \|\phi(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} \\ & + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|\nabla D_\tau\phi(t)\|^2_{L^2(\Omega)} + \|f_N(\phi(t))\|_{L^1(\Omega)} \\ & + \int_t^{t+1} (\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leq \ & C(1+\|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)})e^{-\gamma t}, \end{aligned} \tag{2.48}$$

where the positive constants $\alpha(\alpha > 1/4)$, γ and C are independent of N.

Proof. We consider the nonlinear elliptic problem:

$$\begin{cases} \Delta\phi(t) - f_N(\phi(t)) - \phi(t) = h_1(t) := -\phi(t) + \lambda\phi(t) + A^{-1}\partial_t\phi(t), & x \in \Omega \\ \Delta_\Gamma\psi(t) - \psi(t) - \partial_n\phi(t) = h_2(t) := g_0(\psi(t)) + \partial_t\psi(t), & x \in \Gamma, \end{cases} \tag{2.49}$$

for every fixed t. Note here that the estimates derived above yield the following control of the right-hand side of (2.49):

$$\|h_1(t)\|^2_{L^2(\Omega)} + \|h_2(t)\|^2_{L^2(\Gamma)} \leq C(1 + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)}), \tag{2.50}$$

where C is a positive constant that is independent of N. Due to estimate (2.33), we find that $h_1 \in L^2(\Omega)$ and $h_2 \in L^2(\Gamma)$. Using estimates (2.20), (2.32) and (3.38), we obtain:

$$\begin{aligned} &\|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|f_N(\phi(t))\|_{L^1(\Omega)} \\ &+ \int_t^{t+1} (\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leq\ & C(1 + \|\phi(0)\|^2_{H^1(\Omega)} + \|\psi(0)\|^2_{H^1(\Gamma)} + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)})e^{-\gamma t}. \end{aligned} \tag{2.51}$$

In order to prove the following estimate:

$$\|\phi(t)\|^2_{H^2(\Omega_\delta)} \leq C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}), \tag{2.52}$$

where $C = C_\epsilon$ depends on $\epsilon > 0$. We consider a smooth nonnegative cut-off function θ such that $\theta(x) = 1$ if $d(x, \Gamma) \geq \delta$ and $\theta(x) = 0$ if $d(x, \Gamma) \leq \delta/2$ which satisfies, in addition, the inequality:

$$|\nabla_x\theta(x)| \leq C\theta^{1/2}(x).$$

Then, we multiply equation (2.49) by $\sum_{i=3}^3 \partial_{x_1}(\theta(x)\partial_{x_i}u)$, and we integrate by parts. Using estimate (2.51) and the fact that $f' \geq 0$, we obtain estimate (2.52). In order to prove:

$$\|\nabla_x D_\tau\phi\|^2_{L^2(\Omega)^6} + \|\phi\|^2_{H^2(\Gamma)} \leq C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}), \tag{2.53}$$

we study the function ϕ in a small ϵ-neighborhood of the boundary Γ. To do so, let $x_0 \in \Gamma$ and $y = y(x)$ be a local coordinates in the neighborhood of x_0 such that $y(x_0) = 0$ and Ω is defined, in these coordinates, by the condition $y_1 > 0$. Then, we rewrite problem (2.49) in the variable y and after several transformations we find estimate (2.53). To finish the proof of the theorem, we use the following embedding:

$$L^2(\mathbb{R}, H^2(\mathbb{R}^2)) \cap H^1(\mathbb{R}, H^1(\mathbb{R}^2)) \subset C^\alpha(\mathbb{R}^3),\ \alpha < 1/4,$$

and we deduce the estimate:

$$\|\phi\|^2_{C^\alpha(\Omega)} \leq C(1 + \|h_1\|^2_{L^2(\Omega)} + \|h_2\|^2_{L^2(\Gamma)}). \tag{2.54}$$

Hence, Theorem (2.1) is proved (for more details see Miranville & Zelik [12]). □

In what follows, we will establish the uniform Lipschitz continuity of the solution $(\phi(t), \mu(t), \psi(t))$ of problem (2.9) with respect to the initial data.

Proposition 2.1. *Let the above assumptions hold and let $(\phi_1(t), \mu_1(t), \psi_1(t))$ and $(\phi_2(t), \mu_2(t), \psi_2(t))$ be two solutions of problem (2.9). Then, the following estimate holds:*

$$\begin{aligned} &\|\phi_1(t) - \phi_2(t)\|^2_{H^{-1}(\Omega)} + \|\psi_1(t) - \psi_2(t)\|^2_{L^2(\Gamma)} \\ &+ \int_t^{t+1} (\|\phi_1(s) - \phi_2(s)\|^2_{H^1(\Omega)} + \|\psi_1(s) - \psi_2(s)\|^2_{H^1(\Gamma)})\mathrm{d}s \\ \leq\ & C\left(\|\phi_1(0) - \phi_2(0)\|^2_{H^{-1}(\Omega)} + \|\psi_1(0) - \psi_2(0)\|^2_{L^2(\Gamma)}\right) e^{Kt}, \end{aligned} \tag{2.55}$$

where the constants C and K are independent of t, N and the initial data.

Proof. Let $(\phi(t), \mu(t), \psi(t)) = (\phi_1(t) - \phi_2(t), \mu_1(t) - \mu_2(t), \psi(t) - \psi_2(t))$. Then, this function satisfies the system:

$$\begin{cases} \partial_t \phi = -A\mu, \ \partial_n \mu_{|\partial\Omega} = 0, \\ \mu = -\Delta\phi + \tilde{l}_N(t)\phi, \ \phi_{|t=0} = \phi_0, \\ \partial_t \psi = \Delta_\Gamma \psi - \psi - m(t)\psi - \partial_n \phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi, \end{cases} \tag{2.56}$$

where

$$\tilde{l}_N(t) := \int_0^t \tilde{f}'_N(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s \text{ and } m(t) := \int_0^t g'_0(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s.$$

Multiplying the first equation of (2.31) by $A^{-1}\phi$, the second equation by ϕ and the third one by ψ and taking the sum of the equations that we obtain, we have the following identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}\right) + \|\nabla\phi(t)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\psi(t)\|^2_{L^2(\Gamma)} \\ &+ \|\psi(t)\|^2_{L^2(\Gamma)} + (\tilde{l}_N(t)\phi(t), \phi(t))_\Omega + (m(t)\psi, \psi(t))_\Gamma = 0. \end{aligned} \tag{2.57}$$

Using (2.3), (2.13) and the fact that g'_0 is globally bounded, we obtain:

$$\begin{aligned} &\frac{d}{dt}\left(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\psi(t)\|^2_{L^2(\Gamma)}\right) + \alpha'\left(\|\phi(t)\|^2_{H^1(\Omega)} + \|\psi(t)\|^2_{H^1(\Omega)}\right) \\ \le\ & C(\|\phi(t)\|^2_{L^2(\Omega)} + \|\psi(t)\|^2_{L^2(\Omega)}) \end{aligned} \tag{2.58}$$

for some positive constants α' and C which are independent of N. Using the interpolation inequality $\|u\|_2^2 \le C\|u\|_{H^1(\Omega)}\|u\|_{H^{-1}(\Omega)}$ and applying the Gronwall inequality, we deduce (2.55). □

3. Variational formulation and well-posedness

This section is devoted to the definition of a suitable notion for a solution to the limit problem, that is, the problem obtained by letting $N \to +\infty$ and which coincides with (2.1).To this end, we first fix a constant $L > 0$ such that:

$$\|\nabla\varphi\|^2_{L^2(\Omega)} - \lambda\|\varphi\|^2_{L^2(\Omega)} + L\|\varphi\|^2_{H^{-1}(\Omega)} \ge 1/2\|\varphi\|^2_{H^1(\Omega)}, \tag{3.1}$$

for all $\varphi \in H^1(\Omega)$ and introduce the quadratic form:

$$B(\varphi, \rho) := (\nabla\varphi, \nabla\rho)_\Omega - \lambda(\varphi, \rho)_\Omega + L((-\Delta + I)^{-1}\varphi, \rho)_\Omega + (\nabla_\Gamma\varphi, \nabla_\Gamma\rho)_\Gamma, \tag{3.2}$$

$\forall\varphi, \rho \in H^1(\Omega) \otimes H^1(\Gamma)$. Then, obviously, we have:

$$B(\varphi, \varphi) \ge 0, \quad \forall\varphi \in H^1(\Omega) \otimes H^1(\Gamma). \tag{3.3}$$

The limit problem (2.9), corresponding to $N = +\infty$ formally reads:

$$\begin{cases} A^{-1}\partial_t\phi = \Delta\phi - f(\phi) + \lambda\phi, \ \phi_{|t=0} = \phi_0, \\ \partial_t\psi = \Delta_\Gamma\psi - g(\psi) - \partial_n\phi, \ x \in \partial\Omega, \ \psi_{|t=0} = \psi_0, \\ \phi_{|\partial\Omega} = \psi. \end{cases} \tag{3.4}$$

Multiplying the first equation of (3.4) by the function $\phi - \varphi$, where $\varphi = \varphi(t, x)$ is smooth, and integrating by parts, we obtain:

$$\begin{aligned}(A^{-1}\partial_t\phi, \phi - \varphi)_\Omega + (\partial_t\phi, \phi - \varphi)_\Gamma + (\nabla\phi, \nabla(\phi - \varphi))_\Omega - \lambda(\phi, \phi - \varphi)_\Omega \\ + (\nabla_\Gamma\phi, \nabla_\Gamma(\phi - \varphi))_\Gamma + (f(\phi), \phi - \varphi)_\Omega + (g(\phi), \phi - \varphi)_\Gamma \;=\; 0,\end{aligned} \tag{3.5}$$

which yields:

$$\begin{aligned} & (A^{-1}\partial_t\phi, \phi - \varphi)_\Omega + (\partial_t\phi, \phi - \varphi)_\Gamma + B(\phi, \phi - \varphi)_\Omega + (f(\phi), \phi - \varphi)_\Omega \\ \leq \; & L(A^{-1}\phi, \phi - \varphi)_\Omega - (g(\phi), \phi - \varphi)_\Gamma,\end{aligned} \tag{3.6}$$

$\forall\varphi \in H^1(\Omega) \otimes H^1(\Gamma)$. Finally, since B is positive and f is monotone, we have:

$$B(\phi, \phi - \varphi) \geq B(\varphi, \phi - \varphi), \quad (f(\phi), \phi - \varphi)_\Omega \geq (f(\varphi), \phi - \varphi)_\Omega. \tag{3.7}$$

Consequently, (3.6) can be written as follow:

$$\begin{aligned} & (A^{-1}\partial_t\phi, \phi - \varphi)_\Omega + (\partial_t\phi, \phi - \varphi)_\Gamma + B(\varphi, \phi - \varphi)_\Omega + (f(\varphi), \phi - \varphi)_\Omega \\ \leq \; & L(A^{-1}\phi, \phi - \varphi)_\Omega - (g(\phi), \phi - \varphi)_\Gamma,\end{aligned} \tag{3.8}$$

$\forall\varphi \in H^1(\Omega) \otimes H^1(\Gamma)$. If we consider the solutions of problem (2.9) with initial data belonging to:

$$\Phi := \{(\phi, \psi) \in L^\infty(\Omega) \times L^\infty(\Gamma),\ \|\phi\|_{L^\infty(\Omega)} \leq 1,\ \|\psi\|_{L^\infty(\Gamma)} \leq 1\} \tag{3.9}$$

and then pass to the limit $N \to \infty$, we will find functions living in Φ for all time. These functions are not necessarily solutions to (2.1) in the usual sense. For this, we define a variational solution of the limit problem (3.4) as follows.

Definition 3.1. Let $(\phi_0, \psi_0) \in \Phi$. We say that $(\phi(t), \psi(t))$ is a variational solution to problem (2.1) originating from (ϕ_0, ψ_0) if

1. $\phi(t)|_\Gamma = \psi(t)$ for almost all $t > 0$,
2. $\phi(0) = \phi_0,\ \psi(0) = \psi_0$,
3. $-1 < \phi(t, x) < 1$ for almost all $(t, x) \in \mathbb{R}^+ \times \Omega$,
4. $(\phi, \psi) \in C([0, +\infty), H^{-1}(\Omega) \times L^2(\Gamma)) \cap L^2([0, T], H^1(\Omega) \times H^1(\Gamma))$, for any $T > 0$,
5. $f(\phi) \in L^1([0, T] \times \Omega)$ for any $T > 0$,
6. $(\partial_t\phi, \partial_t\psi) \in L^2([0, T], H^{-1}(\Omega) \times L^2(\Gamma))$, for any $T > 0$,

and the variational inequality

$$\begin{aligned} & (A^{-1}\partial_t\phi(t), \phi(t) - \varphi)_\Omega + (\partial_t\phi(t), \phi(t) - \varphi)_\Gamma + B(\varphi, \phi(t) - \varphi)_\Omega + (f(\varphi), \phi(t) - \varphi)_\Omega \\ \leq \; & L(A^{-1}\phi(t), \phi(t) - \varphi)_\Omega - (g(\phi(t)), \phi(t) - \varphi)_\Gamma,\end{aligned} \tag{3.10}$$

is satisfied for almost all $t > 0$ and any test function $\varphi \in H^1(\Omega) \otimes H^1(\Gamma)$ such that $f(\varphi) \in L^1(\Omega)$.

We emphasize that we do not assume in the definition that ψ_0 is the trace of ϕ_0. In order to show the uniqueness of a variational solution, we consider (3.10) in terms of test functions $\varphi = \varphi(t,x)$ depending on t and x with φ satisfying the regularity assumptions in Definition 3.1. Then, we write inequality (3.10) with $\varphi = \varphi(t,x)$ for almost all $t > 0$. Moreover, due to the regularity assumptions (3.1) on ϕ and φ, we integrate (3.10) with respect to t since all terms are in L^1. This gives, for all $t > s > 0$:

$$\begin{aligned}&\int_s^t \big((A^{-1}\partial_t\phi, \phi-\varphi)_\Omega + (\partial_t\phi, \phi-\varphi)_\Gamma + B(\varphi, \phi-\varphi)_\Omega + (f(\varphi), \phi-\varphi)_\Omega\big)\, d\tau \\ &\leq \int_s^t \big(L(\phi, A^{-1}(\phi-\varphi))_\Omega - (g(\phi), \phi-\varphi)_\Gamma\big)\, d\tau.\end{aligned} \tag{3.11}$$

Arguing as in Miranville & Zelik [12], we set $\varphi_\alpha := (1-\alpha)\phi + \alpha\varphi$, where $\alpha \in (0,1]$. Then, assumption $(2.2)_4$ implies that the function $|f(\phi)|$ is convex and

$$|f(\varphi_\alpha)| \leq |f(\phi)| + |f(\varphi)|, \tag{3.12}$$

which yields that $f(\varphi_\alpha) \in L^1(\Omega)$. Consequently, φ_α is an admissible test function for (3.11). Inserting $\varphi = \varphi_\alpha$ in the variational inequality (3.11), simplifying by α and using the fact that (ϕ,ψ) is absolutely continuous on $[s,t]$ with values in $H^{-1}(\Omega) \times L^2(\Omega)$, we get:

$$\begin{aligned}&\int_s^t ((A^{-1}\partial_t\phi, \phi-\varphi)_\Omega + (\partial_t\phi, \phi-\varphi)_\Gamma + B(\varphi_\alpha, \phi-\varphi)_\Omega + (f(\varphi_\alpha), \phi-\varphi)_\Omega) d\tau \\ &\leq \int_s^t \big(L(\phi, A^{-1}(\phi-\varphi))_\Omega - (g(\phi), \phi-\varphi)_\Gamma\big)\, d\tau.\end{aligned} \tag{3.13}$$

Passing to the limit in (3.13) as $\alpha \to 0$ and using the Lebesgue dominated convergence theorem for the nonlinear term, we obtain:

$$\begin{aligned}&\int_s^t ((A^{-1}\partial_t\phi, \phi-\varphi)_\Omega + (\partial_t\phi, \phi-\varphi)_\Gamma + B(\phi, \phi-\varphi)_\Omega + (f(\phi), \phi-\varphi)_\Omega) d\tau \\ &\leq \int_s^t \big(L(\phi, A^{-1}(\phi-\varphi))_\Omega - (g(\phi), \phi-\varphi)_\Gamma\big)\, d\tau.\end{aligned} \tag{3.14}$$

We can now state the following theorem which gives the uniqueness of such variational solutions.

Theorem 3.1. *Let the nonlinearity f and g satisfy the assumptions of Section 1. Then, the variational solution of problem* (3.4)*(in the sense of Definition 3.1) is unique and is independent of the choice of L satisfying* (3.1)*. Furthermore, for every two variational solutions (ϕ_1,ψ_1) and (ϕ_2,ψ_2), we have the following estimate:*

$$\begin{aligned}&\|\phi_1(t)-\phi_2(t)\|^2_{H^{-1}(\Omega)} + \|\psi_1(t)-\psi_2(t)\|^2_{L^2(\Gamma)} \\ \leq\ & ce^{Kt}(\|\phi_1(0)-\phi_2(0)\|^2_{H^{-1}(\Omega)} + \|\psi_1(0)-\psi_2(0)\|^2_{L^2(\Gamma)}),\end{aligned} \tag{3.15}$$

where the positive constants c and K are independent of t.

Proof. We use (3.11), with $\phi = \phi_1$ and $\varphi = \phi_2$, and we obtain:

$$\begin{aligned}
&\int_s^t ((A^{-1}\partial_t\phi_1, \phi_1 - \phi_2)_\Omega + (\partial_t\phi_1, \phi_1 - \phi_2)_\Gamma)d\tau \\
&\quad + \int_s^t (B(\phi_2, \phi_1 - \phi_2)_\Omega + (f(\phi_2), \phi_1 - \phi_2)_\Omega)d\tau \\
&\leq \int_s^t \left(L(\phi_1, A^{-1}(\phi_1 - \phi_2))_\Omega - (g(\phi_1), \phi_1 - \phi_2)_\Gamma\right) d\tau,
\end{aligned} \tag{3.16}$$

and (3.14) with $\phi = \phi_2$ and $\varphi = \phi_1$, we find:

$$\begin{aligned}
&\int_s^t ((A^{-1}\partial_t\phi_2, \phi_2 - \phi_1)_\Omega + (\partial_t\phi_2, \phi_2 - \phi_1)_\Gamma)d\tau \\
&\quad + \int_s^t (B(\phi_2, \phi_2 - \phi_1)_\Omega + (f(\phi_2), \phi_2 - \phi_1)_\Omega)d\tau \\
&\leq \int_s^t \left(L(\phi_2, A^{-1}(\phi_2 - \phi_1))_\Omega - (g(\phi_2), \phi_2 - \phi_1)_\Gamma\right) d\tau.
\end{aligned} \tag{3.17}$$

Summing the two resulting inequalities (3.16) and (3.17) and using the fact that (ϕ_i, ψ_i) are absolutely continuous on $[s,t]$, $i = 1, 2$, with values in $H^{-1}(\Omega) \times L^2(\Gamma)$, we obtain:

$$\begin{aligned}
&\frac{1}{2}(\|(\phi_1(t), \psi_1(t)) - (\phi_2(t), \psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)} \\
&- \|(\phi_1(s), \psi_1(s)) - (\phi_2(s), \psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)}) \\
\leq &\int_s^t (L\|\phi_1(\tau) - \phi_2(\tau)\|^2_{H^{-1}(\Omega)} - (g(\phi_1(\tau)) - g(\phi_2(\tau)), \phi_1(\tau) - \phi_2(\tau)))_\Gamma d\tau.
\end{aligned} \tag{3.18}$$

Using the fact that g is bounded globally and applying the Gronwall inequality to (3.18), we have:

$$\begin{aligned}
&\|(\phi_1(t), \psi_1(t)) - (\phi_2(t), \psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)} \\
\leq\ & ce^{Kt}\|(\phi_1(s), \psi_1(s)) - (\phi_2(s), \psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)},
\end{aligned} \tag{3.19}$$

where the positive constants c and K are independent of $t > s > 0$ and (ϕ_i, ψ_i), $i = 1, 2$. Passing to the limit as $s \to 0$ and thanks to the continuity of (ϕ_i, ψ_i), $i = 1, 2$ from Definition 3.1, condition 4, we get the desired estimate, which in particular gives the uniqueness.

Now, we need to prove that the above definition of a solution is independent of the choice of L. To do so, we assume that (ϕ_1, ψ_1) is a variational solution for $L = L_1$ and (ϕ_2, ψ_2) is a variational solution for $L = L_2$. Using the following relation:

$$\begin{aligned}
&B_{L_1}(\phi_2, \phi_1 - \phi_2) - B_{L_2}(\phi_2, \phi_1 - \phi_2) \\
=\ & L_1(\phi_1, A^{-1}(\phi_1 - \phi_2)) - L_2(\phi_1, A^{-1}(\phi_1 - \phi_2)) - (L_1 - L_2)\|\phi_1 - \phi_2\|^2_{H^{-1}(\Omega)},
\end{aligned} \tag{3.20}$$

and arguing as in the proof of (3.15), we find:

$$\begin{aligned} &\int_s^t ((A^{-1}(\partial_t\phi_1 - \partial_t\phi_2), \phi_1 - \phi_2)_\Omega + (\partial_t\phi_1 - \partial_t\phi_2, \phi_1 - \phi_2)_\Gamma) d\tau \\ &+ \int_s^t (B_{L_1}(\phi_2, \phi_1 - \phi_2)_\Omega - B_{L_2}(\phi_2, \phi_1 - \phi_2)_\Omega)\, d\tau \\ \leq\ & L_1 \int_s^t \big(\phi_1, A^{-1}(\phi_1 - \phi_2)\big)_\Omega\, d\tau + L_2 \int_s^t \big(\phi_2, A^{-1}(\phi_2 - \phi_1)\big)_\Omega\, d\tau \\ &- \int_s^t (g(\phi_1(\tau)) - g(\phi_2(\tau)), \phi_1(\tau) - \phi_2(\tau))_\Gamma\, d\tau. \end{aligned} \tag{3.21}$$

After simplification, (3.21) gives:

$$\begin{aligned} &\frac{1}{2}(\|(\phi_1(t), \psi_1(t)) - (\phi_2(t), \psi_2(t))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)} \\ &- \|(\phi_1(s), \psi_1(s)) - (\phi_2(s), \psi_2(s))\|^2_{H^{-1}(\Omega)\times L^2(\Gamma)}) \\ \leq & \int_s^t (L_1\|\phi_1(\tau) - \phi_2(\tau)\|^2_{H^{-1}(\Omega)} - (g(\phi_1(\tau)) - g(\phi_2(\tau)), \phi_1(\tau) - \phi_2(\tau)))_\Gamma d\tau. \end{aligned} \tag{3.22}$$

which coincides with (3.18) and also leads to (3.15). Theorem 3.1 is thus proven. □

Theorem 3.2. *For every initial data $(\phi_0, \psi_0) \in \Phi$, problem* (3.4) *possesses a unique variational solution (ϕ, ψ) in the sense of Definition 3.1. Such a solution regularizes as $t > 0$ and all the uniform estimates obtained above hold. In particular, the following estimate is valid for every $\delta > 0$ and $t > 0$:*

$$\begin{aligned} &\|\phi(t)\|^2_{C^\alpha(\Omega)} + \|\phi(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|\phi(t)\|^2_{H^1(\Omega)} + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)} \\ &+ \|\partial_t\psi(t)\|^2_{L^2(\Gamma)} + \|\nabla D_\tau\phi(t)\|^2_{L^2(\Omega)} + \|f(\phi(t))\|_{L^1(\Omega)} \\ &+ \int_t^{t+1} (\|\partial_t\phi(s)\|^2_{H^1(\Omega)} + \|\partial_t\psi(s)\|^2_{H^1(\Gamma)}) \mathrm{d}s \\ \leq\ & C(1 + \|\phi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Gamma)}) e^{-\gamma t}, \end{aligned} \tag{3.23}$$

for some positive constants α and C which are independent of t and ϕ, where D_τ denotes the tangential part of the gradient ∇.

Proof. Repeating the derivation of the variational inequality (3.11), we obtain that (ϕ_N, ψ_N) satisfies:

$$\begin{aligned} &\int_s^t (A^{-1}\partial_t\phi_N, \phi_N - \varphi)_\Omega + (\partial_t\phi_N, \phi_N - \varphi)_\Gamma + B(\varphi, \phi_N - \varphi)_\Omega + (f(\varphi), \phi_N - \varphi)_\Omega d\tau \\ \leq & \int_s^t \big(L(\phi_N, A^{-1}(\phi_N - \varphi))_\Omega - (g(\phi_N), \phi_N - \varphi)_\Gamma\big)\, d\tau, \end{aligned} \tag{3.24}$$

for every admissible test function φ and every $t > s > 0$. Our aim is to pass to the limit $N \to +\infty$. We start with the case when the initial datum ϕ_0 is smooth and satisfies the additional conditions:

$$|\phi_0(x)| \leq 1-\delta,\ \delta > 0,\ \psi := u_0|_\Gamma. \tag{3.25}$$

Then, by (2.48), we have:

$$\begin{aligned}&\|\phi_N(t)\|_{L^\infty([0,T],H^1(\Omega))} + \|\phi_N(t)\|_{L^\infty([0,T],H^2(\Omega_\delta))} \\ &+ \|\psi_N(t)\|_{L^\infty([0,T],H^2(\Gamma))} \ \leq \ C,\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\|\partial_t\phi_N(t)\|_{L^\infty([0,T],H^{-1}(\Omega))} + \|\partial_t\psi_N(t)\|_{L^\infty([0,T],L^2(\Gamma))} \\ &+ \|\partial_t\phi_N(t)\|_{L^2([0,T],H^1(\Omega))} + \|\partial_t\psi_N(t)\|_{L^2([0,T],H^1(\Gamma))} \\ &\leq C,\end{aligned} \tag{3.27}$$

$$\|D^2_\tau\phi_N(t)\|_{L^\infty([0,T],L^2(\Omega))} \leq C, \tag{3.28}$$

$$\|\phi_N(t)\|_{L^\infty([0,T],C^\alpha(\Omega))} \leq C, \tag{3.29}$$

where the positive constant C depends on Ω, Γ, ϕ_0, ψ_0 and T but is independent of N and t. From this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to the sequence ϕ_N entail that there exists a function ϕ such that as $N \to \infty$, the following properties hold:

$$\phi_N \to \phi \text{ weakly star in } L^\infty\left([0,T],(H^1(\Omega)\otimes H^2(\Gamma))\cap H^2(\Omega_\delta)\right), \tag{3.30}$$

$$(\partial_t\phi_N,\partial_t\psi_N) \to (\phi,\psi) \text{ weakly star in } L^\infty\left([0,T],(H^{-1}(\Omega)\times L^2(\Gamma))\right), \tag{3.31}$$

$$(\partial_t\phi_N,\partial_t\psi_N) \to (\phi,\psi) \text{ weakly in } L^2\left([0,T],(H^1(\Omega)\otimes H^1(\Gamma))\right), \tag{3.32}$$

$$D^2_\tau\phi_N \to D^2_\tau\phi \text{ weakly star in } L^\infty([0,T],L^2(\Omega)). \tag{3.33}$$

It follows from (3.27) and (3.29), using the compactness theorem of Aubin-Lions, that:

$$\phi_N \to \phi \text{ strongly in } C^\gamma([0,T]\times\Omega) \text{ for some } \gamma > 0. \tag{3.34}$$

These convergence results allow us to pass to the limit $N \to +\infty$ in (3.24) and prove that the limit function satisfies (3.11) for any admissible test function φ. The only nontrivial term containing the nonlinearity f_N can be treated by using the inequality $|f_N(\varphi)| \leq |f(\varphi)|$, the fact that $f(\varphi) \in L^1([0,T]\times\Omega)$ and the Lebesgue dominated convergence theorem. The crucial point $-1 < \phi(t,x) < 1$, for almost all $(t,x) \in \mathbb{R}\times\Omega$, can be proven as in Miranville & Zelik [12]. Indeed, taking into account the definition of f_N and the fact that the L^1-norm of $f_N(\phi_N)$ is uniformly bounded, we can conclude:

$$\text{meas}\,\{(t,x)\in[T,T+1]\times\Omega,\ |\phi_M(t,x)| > 1+1/N\} \leq \pi(1/N),\ M \geq N, \tag{3.35}$$

where

$$\pi(x) := \frac{C}{\max\{|f(1-x)|,|f(x-1)|\}}, \tag{3.36}$$

for some positive constant C which is independent of $T \in \mathbb{R}^+$, of N and M, with $M \geq N$. Using the fact that $\pi(x) \to 0$ as $x \to 0$ and passing to the limit M, $N \to +\infty$ in (3.35), we conclude that:

$$\text{meas}\,\{(t,x)\in[T,T+1]\times\Omega,\ |\phi(t,x)| = 1\} = 0, \tag{3.37}$$

so that

$$|\phi(t,x)| < 1 \text{ for almost all } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{3.38}$$

Inequality (4.12) and the convergence $\phi_N \to \phi$ strongly in $C^\gamma([0,T]\times\Omega)$ imply the almost everywhere convergence $f_N(\phi_N) \to f(\phi)$. Therefore, Fatou's lemma gives:

$$\|f(\phi)\|_{L^1([0,T]\times\Omega)} \leq \liminf_{N\to+\infty} \|f_N(\phi_N)\|_{L^1([0,T]\times\Omega)} < +\infty. \tag{3.39}$$

Thus, $f(\phi) \in L^1([0,T]\times\Omega)$ and (ϕ,ψ) is a variational solution to problem (3.4). In particular, the L^1-estimate on $f(\phi)$ follows from (3.39). Since the separation from singularities is not ensured on the boundary, we are not allowed to pass to the limit in $\|F_N(\phi_N(t))\|_{L^1(\Gamma)}$.

Finally, we remove assumption (3.25). In that case, we approximate the initial datum $(\phi_0,\psi_0) \in \Phi$ by a sequence (ϕ_0^k,ψ_0^k) of smooth functions satisfying (3.25) such that:

$$\|\phi_0 - \phi_0^k\|_{L^2(\Omega)} \to 0,\ \|\psi_0 - \psi_0^k\|_{L^2(\Gamma)} \to 0, \text{ as } k \to +\infty. \tag{3.40}$$

Let $(\phi_k(t),\psi_k(t))$ be a sequence of variational solutions of problem (3.4) satisfying
$(\phi_k(0),\psi_k(0)) = (\phi_0^k,\psi_0^k)$, where $\phi_k|_\Gamma = \psi_k$. The existence of such a sequence of solutions was proved above. Then, by estimate (3.15) and assumption (3.40), we can see that (ϕ_k,ψ_k) is a Cauchy sequence in $C([0,T],H^{-1}(\Omega)\times L^2(\Gamma))$ and therefore, the limit function exists and

$$(\phi,\psi) := \lim_{k\to+\infty}(\phi_k,\psi_k) \in C([0,T],H^{-1}(\Omega)\times L^2(\Gamma)).$$

Then, the proof of the theorem is finished as above. □

We also have the following result:

Lemma 3.1. *Let $(\phi(t),\psi(t))$ be a variational solution of problem (3.4). Then, $\psi(t) = \phi(t)|_\Gamma$ for $t > 0$. Moreover, this solution solves (3.4) in the usual sense, that is, for any $\varphi \in C_0^\infty((0,T)\times\Omega)$, the following equation holds:*

$$\begin{aligned}\int_{\mathbb{R}^+}(A^{-1}\partial_t\phi(t),\varphi(t))_\Omega \mathrm{d}t = &\int_{\mathbb{R}^+}((\Delta\phi(t),\varphi(t))_\Omega - (f(\phi(t),\varphi(t))_\Omega)\,\mathrm{d}t \\ &+ \int_{\mathbb{R}^+}\lambda(\phi(t),\varphi(t))_\Omega \mathrm{d}t.\end{aligned} \tag{3.41}$$

Furthermore,

$$\phi \in L^\infty([\tau,T],W^{2,1}(\Omega)),\ 0 < \tau < T, \tag{3.42}$$

so that the trace of the normal derivative on the boundary,

$$[\partial_n\phi]_{int} := \partial_n\phi|_\Gamma \in L^\infty([\tau,T],L^1(\Gamma)),\ 0 < \tau < T, \tag{3.43}$$

exists.

Proof. Since ϕ_N is uniformly bounded in $L^\infty([\tau,T],H^2(\Omega_\delta))$, $\forall\delta > 0$, and F_N is uniformly continuous, the sequence $f_N(\phi_N)$ is also uniformly bounded in
$L^\infty([\tau,T],H^2(\Omega_\delta))$. Using this fact and that $f_N(\phi_N) \to f(\phi)$ a.e., we obtain using a weak version of the dominated convergence theorem that $f_N(\phi_N) \to f(\phi)$ weakly

in $L^2([\tau,T],H^2(\Omega_\delta))$. Thus, we are allowed to pass to the limit in the equation corresponding to (3.41) for ϕ_N. We deduce from (3.41), that ϕ is a solution for:

$$A^{-1}\partial_t\phi(t)-\Delta\phi(t)+f(\phi(t))-\lambda\phi(t)=0,\ \text{ in } L^2_{loc}((\tau,T)\times\Omega_\delta). \tag{3.44}$$

Moreover, since $f(\phi)$ and $A^{-1}\partial_t\phi$ belong to $L^\infty((\tau,T),L^1(\Omega))$, we find that

$$\Delta\phi\in L^\infty((\tau,T),L^1(\Omega)).$$

Having the control of $\nabla D_\tau\phi$, we deduce that $\Delta\phi\in L^\infty((\tau,T),W^{2,1}(\Omega))$, which yields the existence of the trace (3.43). □

Concerning the second equation from (3.4), we use Theorem 2.1 and we see that:

$$\|\partial_t\psi_N\|_{L^\infty([\tau,T],L^2(\Gamma))}+\|\psi_N\|_{L^\infty([\tau,T],H^2(\Gamma))}\le c, \tag{3.45}$$

where the constant c is independent of N. We have:

$$\partial_n\phi_N=\partial_t\psi_N-\Delta_\Gamma\psi_N+g(\psi_N). \tag{3.46}$$

Using (3.45), we deduce that $\partial_N\phi_N\in L^\infty([\tau,T],L^2(\Gamma))$. Passing to the limit as $N\to+\infty$, we have the weak-star convergence in $L^\infty([\tau,T],L^2(\Gamma))$

$$[\partial_n\phi]_{ext}:=\lim_{N\to+\infty}\partial_n\phi_N|_\Gamma\in L^\infty([\tau,T],L^2(\Gamma)),\ T>\tau>0, \tag{3.47}$$

and

$$\partial_t\psi-\Delta_\Gamma\psi+g(\psi)+[\partial_n\phi]_{ext}=0,\ \text{ on }\Gamma,\ T>\tau>0. \tag{3.48}$$

In order to verify that the variational solution (ϕ,ψ) satisfies equations (3.4) in the usual sense, there only remains to check that:

$$[\partial_n\phi]_{int}=[\partial_n\phi]_{ext},\ \text{ for almost every } (t,x)\in\mathbb{R}^+\times\Gamma. \tag{3.49}$$

4. Additional regularity results and separation from the singularities

In this section, we formulate several sufficient conditions which ensure that every variational solution satisfies equation (3.4) in the usual sense. We have the following result which gives an additional regularity on ϕ close to the points where $|\phi(t,x)|<1$.

Proposition 4.1. *Let the assumptions of Theorem 3.1 hold and let (ϕ,ψ) be a variational solution to* (3.4)*. For any δ, $T>0$, we set:*

$$\Omega_\delta(T)=\{x\in\Omega,|\phi(T,x)|<1-\delta\}.$$

Then, $\phi\in H^2(\Omega_\delta(T))$ and the following estimate holds:

$$\|\phi\|_{H^2(\Omega_\delta(T))}\le Q_{\delta,T}, \tag{4.1}$$

where the positive constant $Q_{\delta,T}$ depends on T and δ but is independent of the concrete choice of the solution ϕ.

Proof. Since the solution $\phi(T,x)$ is Hölder continuous with respect to x, there exists a smooth nonnegative cut-off function $\theta(x)$ such that:

$$\begin{cases} \theta(x) \equiv 1,\ x \in \Omega_\delta(T), \\ \theta(x) \equiv 0,\ x \in \Omega \backslash \Omega_{\delta/2}(T), \\ \|\theta\|_{C^2(\mathbb{R}^3(\Omega))} \leq K_{\delta,T}, \end{cases} \tag{4.2}$$

where $K_{\delta,T}$ is independent of the concrete choice of the solution ϕ. Let $\phi_N(t,x)$ be a sequence of approximate solutions of problem (2.1) which converges to the variational solution $\phi(t,x)$ as $N \to +\infty$. Then, since the convergence holds in the space $C^\gamma([0,T] \times \Omega)$ for some $\gamma > 0$, we have:

$$|\phi_N(T,x)| < 1 - \delta/4,\ x \in \Omega_{\delta/2}(T), \tag{4.3}$$

for N large enough. Setting $v_N := \theta(x)\phi_N(T,x)$ and $w_N := \theta(x)\psi_N(T,x)$, we have:

$$\begin{cases} \Delta_x \phi_N - f_N(\phi_N(T)) - \phi_N = h_1(T) := -\phi_N + \lambda\phi_N + A^{-1}\partial_t \phi_N, \\ \Delta_\Gamma \psi_N - \psi_N - \partial_n \phi_N = h_2(T) := g_0(\psi_N(T)) + \partial_t \psi_N. \end{cases} \tag{4.4}$$

Multiplying the first equation of (4.4) by θ, we find:

$$\begin{aligned} &\theta\Delta_x\phi_N(T) - \theta\phi_N(T) = \theta h_1(T) + \theta f_N(\phi_N(T)) \\ \Longleftrightarrow &\Delta_x(\theta\phi_N(T)) - \theta\phi_N(T) = \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta \cdot \nabla_x\phi_N(T) \\ &\qquad + \phi_N(T)\Delta_x\theta \\ \Longleftrightarrow &\Delta_x v_N - v_N = \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta \cdot \nabla_x\phi_N(T) + \phi_N(T)\Delta_x\theta. \end{aligned} \tag{4.5}$$

Now, multiplying the second equation of (4.4) by θ, we obtain:

$$\begin{aligned} &\theta\Delta_\Gamma\psi_N(T) - \theta\psi_N(T) - \theta\partial_n\phi_N(T) = \theta h_2(T) \\ \Longleftrightarrow &\Delta_\Gamma w_N - w_N - \partial_n w_N = \theta h_2(T) + 2\nabla_\Gamma\theta \cdot \nabla_\Gamma\psi_N(T) + \psi_N(T)\Delta_\Gamma\theta \\ &\qquad - \psi_N(T)\partial_n\theta. \end{aligned} \tag{4.6}$$

Thus, φ_N satisfies the following elliptic boundary value problem:

$$\begin{cases} \Delta_x v_N - v_N = \tilde{h_1}(\phi_N) := \theta h_1(T) + \theta f_N(\phi_N(T)) + 2\nabla_x\theta \cdot \nabla_x\phi_N(T) + \phi_N(T)\Delta_x\theta, \\ \Delta_\Gamma w_N - w_N - \partial_n w_N = \tilde{h_2}(\psi_N) := \theta h_2(T) + 2\nabla_\Gamma\theta \cdot \nabla_\Gamma\psi_N(T) + \psi_N(T)\Delta_\Gamma\theta \\ \qquad - \psi_N(T)\partial_n\theta. \end{cases} \tag{4.7}$$

Using the estimates (2.33), (2.50), (4.2) and (4.3), we find:

$$\|\tilde{h_1}(\phi_N)\|_{L^2(\Omega)} + \|\tilde{h_2}(\psi_N)\|_{L^2(\Gamma)} \leq Q_{\delta,T}, \tag{4.8}$$

where the positive constant $Q_{\delta,T}$ is independent of N and of the concrete choice of the solution ϕ. Applying an H^2-regularity result to problem (4.7) (see [11]), we deduce that:

$$\|\phi_N(T)\|_{H^2(\Omega_\delta(T))} \leq Q_{\delta,T}. \tag{4.9}$$

By passing to the limit as $N \to +\infty$, we deduce that $\phi(T) \in H^2(\Omega_\delta(T))$ and that $\|\phi(T)\|_{H^2(\Omega_\delta(T))} \leq Q_{\delta,T}$. □

Lemma 4.1. *Let (ϕ, ψ) a variational solution to* (3.4). *Assume, in addition, that we have:*

$$|\phi(t_0, x_0)| < 1, \tag{4.10}$$

for some $(t_0, x_0) \in (0, +\infty) \times \Gamma$. Then, there exists a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon) \times V$ of (t_0, x_0) in $\mathbb{R} \times \Gamma$ such that:

$$[\partial_n \phi]_{int}(t, x) = [\partial_n \phi]_{ext}(t, x),\ \forall (t, x) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times V. \tag{4.11}$$

In particular, if ϕ satisfies:

$$|\phi(t, x)| < 1 \text{ for almost all } (t, x) \in \mathbb{R}^+ \times \Gamma, \tag{4.12}$$

then, the equality $[\partial_n \phi]_{int} = [\partial_n \phi]_{ext}$ holds almost everywhere in $(0, +\infty) \times \Gamma$ and (ϕ, ψ) solves (3.4) *in the usual sense.*

Proof. We know that ϕ is Hölder continuous with respect to t and x. Thus there exists $\varepsilon > 0$ such that $|\phi(t, x)| \leq 1 - \varepsilon$ holds for all (t, x) in a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon) \times V_\varepsilon$ of (t_0, x_0) in $(0, +\infty) \times \Omega$. Thanks to Proposition 4.1, the approximate solution ϕ_N (converging to ϕ) satisfies:

$$\|\phi_N\|_{L^\infty([t_0 - \varepsilon, t_0 + \varepsilon], H^2(\Omega_\varepsilon))} \leq C,$$

where the positive constant C is independent of N. Then, we can assume that $\phi_N \to \phi$ weakly-star in this space, which yields that $\partial_n \phi_N|_\Gamma \to \partial_n \phi|_\Gamma$ weakly in $L^2([t_0 - \varepsilon, t_0 + \varepsilon] \times V)$ for some proper neighborhood V of x_0. This convergence result, together with the definition (3.47), leads to equality (4.11) and relation (4.12) is a consequence of (4.11). □

Thus, in order to prove that any variational solution ϕ is a solution in the usual sense, it is sufficient to verify that ϕ satisfies (4.12).

Corollary 4.1. *Let the assumptions of Theorem 3.1 hold. We assume that:*

$$\lim_{s \to \pm 1} F(s) = +\infty. \tag{4.13}$$

Then, for every variational solution ϕ of problem (3.4), *relation* (4.12) *holds and the potential F verifies:*

$$F(\phi(t)) \in L^1(\Gamma) \text{ and } \|F(\phi(t))\|_{L^1(\Gamma)} \leq C_T, \tag{4.14}$$

for almost all $t \geq T > 0$.

Proof. Let ϕ_N be a sequence of approximate solutions converging to the variational solution ϕ. Applying estimate (6.4) in Miranville & Zelik [12], we obtain:

$$\|F_N(\phi_N)\|_{L^1(\Gamma)} \leq C_T,\ t \geq T, \tag{4.15}$$

where the constant C_T is independent of N. Since $\lim_{s \to \pm 1} F(s) = +\infty$, we deduce that $f(1 - 1/N) \to +\infty$ and $f(1/N - 1) \to -\infty$, as $N \to +\infty$, which yields:

$$\|f_N(\phi_N)\|_{L^1(\Gamma)} \leq C_T,\ t \geq T. \tag{4.16}$$

Arguing as in the proof of Theorem 3.2, we obtain that:

$$\text{meas } \{(t, x) \in [T, T+1] \times \Gamma,\ |\phi_M(t, x)| \geq 1 - 1/N\} \leq \pi(1/N), \tag{4.17}$$

where

$$\pi(x) = \frac{C}{\max\{|f(1-x)|, |f(x-1)|\}}.$$

Passing to the limit $M, N \to +\infty$, we deduce that:

$$\text{meas}\ \{(t,x) \in [T, T+1] \times \Gamma,\ |\phi(t,x)| = 1\} = 0. \tag{4.18}$$

Thus, condition (4.12) holds.

Then, using the convergence $\phi_N \to \phi$ in $C^\gamma([0,T] \times \Omega)$, with $\gamma > 0$ and the already proved statement 4.12, we conclude that $F_N(\phi_N) \to F(\phi)$ almost everywhere in $\mathbb{R}^+ \times \Gamma$ and therefore, thanks to the Fatou lemma,

$$\|F(\phi)\|_{L^1(\Gamma)} \le \liminf_{N \to +\infty} \|F_N(\phi_N)\|_{L^1(\Gamma)} \le C_T. \tag{4.19}$$

□

Corollary 4.2. *Let the assumptions of Theorem 3.1 hold. We assume that:*

$$g(-1) + \varepsilon \le 0 \le g(1) - \varepsilon, \tag{4.20}$$

for some $\varepsilon > 0$. *Then, for every variational solution* ϕ *of problem* (3.4)*, estimate* (4.12) *holds and*

$$\|f(\phi)\|_{L^1([t,t+1]\times\Gamma)} \le C_{\varepsilon,T},\ t \ge T > 0, \tag{4.21}$$

where the constant $C_{\varepsilon,T}$ *is independent of the concrete choice of the variational solution* ϕ.

Proof. We consider the nonlinear elliptic-parabolic system:

$$\begin{cases} \Delta\phi_N(t) - f_N(\phi_N(t)) - \phi_N(t) = h_1(t), \\ \phi_N|\Gamma = \psi_N, \\ \partial_t\psi_N(t) - \Delta_\Gamma\psi_N(t) + \partial_n\phi_N(t) + g(\psi_N(t)) = 0. \end{cases} \tag{4.22}$$

Arguing as in Miranville & Zelik [12], we have:

$$g(s) \cdot f_N(s) \ge \frac{\varepsilon}{2}|f_N(s)| + C_\varepsilon,\ s \in \mathbb{R}, \tag{4.23}$$

where the constant C_ε depends on g and ε but is independent of N. Arguing as in Corollary 4.1, we deduce estimate (4.12). To derive (4.21), we multiply (4.22) by $f_N(\phi_N)$ and we use (4.23). We obtain:

$$\begin{aligned} &\frac{d}{dt}\int_\Gamma F_N(\phi_N(t))\mathrm{d}s + (f_N'(\phi_N(t))\nabla\phi_N(t), \nabla\phi_N(t))_\Omega \\ &+ (f_N'(\phi_N(t))\nabla_\Gamma\phi_N(t), \nabla_\Gamma\phi_N(t))_\Gamma \\ &+ 1/2\|f_N(\phi_N(t))\|^2_{L^2(\Omega)} + \varepsilon/2\|f_N(\phi_N(t))\|_{L^1(\Gamma)} \\ \le\ & C_\varepsilon(1 + \|h_1(t)\|^2_{L^2(\Omega)}). \end{aligned} \tag{4.24}$$

The L^2-norm of $h_1(t)$ is controlled thanks to (2.32) and we find:

$$\begin{aligned} \|h_1(t)\|^2_{L^2(\Omega)} &\le c(1 + \|\partial_t\phi(t)\|^2_{H^{-1}(\Omega)}) \\ &\le c(1 + \|\partial_t\phi(0)\|^2_{H^{-1}(\Omega)} + \|\partial_t\psi(0)\|^2_{L^2(\Gamma)}) \\ &< +\infty. \end{aligned} \tag{4.25}$$

Integrating (4.24) with respect to t and using the fact that $f_N' \geq 0$, we find:

$$\begin{aligned}&\int_\Gamma F_N(\phi_N(t+1))d\sigma + \int_t^{t+1}(\|\nabla\phi_N(s)\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\phi_N(s)\|^2_{L^2(\Gamma)}\mathrm{d}s\\ &+\int_t^{t+1}(\|f_N(\phi_N(s))\|^2_{L^2(\Omega)}) + \varepsilon/2\|f_N(\phi_N(s))\|_{L^1(\Gamma)})\mathrm{d}s \\ \leq\ & C_{\varepsilon,T} + \int_\Gamma F_N(\phi_N(t))d\sigma.\end{aligned} \tag{4.26}$$

We deduce from (4.26):

$$\begin{aligned}\|f_N(\phi_N)\|_{L^1([t,t+1]\times\Gamma)} &\leq 2/\varepsilon(\|F_N(\phi_N(t))\|_{L^1(\Gamma)} + \|F_N(\phi_N(t+1))\|_{L^1(\Gamma)}) + C_{\varepsilon,T}\\ &\leq C'_{\varepsilon,T}.\end{aligned} \tag{4.27}$$

Arguing as in Corollary 4.1, we finish the proof. □

5. Attractors and exponential attractors

In this section, we study the asymptotic behavior of the system. We denote by $\Phi^w := H^{-1}(\Omega)\times L^2(\Gamma)$. The space Φ^w is endowed with the natural norm:

$$\|\varphi\|^2_{\Phi^w} = \|\varphi\|^2_{H^{-1}(\Omega)} + \|\varphi\|^2_{L^2(\Gamma)},\ \text{for all } \varphi\in\Phi^w. \tag{5.1}$$

We have the following result:

Corollary 5.1. *Under the assumptions of Theorem 3.2, equation* (3.4) *generates a solution semigroup* $S(t):\Phi^w\to\Phi^w$*, where* $S(t)(\phi_0,\psi_0) := (\phi(t),\psi(t))$ *is the unique variational solution of problem* (3.4) *departing from* (ϕ_0,ψ_0)*. Furthermore, we have the following Lipschitz continuity property:*

$$\begin{aligned}&\|S(t)(\phi_0^1,\psi_0^1) - S(t)(\phi_0^2,\psi_0^2)\|^2_{\Phi^w}\\ &+\int_t^{t+1}\|S(s)(\phi_0^1,\psi_0^1) - S(s)(\phi_0^2,\psi_0^2)\|^2_{H^1(\Omega)\times H^1(\Gamma)}\mathrm{d}s\\ \leq\ & Ce^{Kt}\|(\phi_0^1-\phi_0^2,\psi_0^1-\psi_0^2))\|^2_{\Phi^w},\end{aligned} \tag{5.2}$$

for all $(\phi_0^1,\psi_0^1),(\phi_0^2,\psi_0^2)\in\Phi^w$.

This corollary is a direct consequence of Proposition 2.1.

The following proposition gives the existence of the global attractor $\mathcal{A}$ for this semigroup. We recall that, by definition, a set $\mathcal{A}\subset H^{-1}(\Omega)$ is the global attractor for the semigroup $S(t)$ if the following properties are satisfied:

1. It is a compact subset of $H^{-1}(\Omega)$;
2. It is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$, $\forall t>0$;
3. It attracts all bounded sets in $H^{-1}(\Omega)$ as $t\to\infty$, i.e., for every bounded set $X\subset H^{-1}(\Omega)$ there exists a neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in $H^{-1}(\Omega)$ and a time $T=T(\mathcal{O})$ such that:
$$S(t)X\subset\mathcal{O}(\mathcal{A}),\ \ t\geq T.$$

Proposition 5.1. *The semigroup $S(t)$ associated with the variational solutions of problem* (3.4) *possesses the global attractor $\mathcal{A}$ which is bounded in the space $C^{\alpha}(\Omega)\times C^{\alpha}(\Gamma)$ for some positive constant $\alpha < 1/4$.*

Proof. The semigroup $S(t)$ is dissipative. Indeed, thanks to estimate (2.27) there exists $R_0 > 0$ such that the ball $B_{\mathcal{H}_1}(R_0)$ centered on zero with radius R_0 in $H^1(\Omega)\times H^1(\Gamma)$ is absorbing in Φ^w and compact in the topology of Φ^w. In particular, there exists a time $t_0 \geq 1$ such that $S(t)B_{\mathcal{H}_1}(R_0) \subset B_{\mathcal{H}_1}(R_0)$, for any $t \geq t_0$. As a consequence, the set:

$$\mathbb{B}_0 := \overline{\cup_{t\geq t_0} S(t)B_{\mathcal{H}_1}(R_0)}^{\Phi^w} \tag{5.3}$$

is absorbing and positively invariant. Thus the existence of the global attractor $\mathcal{A}$ follows from a proper abstract attractor's existence theorem (see Temam [14]). □

In the following theorem, we prove the existence of an exponential attractor which by definition contains the global attractor and has finite fractal dimension. To do this, we first recall the definition of the exponential attractor where $\mathcal{A}$ is the global attractor for the semigroup $\{S(t)\}_{t\geq 0}$:

Definition 5.1. Let X be a compact connected subset of a Banach space E. A compact set $\mathcal{M}$ is called an exponential attractor for the semigroup $\{S(t)\}_{t\geq 0}$ if $\mathcal{A} \subset \mathcal{M} \subset X$ and

1. $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$.
2. $\mathcal{M}$ has finite fractal dimension, $d_F(\mathcal{M}) < \infty$.
3. There exist positive constants c_0 and c_1 such that for every $u_0 \in X$, we have:

$$dist_E(S(t)u_0, \mathcal{M}) \leq c_0 e^{-c_1 t},\ \forall t \geq 0, \tag{5.4}$$

   where the pseudo-distance *dist* is the standard Hausdorff pseudo-distance between two sets, defined by $dist_E(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_E$.

Theorem 5.1. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M}$ which is bounded in $C^{\alpha}(\Omega)\times C^{\alpha}(\Gamma)$, $\alpha < 1/4$.*

Proof. There exists a positive constant $R = R(R_0)$ such that:

$$\begin{aligned}
&\|\phi(t)\|_{C^{\alpha}([t,t+1]\times\Omega)} + \|\phi(t)\|_{H^2(\Gamma)} + \|\partial_t\phi(t)\|_{H^{-1}(\Omega)} + \|\partial_t\phi(t)\|_{L^2(\Gamma)} \\
&+ \|f(\phi(t))\|_{L^1(\Omega)} + \|\partial_t\phi(t)\|_{L^2([t,t+1],H^1(\Omega))} + \|\partial_t\phi(t)\|_{L^2([t,t+1],H^1(\Gamma))} \\
&\leq R,
\end{aligned} \tag{5.5}$$

for any initial datum in $\mathbb{B}_0$ where $\mathbb{B}_0$ is the set defined by (5.3). In particular, for every point $(\phi,\psi) \in \mathbb{B}_0$, there holds $\phi|_{\Gamma} = \psi$. We consider an arbitrary small $\varepsilon-$ball $B(\varepsilon,\phi_0;\Phi^w)$ in the space $\mathbb{B}_0$ and centered on ϕ_0, where $0 < \varepsilon \leq \varepsilon_0 \ll 1$, with the parameter ε_0 that will be fixed below. Let also $\phi^0(t)$, $t \geq 0$, be the solution starting from ϕ_0. We introduce the sets:

$$\Omega_{\delta}(\phi_0) := \{x \in \Omega,\ |\phi_0(x)| \leq 1-\delta\},$$

$$\overline{\Omega}_{\delta}(\phi_0) := \{x \in \Omega,\ |\phi_0(x)| > 1-\delta\},$$

where δ is a sufficiently small positive number. Then, thanks to the Hölder continuity of ϕ_0 with respect to x, we have for all $x_1 \in \partial\Omega_{\delta_1}(\phi_0)$, $x_2 \in \partial\Omega_{\delta_2}(\phi_0)$:

$$0 < |\delta_1 - \delta_2| \leq |\phi_0(x_1) - \phi_0(x_2)| \leq c|x_1 - x_2|^{\alpha}. \tag{5.6}$$

Thus, there is a strict separation between $\partial\Omega_{\delta_1}(\phi_0)$ and $\partial\Omega_{\delta_2}(\phi_0)$ for any $\delta_1 \neq \delta_2$, i.e.

$$d(\partial\Omega_{\delta_1}(\phi_0), \partial\Omega_{\delta_2}(\phi_0)) \geq C_{\delta_1,\delta_2} > 0,\ \delta_1 \neq \delta_2, \tag{5.7}$$

where the constant C_{δ_1,δ_2} depends on $\delta_1,\ \delta_2$.

We note that, since $\phi^0(t)$ is uniformly Hölder continuous with respect to t and x, there exists $T = T(\delta)$ such that:

$$\begin{aligned} &|\phi^0(t)| \leq 1 - \delta/2,\ x \in \Omega_\delta(\phi_0),\ t \in [0,T], \\ &|\phi^0(t)| \geq 1 - 3\delta,\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0,T]. \end{aligned} \tag{5.8}$$

Furthermore, using again the uniform Hölder continuity, we have:

$$\begin{aligned} \|\phi_1(t) - \phi_2(t)\|_{C(\Omega)} &\leq C\|\phi_1(t) - \phi_2(t)\|^{\kappa}_{\Phi^w}\|\phi_1(t) - \phi_2(t)\|^{1-\kappa}_{C^\alpha(\Omega)} \\ &\leq C_T\|\phi_1(0) - \phi_2(0)\|^{\kappa}_{\Phi^w}\|\phi_1(t) - \phi_2(t)\|^{1-\kappa}_{C^\alpha(\Omega)} \\ &\leq C_T\varepsilon^{\kappa}, \end{aligned} \tag{5.9}$$

for every $\phi_1(0),\ \phi_2(0)$ in $B(\varepsilon, \phi_0, \Phi^w)$. We can fix $\varepsilon_0 = \varepsilon_0(\delta)$ such that:

$$\begin{aligned} &|\phi(t)| \leq 1 - \delta/4,\ x \in \Omega_\delta(\phi_0),\ t \in [0,T], \\ &|\phi(t)| \geq 1 - 4\delta,\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0,T], \end{aligned} \tag{5.10}$$

for all trajectories $\phi(t)$ starting from the ball $B(\varepsilon, \phi_0, \Phi^w),\ \varepsilon \leq \varepsilon_0$.

Due to (5.7), there exists a smooth cut-off function $\theta \in C^\infty(\mathbb{R}^3, [0,1])$ such that:

$$\theta(x) = \begin{cases} 0, & \text{if } x \in \overline{\Omega}_\delta(\phi_0), \\ 1, & \text{if } x \in \Omega_{2\delta}(\phi_0). \end{cases} \tag{5.11}$$

Furthermore, θ satisfies the additional condition:

$$\|\theta\|_{C^k(\mathbb{R}^3)} \leq C_k, \tag{5.12}$$

where $k \in \mathbb{N}$ is arbitrary and the constant C_k depends on δ, but is independent of the choice of $\phi_0 \in \mathbb{B}_0$. The second estimate of (5.10) yields:

$$f'(\phi(t,x)) \geq \Lambda(\delta),\ x \in \overline{\Omega}_{2\delta}(\phi_0),\ t \in [0,T], \tag{5.13}$$

for all trajectories $\phi(t)$ starting from the ball $B(\varepsilon, \phi_0, \Phi^w)$, where

$$\Lambda(\delta) := \min\left\{f'(1 - 4\delta), f'(-1 + 4\delta)\right\}. \tag{5.14}$$

Indeed, for every $x \in \overline{\Omega}_{2\delta}(\phi_0)$, we have that $|\phi(t)| \geq 1 - 4\delta$. Then, using the fact that $\operatorname{sgn} s \cdot f''(s) \geq 0$, we obtain:

$$\begin{aligned} &\text{if } \phi(t) \geq 1 - 4\delta \Rightarrow f''(\phi(t)) \geq 0 \Rightarrow f'(\phi(t)) \geq f'(1 - 4\delta), \\ &\text{if } \phi(t) \leq -1 + 4\delta \Rightarrow f''(\phi(t)) \leq 0 \Rightarrow f'(\phi(t)) \geq f'(-1 + 4\delta), \end{aligned} \tag{5.15}$$

which yields (5.13). Since $f'(s) \xrightarrow[s \to \pm 1]{} +\infty$, then, $\Lambda(\delta) \xrightarrow[\delta \to 0]{} +\infty$ and we can fix $\delta > 0$ close enough to zero such that $\Lambda(\delta)$ is arbitrarily large. The next lemma gives some kind of smoothing property for the difference of two solutions.

Lemma 5.1. *Let the above assumptions hold. Then, there exists $\delta > 0$ such that:*

$$\|\phi_1(T) - \phi_2(T)\|^2_{\Phi^w} \leq e^{-\beta T}\|\phi_1(0) - \phi_2(0)\|^2_{\Phi^w} + C\int_0^T \|\theta\left(\phi_1(s) - \phi_2(s)\right)\|^2_{L^2(\Omega)}\mathrm{d}s, \tag{5.16}$$

where the positive constants β and C are independent of $\phi_1(0),\ \phi_2(0) \in B(\varepsilon, \phi_0, \Phi^w)$ and $\phi_0 \in \mathbb{B}_0$.

Proof. We set $\phi(t) = \phi_1(t) - \phi_2(t)$. Then, ϕ solves the following problem:

$$\begin{cases} A^{-1}\partial_t\phi = \Delta\phi - l(t)\phi + \lambda\phi, \text{ in } \Omega, \\ \partial_n(\Delta\phi - l(t)\phi + \lambda\phi)|_\Gamma = 0, \\ \partial_t\phi - \Delta_\Gamma\phi + \partial_n\phi + \phi(t) + m(t)\phi = 0, \quad \text{on } \Gamma, \end{cases} \tag{5.17}$$

where

$$l(t) := \int_0^t f'(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s \text{ and } m(t) := \int_0^t g_0'(s\phi_1(t) + (1-s)\phi_2(t))\mathrm{d}s.$$

Multiplying (5.17) by $\phi(t)$ and integrating over Ω, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\phi(t)\|^2_{H^{-1}(\Omega)} + \|\phi(t)\|^2_{L^2(\Gamma)}) + \|\nabla\phi(t)\|^2_{L^2(\Omega)} \\ &+\|\nabla_\Gamma\phi(t)\|^2_{L^2(\Gamma)} + \|\phi(t)\|^2_{L^2(\Gamma)} + (l(t)\phi(t), \phi(t))_\Omega \\ = \ &\lambda\|\phi(t)\|^2_{L^2(\Omega)} - (m(t)\phi(t), \phi(t))_\Gamma \\ \leq \ &\lambda\|\phi(t)\|^2_{L^2(\Omega)} + C_0\|\phi(t)\|^2_{L^2(\Gamma)}, \end{aligned} \tag{5.18}$$

where $C_0 = \|g'\|_{C([-1,1])}$.

Due to (5.13), we have:

$$\begin{aligned} \int_\Omega l(t,x)|\phi(t,x)|^2\mathrm{d}x \geq &\int_{\overline{\Omega}_{2\delta}} l(t,x)|\phi(t,x)|^2\mathrm{d}x \\ \geq &\Lambda\|\phi\|^2_{L^2(\overline{\Omega}_{2\delta})} \\ = &\Lambda\|\phi\|^2_{L^2(\Omega)} - \Lambda\|\phi\|^2_{L^2(\Omega_{2\delta})} \\ \geq &\Lambda\|\phi\|^2_{L^2(\Omega)} - \Lambda\|\theta\phi\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.19}$$

Thus, we obtain:

$$\begin{aligned} &\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + 2\|\nabla\phi(t)\|^2_{L^2(\Omega)} + 2\|\phi(t)\|^2_{H^1(\Gamma)} + 2(\Lambda - \lambda)\|\phi\|^2_{L^2(\Omega)} \\ \leq \ &2C_0\|\phi(t)\|^2_{L^2(\Gamma)} + 2\Lambda\|\theta\phi\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.20}$$

For the first term in the right hand-side of (5.20), we use the following trace inequality:

$$2\|\phi\|^2_{L^2(\Gamma)} \leq 2C\|\phi\|_{H^1(\Omega)}\|\phi\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\Lambda - \lambda}}\|\phi\|^2_{H^1(\Omega)} + C\sqrt{\Lambda - \lambda}\|\phi\|^2_{L^2(\Omega)}, \tag{5.21}$$

and the fact that for some $\omega \in (0,2)$, we have:

$$2\|\nabla\phi(t)\|^2_{L^2(\Omega)} + 2\|\phi(t)\|^2_{H^1(\Gamma)} \geq \omega\|\phi(t)\|^2_{H^1(\Omega)} + \omega\|\phi(t)\|^2_{H^1(\Gamma)}. \tag{5.22}$$

Thus, fixing δ in such a way that $C_0C \leq \omega\frac{\sqrt{\Lambda-\lambda}}{2}$ and using (5.21) and (5.22) in (5.20), we find:

$$\begin{aligned}
&\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega\|\phi(t)\|^2_{H^1(\Omega)} + \omega\|\phi(t)\|^2_{H^1(\Gamma)} + 2(\Lambda-\lambda)\|\phi(t)\|^2_{L^2(\Omega)} \\
\leq\ & \frac{C_0C}{\sqrt{\Lambda-\lambda}}\|\phi(t)\|^2_{H^1(\Omega)} + C_0C\sqrt{\Lambda-\lambda}\|\phi(t)\|^2_{L^2(\Omega)} + 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)} \\
\leq\ & \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \omega/2(\Lambda-\lambda)\|\phi(t)\|^2_{L^2(\Omega)} + 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.23}$$

Taking $\beta' > 0$ small enough, estimate (5.23) leads to:

$$\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \beta'\left(\|\phi(t)\|^2_{H^1(\Gamma)} + \|\phi(t)\|^2_{L^2(\Omega)}\right) \leq 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}. \tag{5.24}$$

Using the inequalities $\|\phi\|_{H^{-1}(\Omega)} \leq C\|\phi\|_{L^2(\Omega)}$ and $\|\phi\|_{L^2(\Gamma)} \leq C\|\phi\|_{H^1(\Gamma)}$, we end up with:

$$\frac{d}{dt}\|\phi(t)\|^2_{\Phi^w} + \omega/2\|\phi(t)\|^2_{H^1(\Omega)} + \beta\left(\|\phi(t)\|^2_{L^2(\Gamma)} + \|\phi(t)\|^2_{H^{-1}(\Omega)}\right) \leq 2\Lambda\|\theta\phi(t)\|^2_{L^2(\Omega)}, \tag{5.25}$$

for some positive constant β. Applying the Gronwall lemma, we obtain:

$$\begin{aligned}
\|\phi(T)\|^2_{\Phi^w} &\leq \|\phi(0)\|^2_{\Phi^w}e^{-\beta T} + C\int_0^T e^{-\beta(T-s)}\|\theta\phi(s)\|^2_{L^2(\Omega)}\mathrm{d}s \\
&\leq \|\phi(0)\|^2_{\Phi^w}e^{-\beta T} + C\int_0^T \|\theta\phi(s)\|^2_{L^2(\Omega)}\mathrm{d}s,
\end{aligned} \tag{5.26}$$

and estimate (5.16) is proven. □

Lemma 5.2. *Let the nonlinearities f and g satisfy the assumptions of Section 2. Then, there exists positive constants C and K independent of $\phi_i(0)$ in $B(\varepsilon, \phi_0, \Phi^w)$, $i = 1, 2$, and ϕ_0 in $\mathbb{B}_0$ such that the following estimate holds:*

$$\begin{aligned}
&\|\partial_t(\theta(\phi_1-\phi_2))\|_{L^2([0,T],H^{-3}(\Omega))} + \|\theta(\phi_1-\phi_2)\|_{L^2([0,T],H^1(\Omega))} \\
\leq & Ce^{KT}\|\phi_1(0)-\phi_2(0)\|_{\Phi^w}.
\end{aligned} \tag{5.27}$$

Proof. Due to (5.2) and the fact that $\nabla_x\theta$ is uniformly bounded, we have:

$$\begin{aligned}
&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],H^1(\Omega))} \\
=&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))} + \|\nabla_x(\theta(\phi_1-\phi_2))\|^2_{L^2([0,T],L^2(\Omega))} \\
\leq&\|\theta(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))} + \|\nabla_x\theta\cdot(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))} \\
&+ \|\theta\cdot\nabla_x(\phi_1-\phi_2)\|^2_{L^2([0,T],L^2(\Omega))} \\
\leq&Ce^T\|\phi_1(0)-\phi_2(0)\|^2_{\Phi^w}.
\end{aligned} \tag{5.28}$$

In order to find estimates on the time derivative, we first recall that $\partial_t\phi$ verifies:

$$\partial_t\phi = (-\Delta_x + I)(\Delta_x\phi - l(t)\phi + \lambda\phi), \tag{5.29}$$

where $\phi = \phi_1 - \phi_2$. Testing this equation by $\theta\varphi$ for any test function $\varphi \in C_0^\infty(\Omega)$, using that supp $\theta \subset \Omega_\delta(\phi_0)$, we obtain:

$$\begin{aligned}
&(\partial_t(\theta\phi(t)), \varphi)_\Omega \\
=& -(\Delta_x\phi(t) - l(t)\phi(t) + \lambda\phi(t), \Delta_x(\theta\varphi(t)) - \theta\varphi(t))_\Omega \\
=&(\nabla_x\phi(t), \nabla_x\Delta_x(\theta\varphi(t)))_\Omega - (l(t)\phi(t), \theta\varphi(t))_\Omega + (\lambda\phi(t), \theta\varphi(t))_\Omega \\
&-(\nabla_x\phi(t), \nabla_x\theta\varphi(t))_\Omega + (l(t)\phi(t), \Delta_x(\theta\varphi(t)))_\Omega - (\lambda\phi(t), \Delta_x(\theta\varphi(t)))_\Omega \\
\leq& C\|\phi\|_{H^1(\Omega)}\|\varphi\|_{H^3(\Omega)}.
\end{aligned} \tag{5.30}$$

Estimate (5.30) yields that $\|\partial_t(\theta(\phi_1(t) - \phi_2(t)))\|_{H^{-3}(\Omega)} \leq C\|\phi\|_{H^1(\Omega)}$ and using (5.2), we obtain (5.27). □

To conclude the proof of the theorem, we introduce the functional spaces:

$$\begin{aligned}
&\mathbb{H}_1 := L^2([0,T], H^1(\Omega)) \cap H^1([0,T], H^{-3}(\Omega)), \\
&\mathbb{H} := L^2([0,T], L^2(\Omega)).
\end{aligned} \tag{5.31}$$

We have that $\mathbb{H}_1$ is compactly embedded into $\mathbb{H}$. For every $\phi_0 \in \mathbb{B}_0$, we define the following operator:

$$\begin{aligned}
\mathbb{K}_{\phi_0} : B(\varepsilon, \phi_0, \Phi^w) &\longmapsto \mathbb{H}_1 \\
\phi(0) &\longmapsto \mathbb{K}_{\phi_0}\phi(0) := \theta\phi(\cdot),
\end{aligned}$$

where $\phi(t)$ is the variational solution departing from $\phi(0)$. Due to Lemma 5.2, the map $\mathbb{K}_{\phi_0}$ is uniformly Lipschitz continuous:

$$\|\mathbb{K}_{\phi_0}(\phi_1 - \phi_2)\|_{\mathbb{H}_1} \leq L\|\phi_1 - \phi_2\|_{\Phi^w},\ \phi_1, \phi_1 \in B(\varepsilon, \phi_0, \Phi^w),\ \varepsilon \leq \varepsilon_0 \tag{5.32}$$

and thanks to Lemma 5.16, we have:

$$\|S(T)\phi_1 - S(T)\phi_2\|_{\Phi^w} \leq \gamma\|\phi_1 - \phi_2\|_{\Phi^w} + c\|\mathbb{K}_{\phi_0}(\phi_1 - \phi_2)\|_{\mathbb{H}}, \tag{5.33}$$

where $\gamma < 0$, $c > 0$ are independent of $\phi_0 \in \mathbb{B}_0$, $\varepsilon \leq \varepsilon_0$ and $\phi_1, \phi_1 \in B(\varepsilon, \phi_0; \Phi^w)$. Arguing as in Miranville & Zelik[12], we have that inequalities (5.32) and (5.33), together with the compactness of the embedding $\mathbb{H}_1 \subset \mathbb{H}$, guarantee the existence of an exponential attractor $\mathcal{M}_d \subset \mathbb{B}_0$ for the discrete semigroup $S(nT)$ acting on the phase space $\mathbb{B}_0$. Since the semigroup $S(t)$ is uniformly Hölder continuous with respect to time and space in $[0,T] \times \mathbb{B}_0$, we deduce the existence of an exponential attractor $\mathcal{M}$ for the continuous semigroup $S(t)$ on $\mathbb{B}_0$ which can be obtained by the standard formula

$$\mathcal{M} := \cup_{t\in[0,T]}\mathcal{M}_d.$$

□

Acknowledgements. The author wishes to thank Prof. Alain Miranville and Dr. Madalina Petcu for many stimulating discussions and useful comments on the subject of the paper.

References

[1] L. Cherfils, S. Gatti and A. Miranville, *Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials*, J. Math. Anal. Appl., 243 (2008), 557–566.

[2] L. Cherfils, S. Gatti and A. Miranville, *Long time behavior of the Caginalp system with singular potentials and dynamic boundary conditions*, Commun. Pure Appl. Anal., to appear.

[3] C. Gal, *Well-posedness and long time behavior of the non-isothermal viscous Cahn-Hilliard equation with dynamic boundary conditions*, Dyn. Partial Differ. Equ., 5 (2008), 39-67.

[4] C. Gal, *Global well-posedness for the non-isothermal Cahn-Hilliard equation with dynamic boundary conditions*, Adv. Differential Equations, 12 (2007), 1241-1274.

[5] G. Gilardi, A. Miranville and G. Schimperna, *Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, 31 (2007), 679-712.

[6] H. Israel, *Well-posedness and long time behavior of an Allen-Cahn type equation*, Commun. Pure Appl. Anal., to appear.

[7] G. Karali and M. Katsoulakis, *The role of multiple microscopic mechanisms in cluster interface evolution*, J. Differential Equations, 235 (2007), 418-438.

[8] G. Karali and T. Ricciardi, *On the convergence of a fourth order evolution equation to the Allen-Cahn equation*, Nonlinear Anal., 72 (2010), 4271–4281.

[9] M.A. Katsoulakis and D.G. Vlachos, *From microscopic interactions to macroscopic laws of cluster evolution*, 83 (2000), 1511-1514.

[10] A. Mikhailov, M. Hildebrand and G. Ertl, *Nonequilibrium nanostructures in condensed reactive systems*, Coherent structures in complex systems (Sitges, 2000), 567 (2001), 252-269.

[11] A. Miranville and S. Zelik, *Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions*, Math. Methods Appl. Sci., 28 (2005), 709-735.

[12] A. Miranville and S. Zelik, *The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions*, 28 (2010), 275-310.

[13] R. Racke and S. Zheng, *The Cahn-Hilliard equation with dynamic boundary conditions*, Adv. Differential Equations,8 (2003), 83-110.

[14] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, Springer-Verlag, New York, 1997.

[15] R. Walter, *Functional analysis*, International Series in Pure and Applied Mathematics, New York, 1991.