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NUMERICAL SOLUTION OF LR FUZZY HUNTER-SAXETON EQUATION BY USING HOMOTOPY ANALYSIS METHOD

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Abstract In this paper, a LR fuzzy Hunter-Saxton equation is solved by using the homotopy analysis method (HAM). The approximation solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed method are proved. A numerical example is studied to demonstrate the accuracy of the presented method.

Keywords Hunter-Saxton equation, homotopy analysis method, fuzzy number, L-R fuzzy number, generalized differentiability, fuzzy-valued function, h-difference.

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1. Introduction

The Hunter-Saxton equation

$$u_{txx} \oplus 2 \odot u_x \odot u_{xx} \oplus u \odot u_{xxx} = 0, \quad t > 0. \tag{1.1}$$

Models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field, x being the space variable in a reference frame moving with the unperturbed wave speed and t being a slow time variable [15]. In this work, we develop the HAM to solve the Eq.(1.1) with the LR fuzzy initial conditions as follows:

$$\begin{aligned} &u(0,0) = (0,0,0),\\ &u_{tx}(a,x) = (2xe^{x^2}, e^x, -e^{-x}),\\ &u_t(a,x) = (2axe^{x^2}, ae^x, -ae^{-x}). \end{aligned} \tag{1.2}$$

The fuzzy exact solution is $u(x,t) = (xe^{t^2}, xe^t, xe^{-t})$.

The paper is organized as follows. In section 2, the homotopy analysis method is introduced for solving Eq.(1.1). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved in section 3. An example is presented in section 4 to illustrate the accuracy of this method.

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To obtain the approximate solution of Eq.(1.1), by integrating three times from Eq.(1.1) with respect to x, t and using the fuzzy initial conditions we obtain

$$\begin{aligned}&\widetilde{u}(x,t) \oplus (-1) \odot \widetilde{F}(x,t) \oplus 2 \odot \int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}(x,t)) \ dt \ dx \\ &\oplus \int_0^t \int_a^x (x-t) \ \odot F_2(\widetilde{u}(x,t)) \ dt \ dx,\end{aligned} \tag{1.3}$$

where,

$$\begin{aligned}&D^i(\widetilde{u}(x,t)) = \frac{\partial^i \widetilde{u}(x,t)}{\partial x^i}, \quad i = 1,2,3, \\ &\widetilde{F}(x,t) = \int_0^t \int_a^x (2xe^{x^2}, e^x, -e^{-x}) dx \ dt \oplus \int_0^t (2axe^{x^2}, ae^x, -ae^{-x}) \ dt, \\ &F_1(u(x,t)) = D(\widetilde{u}(x,t)) \odot D^2(\widetilde{u}(x,t)), \\ &F_2(u(x,t)) = \widetilde{u}(x,t) \odot D^3(\widetilde{u}(x,t)).\end{aligned}$$

In Eq.(1.3),we assume $\widetilde{F}(x,t)$ is bounded for all t in $J = [0,T]$ and x in $[a,b]$ $(T, a, b \in \mathbb{R})$.

The terms $F_1(\widetilde{u}(x,t)), F_2(\widetilde{u}(x,t))$ are Lipschitz continuous with $\widehat{D}(F_i(\widetilde{u}), F_i(\widetilde{u}^*)) \leq L_i \ \widehat{D}(\widetilde{u}, \widetilde{u}^*)(i = 1,2)$, where $\widehat{D}$ is the Hausdorff metric [7] and

$$\begin{aligned}&\mid x - t \mid \leq M, \\ &\alpha = T(b-a)M(2L_1 + L_2).\end{aligned}$$

2. Definitions

The basic definitions of a fuzzy number are given in [1, 4, 6, 7, 8, 12, 13, 14, 15] as follows:

Definition 2.1. A fuzzy number is a fuzzy set like $u : \mathbb{R} \rightarrow [0,1]$ which satisfies:

1. u is an upper semi-continuous function
2. $u(x) = 0$ outside some interval [a,d]
3. There are real numbers b, c such as $a \leq b \leq c \leq d$ and
   - 3.1 $u(x)$ is a monotonic increasing function on [a, b]
   - 3.2 $u(x)$ is a monotonic decreasing function on [c, d]
   - 3.3 $u(x) = 1$ for all $x \in [b, c]$.

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(r) \leq \overline{u}(r), \ 0 \leq r \leq 1$.

Definition 2.3. The membership function u is presented as

$$u(x) = \begin{cases} u_L(x) & if \quad x \in [a,b] \\ 1 & if \quad x \in [b,c] \\ u_R(x) & if \quad x \in [c,d] \\ 0 & otherwise \end{cases} \tag{2.1}$$

where $u_L : [a,b] \to [0,1]$ and $u_R : [c,d] \to [0,1]$ are left and right membership functions of fuzzy number u. Another definition for a fuzzy number is as follows.

Definition 2.4. A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \geq 0, \beta \geq 0$ with

$$\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \leq a \\ R(\frac{x-b}{\beta}) & x \geq a \end{cases} \tag{2.2}$$

the mean value of $\tilde{A}, a$ is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.5. Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,

$(1) : \lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$

$(2) : -\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$

$(3) : \tilde{M} \oplus \tilde{N} = (m+n, \alpha+\gamma, \beta+\delta)_{LR}$

$$(4) : \tilde{M} \odot \tilde{N} \simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR} & \tilde{M}, \tilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR} & \tilde{M} > 0, \tilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR} & \tilde{M}, \tilde{N} < 0 \end{cases} \tag{2.3}$$

Definition 2.6. For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$, we use the distance (Hausdorff metric) [16]

$$D(u(r), v(r)) = \max\{\sup_{r\in[0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup\ |\overline{u}(r) - \overline{v}(r)|\},$$

and it is shown [20] that (E^1 , D) is a complete metric space and the following properties are well known:

$$D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\ \widetilde{u}, \widetilde{v} \in E^1,$$
$$D(k\widetilde{u}, k\widetilde{v}) = \mid k \mid D(\widetilde{u}, \widetilde{v}), \forall\ k \in R, \widetilde{u}, \widetilde{v} \in E^1,$$
$$D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \forall\ \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E^1.$$

Definition 2.7. A triangular fuzzy number is defined as a fuzzy set in E^1, that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \leq b \leq c$ such that $[u]^r = [u^r_-, u^r_+]$ are the endpoints of r-level sets for all $r \in [0,1]$, where $u^r_- = a + (b-a)r$ and $u^r_+ = c - (c-b)r$. Here, $u^0_- = a, u^0_+ = c, u^1_- = u^1_+ = b$, which is denoted by u^1. The set of triangular fuzzy numbers will be denoted by E^1.

Definition 2.8. Consider x, $y \in E$. If there exists $z \in E$ such that $x = y + z$ then z is called the H- difference of x and y, and is denoted by $x \ominus y$. [7]

Definition 2.9. (see[7]) Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f^{'}(x_0) \in E$, such that:

i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0)$$

or

ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f^{'}(x_0)$$

or

iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0)$$

Definition 2.10. Let $f : (a, b) \to E$. We say f is (i)-differentiable on (a, b) if f is differentiable in the sense (i) of Definition (2.9) and similarly for (ii), (iii) and (iv) differentiability.

Definition 2.11. (see[15]) The mapping $f : T \to E^n$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:

$$[f(t)]^r = [f^r_-(t), f^r_+(t)], \quad t \in T, \quad r \in [0, 1].$$

Definition 2.12. (see[15]) Let $f : T \to E^n$ be Hukuhara differentiable and denote $[f(t)]^r = [f^r_-, f^r_+]$. Then, the boundary function f^r_- and f^r_+ are differentiable (or Seikkala differentiable) and

$$[f^{'}(t)]^r = [(f^r_-)^{'}(t), (f^r_+)^{'}(t)], \quad t \in T, \quad r \in [0, 1].$$

2.1. Description of the HAM

Consider,

$$N[\widetilde{u}] = 0.$$

Where N is a nonlinear operator, $\widetilde{u}(x, t)$ is unknown function and x is an independent variable. let $\widetilde{u}_0(x, t)$ denote an initial guess of the exact solution $\widetilde{u}(x, t)$,

$h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned} & (1-q)L[\widetilde{\phi}(x,t;q) \oplus (-1) \odot \widetilde{u}_0(x,t)] \oplus (-1)qhH_1(x,t) \odot N[\widetilde{\phi}(x,t;q)] \\ = \; & \hat{H}[\widetilde{\phi}(x,t;q); \widetilde{u}_0(x,t), H_1(x,t), h, q]. \end{aligned} \tag{2.4}$$

It should be emphasized that we have great freedom to choose the initial guess $\widetilde{u}_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$.

Enforcing the homotopy (2.4) to be zero, i.e.

$$\hat{H}_1[\widetilde{\phi}(x,t;q); \widetilde{u}_0(x,t), H_1(x,t), h, q] = 0, \tag{2.5}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\widetilde{\phi}(x,t;q) \oplus (-1) \odot \widetilde{u}_0(x,t)] = qhH_1(x,t) \odot N[\widetilde{\phi}(x,t;q)]. \tag{2.6}$$

When $q = 0$, the zero-order deformation Eq.(2.5) becomes

$$\widetilde{\phi}(x;0) = \widetilde{u}_0(x,t), \tag{2.7}$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(2.5) is equivalent to

$$\widetilde{\phi}(x,t;1) = \widetilde{u}(x,t). \tag{2.8}$$

Thus, according to (2.7) and (2.8), as the embedding parameter q increases from 0 to 1, $\widetilde{\phi}(x,t;q)$ varies continuously from the initial approximation $\widetilde{u}_0(x,t)$ to the exact solution $\widetilde{u}(x,t)$. Such a kind of continuous variation is called deformation in homotopy [2, 3, 5, 9, 10, 11, 18, 19].

Due to Taylor's theorem, $\widetilde{\phi}(x,t;q)$ can be expanded in a power series of q as follows

$$\widetilde{\phi}(x,t;q) = \widetilde{u}_0(x,t) \oplus \sum_{m=1}^{\infty} \widetilde{u}_m(x,t) \odot q^m, \tag{2.9}$$

where,

$$\widetilde{u}_m(x,t) = \frac{1}{m!} \odot \frac{\partial^m \widetilde{\phi}(x,t;q)}{\partial q^m} \,|_{q=0} .$$

Let the initial guess $\widetilde{u}_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series (2.9) of $\widetilde{\phi}(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$\widetilde{u}(x,t) = \widetilde{\phi}(x,t;1) = \widetilde{u}_0(x,t) \oplus \sum_{m=1}^{\infty} \widetilde{u}_m(x,t). \tag{2.10}$$

From Eq.(2.9), we can write Eq.(2.6) as follows

$$\begin{aligned}
& (1-q)\odot L[\widetilde{\phi}(x,t,q)\oplus(-1)\odot\widetilde{u}_0(x,t)] \\
= \ & (1-q)\odot L[\sum_{m=1}^{\infty}\widetilde{u}_m(x,t)\odot q^m] \\
= \ & q\ h\ H_1(x,t)\odot N[\widetilde{\phi}(x,t,q)] \qquad (2.11)\\
\Rightarrow \ & L[\sum_{m=1}^{\infty}\widetilde{u}_m(x,t)\odot q^m]-q\odot L[\sum_{m=1}^{\infty}\widetilde{u}_m(x,t)\odot q^m] \\
= \ & q\ h\ H_1(x,t)\odot N[\widetilde{\phi}(x,t,q)].
\end{aligned}$$

By differentiating (2.11) m times with respect to q, we obtain

$$\begin{aligned}
& \{L[\sum_{m=1}^{\infty}\widetilde{u}_m(x,t)\odot q^m]\oplus(-1)\odot q\odot L[\sum_{m=1}^{\infty}\widetilde{u}_m(x,t)\odot q^m]\}^{(m)} \\
= \ & \{q\ h\ H_1(x,t)\odot N[\widetilde{\phi}(x,t,q)]\}^{(m)} \\
= \ & m!\odot L[\widetilde{u}_m(x,t)\oplus(-1)\odot\widetilde{u}_{m-1}(x,t)] \\
= \ & h\ H_1(x,t)\ m\odot\frac{\partial^{m-1}N[\widetilde{\phi}(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .
\end{aligned}$$

Therefore,

$$L[\widetilde{u}_m(x,t)\oplus(-1)\odot\chi_m\widetilde{u}_{m-1}(x,t)]=hH_1(x,t)\odot\Re_m(\widetilde{u}_{m-1}(x,t)), \qquad (2.12)$$

where,

$$\Re_m(\widetilde{u}_{m-1}(x,t))=\frac{1}{(m-1)!}\odot\frac{\partial^{m-1}N[\widetilde{\phi}(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \qquad (2.13)$$

and,

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & m>1.\end{cases}$$

Note that the high-order deformation Eq.(2.12) is governing the linear operator L, and the term $\Re_m(\widetilde{u}_{m-1}(x,t))$ can be expressed simply by (2.13) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1.3), according to HAM, let

$$\begin{aligned}
N[\widetilde{u}(x,t)]=&\widetilde{u}(x,t)\oplus(-1)\odot\widetilde{F}(x,t) \\
&\oplus 2\odot\int_0^t\int_a^x(x-t)\odot F_1(\widetilde{u}(x,t))\ dt\ dx \\
&\oplus\int_0^t\int_a^x(x-t)\odot F_2(\widetilde{u}(x,t))\ dt\ dx,
\end{aligned}$$

so,

$$\begin{aligned}\Re_m(\widetilde{u}_{m-1}(x,t)) =& \widetilde{u}_{m-1}(x,t) \oplus (-1) \odot \widetilde{F}(x,t)\\ &\oplus 2 \odot \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}(x,t))\ dt\ dx \\ &\oplus \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}(x,t))\ dt\ dx.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.12)

$$\begin{aligned}& L[\widetilde{u}_m(x,t) \oplus (-1) \odot \chi_m \widetilde{u}_{m-1}(x,t)] \\ =\ & hH_1(x,t) \odot [\widetilde{u}_{m-1}(x,t) \oplus 2 \odot \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}(x,t))\ dt\ dx \\ & \oplus \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}(x,t))\ dt\ dx \oplus (\chi_m - 1) \odot \widetilde{F}(x,t)].\end{aligned} \tag{2.15}$$

We take an initial guess $\widetilde{u}_0(x,t) = -\widetilde{F}(x,t)$, an auxiliary linear operator $L\widetilde{u} = \widetilde{u}$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into (2.15) to give the recurrence relation

$$\begin{aligned}\widetilde{u}_0(x,t) = \widetilde{F}(x,t) =& \int_0^t \int_a^x (2xe^{x^2}, e^x, -e^{-x}) dx\ dt \\ &\oplus \int_0^t (2axe^{x^2}, ae^x, -ae^{-x})\ dt, \\ \widetilde{u}_{n+1}(x,t) = 2 \odot & \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}_n(x,t))\ dt\ dx \\ &\oplus \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}_n(x,t))\ dt\ dx, \quad n \geq 1.\end{aligned} \tag{2.16}$$

3. Existence solution and convergence of HAM

Theorem 3.1. *Let $0 < \alpha < 1$, then equation (1.3), has a unique solution.*

Proof. Let $\widetilde{u}$ and $\widetilde{u}^*$ be two different solutions of (1.3) then

$$\begin{aligned}D(\widetilde{u}, \widetilde{u}^*) = D(&\widetilde{F}(x,t) \oplus (-2) \odot \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}(x,t))\ dt\ dx \\ &\oplus (-1) \odot \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}(x,t))\ dt\ dx, \\ &\widetilde{F}(x,t) \oplus (-2) \odot \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}^*(x,t))\ dt\ dx \\ &\oplus (-1) \odot \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}^*(x,t))\ dt\ dx) \\ \leq\ & TM(b-a)(2L_1 + L_2)\ \widehat{D}(\widetilde{u}, u^*)\ =\ \alpha \widehat{D}(\widetilde{u}, u^*).\end{aligned}$$

From which we get $(1-\alpha)D(\widetilde{u},\widetilde{u}^*) \leq 0$. Since $0<\alpha<1$, then $D(\widetilde{u},\widetilde{u}^*)=0$. Implies $\widetilde{u}=\widetilde{u}^*$ and completes the proof.

□

Theorem 3.2.

$$\phi_{k+1}(x,t) = \widetilde{F}(x,t) \oplus \sum_{i=1}^{k+1}[2\odot \int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}_i(x,t)) \ dt \ dx \\ \oplus \int_0^t \int_a^x (x-t) \ \odot F_2(\widetilde{u}_i(x,t)) \ dt \ dx], \quad k \geq 0.$$

$$\begin{aligned}
& D(\phi_{k+1}(x,t), \phi_k(x,t)) \\
= \quad & D(\widetilde{F}(x,t) \oplus \textstyle\sum_{i=1}^{k+1}[2 \odot \int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}_i(x,t)) \ dt \ dx \\
& \qquad \oplus \textstyle\int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}_i(x,t)) \ dt \ dx], \\
& \quad \widetilde{F}(x,t) \oplus \textstyle\sum_{i=1}^{k+1}[2 \odot \int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}_{i-1}(x,t)) \ dt \ dx \\
& \qquad \oplus \textstyle\int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}_{i-1}(x,t)) \ dt \ dx]) \\
= \quad & D(\phi_k(x,t) \oplus 2 \odot \textstyle\int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}_k(x,t)) \ dt \ dx \\
& \quad \oplus \textstyle\int_0^t \int_a^x (x-t) \ \odot F_2(\widetilde{u}_k(x,t)) \ dt \ dx \ , \ \phi_k(x,t)) \\
= \quad & D(2 \odot \textstyle\int_0^t \int_a^x (x-t) \ \odot F_1(\widetilde{u}_k(x,t)) \ dt \ dx \\
& \quad \oplus \textstyle\int_0^t \int_a^x (x-t) \ \odot F_2(\widetilde{u}_k(x,t)) \ dt \ dx \ , \ \widetilde{0}) \\
\leq \quad & D(\widetilde{u}_k(x,t), \widetilde{0}) \leq \alpha^k D(\widetilde{F}, \widetilde{0}) \\
\Longrightarrow \quad & D(\phi_{k+1}(x,t), \phi_k(x,t)) \leq \alpha^{k+1} D(\widetilde{F}, \widetilde{0}) \\
\Longrightarrow \quad & \textstyle\sum_{k=0}^{\infty} D(\phi_{k+1}(x,t), \phi_k(x,t)) \leq \alpha^{k+1} D(\widetilde{F}, \widetilde{0}) \sum_{k=0}^{\infty} \alpha^k.
\end{aligned}$$

4. Numerical example

In this section, we compute a numerical example which is solved by the HAM. The program has been provided with Mathematica 6.

Algorithm:

$$u_0 = \widetilde{F}(x,t) = \int_0^t \int_a^x (2xe^{x^2}, e^x, -e^{-x}) dx \ dt \oplus \int_0^t (2axe^{x^2}, ae^x, -ae^{-x}) \ dt$$
$$= (f_{0_1}, f_{0_2}, f_{0_3}).$$

For $i=0, \ i \leq n, \ i++$, if $[D^2(f_{i_1}) > 0$ and $D^2(f_{i_3}) > 0$,

$D(\widetilde{u}_i(x,t)) = (\frac{\partial f_{i_1}}{\partial x}, \frac{\partial f_{i_2}}{\partial x}, \frac{\partial f_{i_3}}{\partial x}), D(\widetilde{u}_i(x,t)) = (\frac{\partial f_{i_3}}{\partial x}, \frac{\partial f_{i_2}}{\partial x}, \frac{\partial f_{i_1}}{\partial x})]$;

If$[D^4(f_{i_1}) > 0$ and $D^4(f_{i_3}) > 0, D^2(\widetilde{u}_i(x,t)) = (\frac{\partial^2 f_{i_1}}{\partial x^2}, \frac{\partial^2 f_{i_2}}{\partial x^2}, \frac{\partial^2 f_{i_3}}{\partial x^2})$,

$D^2(\widetilde{u}_i(x,t)) = (\frac{\partial^2 f_{i_3}}{\partial x^2}, \frac{\partial^2 f_{i_2}}{\partial x^2}, \frac{\partial^2 f_{i_1}}{\partial x^2})]; F_1(u_i(x,t)) = D(\widetilde{u}_i(x,t)) \odot D^2(\widetilde{u}_i(x,t));$

If $[\frac{\partial^5 f_{i_1}}{\partial x^5} > 0$ and $\frac{\partial^5 f_{i_3}}{\partial x^5} > 0, D^3(\widetilde{u}_i(x,t)) = (\frac{\partial^5 f_{i_1}}{\partial x^5}, \frac{\partial^5 f_{i_2}}{\partial x^5}, \frac{\partial^5 f_{i_3}}{\partial x^5}),$

$D^3(\widetilde{u}_i(x,t)) = (\frac{\partial^5 f_{i_3}}{\partial x^5}, \frac{\partial^5 f_{i_2}}{\partial x^5}, \frac{\partial^5 f_{i_1}}{\partial x^5})]; F_2(u(x,t)) = \widetilde{u}(x,t) \odot D^3(\widetilde{u}(x,t));$

$$\widetilde{u}_{i+1}(x,t) = 2 \odot \int_0^t \int_a^x (x-t) \odot F_1(\widetilde{u}_i(x,t))\ dt\ dx \oplus \int_0^t \int_a^x (x-t) \odot F_2(\widetilde{u}_i(x,t))\ dt\ dx;$$

Example 4.1. Consider the fuzzy hunter-Saxeton equation as follows:

$$u_{txx} \oplus 2 \odot u_x \odot u_{xx} \oplus u \odot u_{xxx} = \widetilde{0}.$$

Table 1 Numerical results for Example 4.1

x	App.Sol ($n = 3$)	Errors($\widehat{D}$)
0.01	0.0101130	0.00001253
0.02	0.0204544	0.00005032
0.05	0.0528836	0.00032007
0.1	0.1118610	0.00134369
0.12	0.1372840	0.00198486
0.15	0.1775160	0.00324104
0.2	0.2505650	0.00628427
0.25	0.3318580	0.01851500
0.3	0.4223900	0.01743260
0.35	0.5233110	0.02663750
0.4	0.6359360	0.03920560
0.45	0.7617640	0.05602370
0.5	0.9024930	0.07813260

Table 1 shows that, the approximation solution of the fuzzy Hunter-Saxeton equation is convergent with 3 iterations by using the HAM.

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate analytical solution of the fuzzy Hunter-Saxton equation.

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