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DUALITY THEORY OF REGULARIZED RESOLVENT OPERATOR FAMILY*

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Abstract Let $k \in C(\mathbb{R}^+)$, A be a closed linear densely defined operator in the Banach space X and $\{R(t)\}_{t\geq 0}$ be an exponentially bounded k-regularized resolvent operator families generated by A. In this paper, we mainly study pseudo k-resolvent and duality theory of k-regularized resolvent operator families. The conditions that pseudo k-resolvent become k-resolvent of the closed linear densely defined operator A are given. The some relations between the duality of the regularized resolvent operator families and the generator A are gotten. In addition, the corresponding results of duality of k-regularized resolvent operator families in Favard space are educed.

Keywords k-regularized resolvent operator family, pseudo k-resolvent, duality, Favard space.

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1. Introduction

The notion of the regularized resolvent operator family was recently introduced by Lizama [4] and it is a nature extension of resolvent operator family [3, 10]. For example, the notions of C_0-semigroup, integrated semigroup, strongly continuous cosine operator function, integrated cosine function, integrated resolvent family and convolution semigroup and so on may be unified under the frame of regularized resolvent operator family. Thus it is a very significative thing that some problems of the regularized resolvent operator family are studied. The main purpose of this paper is to study duality problem of k-regularized resolvent operator families. We must first study the pseudo k-resolvent for the sake of this goal. The conditions that pseudo k-resolvent become k-resolvent of the closed linear densely defined operator A are given. The some relations between the duality of the regularized resolvent operator families and the generator A are gotten. The results that we obtain generalize and include the correspondingly case of the pseudo-resolvent and the duality problem of the C_0-semigroup and other operator families(see [1, 2, 8, 9]). In addition, the Favard class and its norm of the operator family are introduced. The correlating problems between the Favard class of the operator family and the duality of the generator A are studied. Recently, Lizana and Humberto [7] have construct a duality theory for k-regularized resolvents and ex-tending some of the

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known theorems for dual semigroups. But our results in this paper are different with Lizana and Humberto in [7].

Let $\mathbb{C}$ denote the complex plane. Let X be a complex Banach space with norm $\|\cdot\|$ and $B(X)$ be the set of all bounded linear operators from X to itself. We will use that X^* denotes the dual space of the X and write that $< x^*, x >= x^*(x)$ denotes the value of $x^* \in X^*$ at $x \in X$. If A is a closed linear densely defined operator in X, we will write $\mathcal{D}(A)$ for its domain, $\mathcal{R}(A)$ for its range and $\mathcal{N}(A)$ for its zero space, $\rho(A)$ for its resolvent set and A^* for the adjoint operator of A.

Let $\mathbb{R}^+ = [0, \infty)$. Consider the linear Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \tag{1.1}$$

where $a \in L^1_{loc}(\mathbb{R}^+)$ is scalar kernel $\neq 0$, $f \in \mathbf{C}(J, X)$ and $J = [0, T]$ $(T > 0)$.

Definition 1.1 Let A be closed and densely defined operator on X and $k \in \mathbb{C}(\mathbb{R}^+)$ be a scalar kernel. A family $\{R(t)\}_{t\geq 0}$ is called a k-regularized resolvent family if the following conditions are satisfied.

(a) $R(t)$ is strongly continuous on $\mathbb{R}^+$ and $R(0) = k(0)I$;

(b) $R(t)x \in \mathcal{D}(A)$ and $AR(t)x = R(t)Ax$ for all $x \in \mathcal{D}(A)$ and $t \geq 0$;

(c) the k-regularized resolvent equation holds

$$R(t)x = k(t)x + \int_0^t a(t-s)AR(s)xds, \quad \forall x \in \mathcal{D}(A), t \geq 0.$$

In addition, if there exists constants $M \geq 1, \omega \in \mathbb{R}$, such that $\|R(t)\| \leq Me^{\omega t}$ $(t \geq 0)$, then $\{R(t)\}_{t\geq 0}$ is called an exponentially bounded regularized resolvent family. If $k(t) \equiv 1$, then $\{R(t)\}_{t\geq 0}$ is called a resolvent family (see [3, 10]).

If $a \in L^1_{loc}(\mathbb{R}^+)$ and there exists constant $\omega > 0$ such that $\int_0^\infty e^{-\omega s}|a(s)|ds < \infty$, then the Laplace transform $\hat{a}(\lambda)$ exists for all $Re\lambda > 0$. Throughout this paper, we suppose that the Laplace transform of $a(t), k(t)$ and $R(t)$ exists for all $Re\lambda > 0$. $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$ and $\hat{R}(\lambda)$ denote the Laplace transform of $a(t), k(t)$ and $R(t)$, respectively.

We will have the following assumption for $a \in L^1_{loc}(\mathbb{R}^+)$ and $k \in C(\mathbb{R}^+)$.

(H) There exists that $\epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that for all $0 < t < t_{a,k}$,

$$|\int_0^t a(t-s)k(s)ds| \geq \epsilon_{a,k} \int_0^t |a(t-s)k(s)|ds.$$

and $\frac{1}{|k(t)|}\int_0^t |a(s)|ds \to 0$ $(t \to 0^+)$.

Lemma 1.2 ([6]) *A generates an exponentially bounded k-regularized resolvent family $\{R(t)\}_{t\geq 0}$, and*

$$\lim_{t\to 0^+} \inf \frac{\|R(t)\|}{|k(t)|} < \infty.$$

If the assumption (H) holds, then

(a) For $x \in X$, $\lim_{t\to 0^+}[(k*a)(t)]^{-1}\int_0^t a(t-s)R(s)xds = x$,

(b) For all $x \in \mathcal{D}(A)$, $Ax = \lim_{t\to 0^+}\frac{R(t)x-k(t)x}{(k*a)(t)}$, where

$$\mathcal{D}(A) = \{x \in X : \lim_{t\to 0^+}\frac{R(t)x-k(t)x}{(k*a)(t)} \quad exists\}.$$

Lemma 1.3 *Let A be closed and densely defined operator in the Banach space X. If the resolvent set $\rho(A)$ of A contains $\mathbb{R}_\omega^+ = \{\lambda \in \mathbb{R}, \lambda > \omega\}$, and there exists the constant $M > 0$ such that for $\lambda \in \mathbb{R}_\omega^+$,*

$$\|(I - \hat{a}(\lambda)A)^{-1}\| \leq M(\lambda - \omega)^{-1}, \tag{1.2}$$

then we have

$$\lim_{\lambda\to\infty}(I - \hat{a}(\lambda)A)^{-1}x = x. \tag{1.3}$$

Proof. For all $x \in \mathcal{D}(A)$ and $\lambda > \omega$, we have that

$$\begin{aligned}\|(I - \hat{a}(\lambda)A)^{-1}x - x\| &= \|\hat{a}(\lambda)A(I - \hat{a}(\lambda)A)^{-1}x\| \\ &= \|\hat{a}(\lambda)(I - \hat{a}(\lambda)A)^{-1}Ax\| \leq M(\lambda - \omega)^{-1}\|Ax\| \to 0 \ (\lambda \to \infty).\end{aligned}$$

By (1.2), there exists the constant $M', \tilde{\omega} > \omega$ such that

$$\|(I - \hat{a}(\lambda)A)^{-1}\| \leq M',$$

for $\lambda \in [\tilde{\omega}, +\infty)$. This show that $(I - \hat{a}(\lambda)A)^{-1}$ is the uniform bounded on $\lambda \geq \tilde{\omega}$. Thus, by $\overline{\mathcal{D}(A)} = X$, (1.3) holds for all $x \in X$. □

The following generation theorem of the k-regularized resolvent families will play an important role in the latter proofs (see [4]).

Lemma 1.4 *Let A be closed and densely defined operator in the Banach space X. Then A generates an exponentially bounded k-regularized resolvent family $\{R(t)\}_{t\geq 0}$ if and only if*

(a) $\hat{a}(\lambda) \neq 0$ and $1/\hat{a}(\lambda) \in \rho(A)$, $\forall \lambda > \omega$;

(b) $H(\lambda, A) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ calls the k-resolvent of A and it satisfies the estimates

$$\|H^{(n)}(\lambda, A)\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}} \ \forall \lambda > \omega, \ n \in \mathbb{N}$$

The following Lemma may be found in [9].

Lemma 1.5 *If S is a bounded operator on X, then S^* is a bounded operator on X^* and $\|S\| = \|S^*\|$.*

Lemma 1.6 *Let A be closed and densely defined operator on X. If $1/\hat{a}(\lambda) \in \rho(A)$, then $1/\hat{a}(\lambda) \in \rho(A^*)$ and $H(\lambda, A^*) = H(\lambda, A)^*$.*

Proof. By the definition of the adjoint, we have

$$(I - \hat{a}(\lambda)A)^* = I^* - \hat{a}(\lambda)A^*,$$

where I^* is an unit operator in X^*. Because $H(\lambda, A)$ is a bounded operator, $H(\lambda, A)^*$ also is a bounded operator in X^* by Lemma 1.5. In the following, we will prove that $H(\lambda, A^*)$ exists and is equal to $H(\lambda, A)^*$. We first prove that $I^* - \hat{a}(\lambda)A^*$ is one-to-one. In fact, if $(I^* - \hat{a}(\lambda)A^*)x^* = 0$ for some $x^* \neq 0$, then

$$< (I^* - \hat{a}(\lambda)A^*)x^*, x >=< (I - \hat{a}(\lambda)A)x, x^* >= 0, \quad x \in \mathcal{D}(A).$$

Because $1/\hat{a}(\lambda) \in \rho(A)$,$R(I-\hat{a}(\lambda)A) = X$, $x^* = 0$ and $(I^* - \hat{a}(\lambda)A^*)$ are one-to-one. For $x \in X,\ x^* \in \mathcal{D}(A^*)$, we have

$$\begin{aligned} < x^*, x > &= < x^*, \hat{k}(\lambda)^{-1}(I - \hat{a}(\lambda)A)H(\lambda, A)x > \\ &= < \hat{k}(\lambda)^{-1}(I^* - \hat{a}(\lambda)A^*)x^*, H(\lambda, A)x > \end{aligned}$$

Thus,

$$H(\lambda, A)^*\hat{k}(\lambda)^{-1}(I^* - \hat{a}(\lambda)A^*)x^* = x^*, \quad \forall x^* \in \mathcal{D}(A^*). \tag{1.4}$$

On the other hand, if $x^* \in X^*$ and $x \in \mathcal{D}(A)$, then

$$\begin{aligned} < x^*, x > &= < x^*, \hat{k}(\lambda)^{-1}(I - \hat{a}(\lambda)A)H(\lambda, A)x > \\ &= < \hat{k}(\lambda)^{-1}(I^* - \hat{a}(\lambda)A^*)x^*, H(\lambda, A)x > . \end{aligned}$$

This implies that

$$\hat{k}(\lambda)^{-1}(I^* - \hat{a}(\lambda)A^*)H(\lambda, A)^*x^* = x^*, \quad \forall x^* \in X^*. \tag{1.5}$$

By (1.4) and (1.5), we obtain that $1/\hat{a}(\lambda) \in \rho(A^*)$ and $H(\lambda, A^*) = H(\lambda, A)^*$. □

In the following, we give out an exponential representation formula of the k-regularized resolvent families $\{R(t)\}_{t\geq 0}$(see [5]).

Lemma 1.7 *Let A generate an exponentially bounded k-regularized resolvent family $\{R(t)\}_{t\geq 0}$. Then for $x \in X$, uniformly for t in bounded intervals of $\mathbb{R}^+$ we have*

$$R(t)x = \lim_{n\to\infty} J(\frac{n}{t}, A) = \lim_{n\to\infty} \sum_{k=0}^{n} \sum \begin{pmatrix} n \\ k \end{pmatrix} \frac{k!m!(-1)^{n+m}}{n!i!j!\cdots h!} (\dot{y}(\frac{n}{t})/1!)^i (\ddot{y}(\frac{n}{t})/2!)^j \cdots$$

$$(y^{(l)}(\frac{n}{t})/l!)^h y^{-m-1}(\frac{n}{t}) x^{(n-k)}(\frac{n}{t})(\frac{n}{t})^{n+1}(I - \hat{a}(\frac{n}{t})A)^{-m-1}x, \tag{1.6}$$

where $x(\lambda) = \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}, y(\lambda) = \frac{1}{\hat{a}(\lambda)}, i + 2j + \cdots + lh = k\ ; i + j + \cdots + h = m.$

2. Pseudo-k-resolvent

Definition 2.1 Let Δ be a subset of the complex plane $\mathbb{C}$, $a \in L^1_{loc}(\mathbb{R}^+)$ and $k \in C(\mathbb{R}^+)$. If the bounded linear operator families $L(\lambda) = \hat{k}(\lambda)J(\lambda)$ $(\lambda \in \Delta)$ on X and $J(\lambda)$ satisfies

$$\widehat{a}(\lambda)J(\lambda) - \widehat{a}(\mu)J(\mu) = (\widehat{a}(\lambda) - \widehat{a}(\mu))J(\lambda)J(\mu), \tag{2.1}$$

then $L(\lambda)$ is called a pseudo k-resolvent on Δ.

The main aim in this section is to determine conditions under which there exists a densely defined closed linear operator A such that $L(\lambda)$ is the k-resolvent family of A.

Lemma 2.2. *Let Δ be a subset of the complex plane $\mathbb{C}$. If $L(\lambda)$ is a pseudo k-resolvent on Δ, then $L(\lambda)L(\mu) = L(\mu)L(\lambda)$. The null space $\mathcal{N}(L(\lambda))$ and the range $\mathcal{R}(L(\lambda))$ are independent of $\lambda \in \Delta$, and $\mathcal{N}(L(\lambda))$ is a closed subspace of X.*

Proof. It is evident from (2.1) that $L(\lambda)$ and $L(\mu)$ commute for $\lambda, \mu \in \Delta$ and

$$\widehat{a}(\lambda)J(\lambda) = J(\mu)[\widehat{a}(\mu) + (\widehat{a}(\lambda) - \widehat{a}(\mu))J(\lambda)].$$

Hence $\mathcal{R}(J(\mu)) \subset \mathcal{R}(J(\lambda))$ and $\mathcal{R}(J(\mu)) = \mathcal{R}(J(\lambda))$. Similarly, we may prove that $\mathcal{N}(J(\mu)) = \mathcal{N}(J(\lambda))$. By Definition 2.1, our claims hold. The closedness of $\mathcal{N}(L(\lambda))$ is evident. □

Theorem 2.3. *Let Δ be a subset of the complex plane $\mathbb{C}$. If $L(\lambda)$ is a pseudo k-resolvent on Δ and $\lim_{\lambda\to\infty} J(\lambda)x = x$ for any $x \in X$, then $L(\lambda)$ is the k-resolvent of a unique densely defined closed linear operator A if and only if $\mathcal{N}(L(\lambda)) = \{0\}$ and $\mathcal{R}(L(\lambda))$ is dense in X.*

Proof. We first prove the necessity. If $L(\lambda)$ is a k-resolvent of the closed densely defined linear operator A, then $x = \lim_{n\to\infty} y_n$ for any $x \in X$, where $y_n = J(n)x \in \mathcal{R}(L(n))$. Let $L(\lambda)x = 0$ for $\lambda \in \Delta$. Then $x = \lim_{\lambda\to\infty} J(\lambda)x = 0$ for any $x \in X$. This means that $\mathcal{R}(L(\lambda))$ is dense in X and $\mathcal{N}(L(\lambda)) = \{0\}$.

Now we assume that $\mathcal{N}(L(\lambda)) = \{0\}$ and $\mathcal{R}(L(\lambda))$ is dense in X. From $\mathcal{N}(L(\lambda)) = \{0\}$ we see that $L(\lambda)$ is one-to-one. Let $\lambda_0 \in \Delta$ and set

$$A = \widehat{a}(\lambda_0)^{-1}(I - J(\lambda_0)^{-1}).$$

Then A is linear, closed and $\mathcal{D}(A) = \mathcal{R}(L(\lambda_0))$ is dense in X. Further, we have

$$(I - \widehat{a}(\lambda_0)A)J(\lambda_0) = J(\lambda_0)(I - \widehat{a}(\lambda_0)A) = I.$$

Thus $L(\lambda_0) = \widehat{k}(\lambda_0)(I - \widehat{a}(\lambda_0)A)^{-1}$. If $\lambda \in \Delta$, then

$$\begin{aligned}
&\widehat{a}(\lambda_0)(I - \widehat{a}(\lambda)A)\widehat{a}(\lambda)J(\lambda) \\
=\;& [(\widehat{a}(\lambda_0) - \widehat{a}(\lambda) + \widehat{a}(\lambda)(I - \widehat{a}(\lambda_0)A)]J(\lambda_0)[\widehat{a}(\lambda_0) - (\widehat{a}(\lambda_0) - \widehat{a}(\lambda))J(\lambda)] \\
=\;& \widehat{a}(\lambda_0)\widehat{a}(\lambda) + (\widehat{a}(\lambda_0) - \widehat{a}(\lambda))[\widehat{a}(\lambda_0)J(\lambda_0) - \widehat{a}(\lambda)J(\lambda) \\
&-(\widehat{a}(\lambda_0) - \widehat{a}(\lambda))J(\lambda_0)J(\lambda)] \\
=\;& \widehat{a}(\lambda_0)\widehat{a}(\lambda).
\end{aligned}$$

$\widehat{a}(\lambda_0)^{-1}\widehat{a}(\lambda)^{-1}$ is multiplied in above two side and we obtain that $(I - \widehat{a}(\lambda)A)L(\lambda) = \widehat{k}(\lambda)$. Similarly, we may prove that $L(\lambda)(I - \widehat{a}(\lambda)A) = \widehat{k}(\lambda)$. Thus we have that $L(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ for every $\lambda \in \Delta$. In particular, A is independent of λ_0 and is uniquely by $L(\lambda)$. □

In the following, we give out two useful sufficient conditions for a pseudo k-resolvent to be a k-resolvent.

Theorem 2.4. *Let Δ be an unbounded subset of the complex plane $\mathbb{C}$ and let $L(\lambda)$ be pseudo k-resolvent on Δ. If $\mathcal{R}(L(\lambda))$ is dense in X and there is a sequence $\lambda_n \in \Delta$ such that $|\lambda_n| \to \infty$ and*

$$\|J(\lambda_n)\| \le M \tag{2.2}$$

for some constant M, then $L(\lambda)$ is the k-resolvent of a unique densely defined closed linear operator A.

Proof. From (2.2) it follows that $\|J(\lambda_n)\| \to 0$ as $n \to \infty$. Let $\mu \in \Delta$. From (2.1) we have that

$$\|\widehat{a}(\lambda_0)(J(\lambda_n) - I)J(\mu)\| \to 0 \qquad \text{as } n \to \infty.$$

Therefore if x is in the range of $L(\mu)$,then we obtain that

$$J(\lambda_n)x \to x \qquad \text{as } n \to \infty. \tag{2.3}$$

Since $\mathcal{R}(L(\lambda_n))$ is dense in X and $J(\lambda_n)$ are uniformly bounded, we see that (2.3) holds for every $x \in X$. If $x \in \mathcal{N}(L(\lambda))$, then $J(\lambda_n)x = 0$ and from (2.3) we deduce that $x = 0$. Thus $\mathcal{N}(L(\lambda)) = \{0\}$ and $L(\lambda)$ is the k-resolvent of a densely defined closed operator A by Theorem 2.3. □

Corollary 2.5. Let Δ be an unbounded subset of the complex plane $\mathbb{C}$ and let $L(\lambda)$ be pseudo k-resolvent on Δ. If there is a sequence $\lambda_n \in \Delta$ such that $|\lambda_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} J(\lambda_n)x = x \quad \forall x \in X, \tag{2.4}$$

then $L(\lambda)$ is the k-resolvent of a unique densely defined closed operator A.

Proof. From the uniform boundedness theorem and (2.4) it follows that (2.2) holds. From Lemma 2.2 we know that $\mathcal{R}(L(\lambda))$ is independent of $\lambda \in \Delta$ and from (2.4) we deduce that $\mathcal{R}(L(\lambda))$ is dense in X. Thus, the conditions of Theorem 2.4 hold and $L(\lambda)$ is the k-resolvent of an operator A. □

Remark: In above theorem 2.3 and 2.4, if we take that $k(t) \equiv 1$ and $a(t) \equiv 1$, then we obtain taht the corresponding results on the C_0-semigroup (see [9]).

3. The adjoint of the k-regularized resolvent families

Let $\{R(t)\}_{t\geq 0}$ be a k-regularized resolvent family on a Banach space X. Its adjoint $\{R^*(t)\}_{t\geq 0}$ is the k-regularized resolvent family on the dual space X^* which is obtained from $\{R(t)\}_{t\geq 0}$ by taking pointwise in t the adjoint operators $R^*(t) \equiv (R(t))^*$. It is easy to see that $\{R^*(t)\}_{t\geq 0}$ is also a k-regularized resolvent family. In general, if $\{R(t)\}_{t\geq 0}$ be a k-regularized resolvent family, then

$$| < x, R^*(t)x^* - k(t)x^* > | = | < (R(t)x - k(t)x, x^* > | \leq \|x^*\|\|R(t)x - k(t)x\|$$

shows that $R^*(t)$ is weak*-continuous but it needn't be strongly continuous. It is well-known that if A is closed and densely defined, then A^* is a weak*-densely defined, weak*-closed operator (see[7]). But we have the following results.

Proposition 3.1 If A generate an exponentially bounded k-regularized resolvent family $\{R(t)\}_{t\geq 0}$ on X and $\{R^*(t)\}_{t\geq 0}$ be its dual operator on x^*, then $\{R^*(t),\ t \geq 0\}$ is the strongly continuous on $\mathcal{D}(A^*)$ and $R^*(0) = k(0)I^*$.

Proof. Let $x^* \in \mathcal{D}(A^*)$. Then by Definition 1.1(c) we have

$$| < x, R^*(t)x^* - k(t)x^* > | = | < (a * R)(t)x, A^*x^* > |.$$

Because $a(t)$ and $R(t)$ are the exponentially bounded and strongly continuous, $\{R^*(t),\ t \geq 0\}$ is the strongly continuous on $\mathcal{D}(A^*)$ and $R^*(0)x^* = k(0)I^*$. □

Proposition 3.2 Let A generate a k-regularized resolvent family $\{R(t)\}_{t\geq 0}$ on X and $\{R^*(t)\}_{t\geq 0}$ be its dual operator on X^*. Then the following results hold.

(a) $R^*(t)x^* \in \mathcal{D}(A^*)$ and $A^*R^*(t)x^* = R^*(t)A^*x^*,\ \forall x^* \in \mathcal{D}(A^*),\ t \geq 0$;

(b) For all $t > 0$ and $x^* \in X^*$, $\omega^* - \int_0^t a(t-s)R^*(s)x^*ds \in \mathcal{D}(A^*)$ and

$$A^*(\omega^* - \int_0^t a(t-s)R^*(s)x^*ds) = R^*(t)x^* - k(t)x^*. \tag{3.1}$$

In addition, if $x^* \in \mathcal{D}(A^*)$, then

$$A^*(\omega^* - \int_0^t a(t-s)R^*(s)x^* ds) = \omega^* - \int_0^t a(t-s)R^*(s)A^*x^* ds. \qquad (3.2)$$

Proof.(a) If $\forall x^* \in D(A^*), x \in \mathcal{D}(A)$ and $t \geq 0$, then by Definition 1.1(b) we obtain

$$\begin{aligned} < R^*(t)x^*, Ax > &= < x^*, R(t)Ax > = < x^*, AR(t)x > \\ &= < A^*x^*, R(t)x > = < R^*(t)A^*x^*, x > . \end{aligned}$$

Hence $R^*(t)x^* \in \mathcal{D}(A^*)$ and $A^*R^*(t)x^* = R^*(t)A^*x^*$.

(b) $\forall x \in \mathcal{D}(A)$, by Definition 1.1(c) we have

$$\begin{aligned} < \omega^* - \int_0^t a(t-s)R^*(s)x^* ds, Ax > &= \int_0^t < a(t-s)R^*(s)x^*, Ax > ds \\ &= \int_0^t < x^*, a(t-s)R(s)Ax > ds \\ &= < x^*, \int_0^t a(t-s)R(s)Axds > \\ &= < x^*, A\int_0^t a(t-s)R(s)xds > \\ &= < x^*, R(t)x - k(t)x > \\ &= < R^*(t)x^* - k(t)x^*, x > . \end{aligned}$$

This means that $\omega^* - \int_0^t a(t-s)R^*(s)x^* ds \in \mathcal{D}(A^*)$ and (3.1) holds. Similarly, we may prove that (3.2) holds. □

It is simple to see that $R^*(t)$ leaves $\mathcal{D}(A^*)$ invariant from Proposition 3.2(a).

Let $U^*(t)$ be an exponentially bounded $\omega^* - k$-regularized resolvent family on X^*. The ω^*-generator of $U^*(t)$ is the linear operator B on X^* defined by

$$\begin{cases} \mathcal{D}(B) = \{x^* \in X^*, \omega^* - \lim_{t\to 0} \frac{U^*(t)x^* - k(t)x^*}{(k*a)(t)} \text{ exist}\}, \\ Bx^* = \omega^* - \lim_{t\to 0} \frac{U^*(t)x^* - k(t)x^*}{(k*a)(t)}, \forall x^* \in \mathcal{D}(B). \end{cases}$$

Theorem 3.2 *Let $\{R(t), t \geq 0\}$ be an exponentially bounded k-regularized resolvent family on X with the generator A and $\{R^*(t), t \geq 0\}$ be $\omega^* - k$-regularized resolvent family on x^*. If the assumption condition (H) holds, then A^* is the ω^*-generator of $R^*(t)$.*

Proof. Let B be the ω^*-generator of $R^*(t)$ and for the fixed $x^* \in \mathcal{D}(A^*)$ and any $x \in X$. From Proposition 3.1 and Lemma 1.2 we have

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{(k*a)(t)} < R^*(t)x^* - k(t)x^*, x > \\ &= \lim_{t\downarrow 0} \frac{1}{(k*a)(t)} < A^*(\omega^* - \int_0^t a(t-s)R^*(s)x^* ds), x > \\ &= \lim_{t\downarrow 0} \frac{1}{(k*a)(t)} < A^*x^*, \int_0^t a(t-s)R(s)xds > \\ &= < A^*x^*, x > . \end{aligned}$$

Hence $\omega^* - \lim\limits_{t\downarrow 0} \frac{1}{(k*a)(t)}(R^*(t)x^* - k(t)x^*)$ exists and equals A^*x^*. This shows that $x^* \in \mathcal{D}(B)$ and $Bx^* = A^*x^*$, and therefore $A^* \subset B$. To prove the converse inclusion, for the fixed $x^* \in \mathcal{D}(B)$ and any $x \in \mathcal{D}(A)$,

$$< Bx^*, x >= \lim_{t\to 0} \frac{1}{(k*a)(t)} < R^*(t)x^* - k(t)x^*, x >=< x^*, Ax > .$$

This shows that $x^* \in \mathcal{D}(A^*)$ and $A^*x^* = Bx^*$, and therefore $B \subset A^*$. Thus $B = A^*$. □

We will write that

$$X^{\odot} = \{x^* \in X^*, \lim_{t\to 0} ||R^*(t)x^* - k(t)x^*|| = 0\}.$$

It follows trivially from this definition that $X^{\odot}$ is $R^*(t)$-invariant, which by above definition means that $R^*(t)X^{\odot} \subset X^{\odot}$ holds for all $t \geq 0$. Also, $X^{\odot}$ is a closed subspace of X^*.

Theorem 3.3 *Let $\{R(t), t \geq 0\}$ be an exponentially bounded k-regularized resolvent family on X with the generator A and $\{R^*(t), t \geq 0\}$ be its adjoint operator family. If the assumption condition (H) holds, then $X^{\odot} = \overline{\mathcal{D}(A^*)}$.*

Proof. Let $x^* \in \mathcal{D}(A^*)$. Then for any $x \in X$ we have from (3.2)

$$\begin{aligned} < R^*(t)x^* - k(t)x^*, x > &= \Big\langle A^*(\omega^* - \int_0^t a(t-s)R^*(s)x^* ds), x\Big\rangle \\ &= \int_0^t < a(t-s)R^*(s)A^*x^*, x > ds. \end{aligned}$$

Hence

$$\begin{aligned} \|R^*(t)x^* - k(t)x^*\| &= \sup\{| < R^*(t)x^* - k(t)x^*, x > |; x \in X, ||x|| \leq 1\} \\ &= \sup\{| \int_0^t < a(t-s)R^*(s)A^*x^*, x > ds|; x \in X, ||x|| \leq 1\} \\ &\leq t(\sup_{0\leq s\leq t} ||a(t-s)R(s)||)||A^*x^*|| \to 0 \ \ (t \to 0) \end{aligned}$$

which shows that $\mathcal{D}(A^*) \subset X^{\odot}$. Since $X^{\odot}$ is the closed subset of X^*, $\overline{\mathcal{D}(A^*)} \subset X^{\odot}$.

For the converse inclusion, let $x^{\odot} \in X^{\odot}$. Then for the sufficiently small $\varepsilon > 0$ and $t \in (0, 1)$ we have $||\frac{R^*(t)}{k(t)}x^{\odot} - x^{\odot}|| < \varepsilon$. Thus for any $x \in X$ we have

$$\begin{aligned} &\Big\langle \frac{1}{(k*a)(t)} \int_0^t a(t-s)R^*(s)x^{\odot} ds - x^{\odot}, x\Big\rangle \\ = \ &\frac{1}{(k*a)(t)} \int_0^t a(t-s)k(s)\Big\langle \frac{R^*(s)}{k(s)}x^{\odot} - x^{\odot}, x\Big\rangle ds. \end{aligned}$$

Hence by the assumption condition (H) we obtain

$$||\frac{1}{(k*a)(t)} \int_0^t a(t-s)R^*(s)x^{\odot} ds - x^{\odot}|| \leq \frac{\varepsilon}{|(k*a)(t)|} \int_0^t |a(t-s)k(s)| ds < \frac{\varepsilon}{\varepsilon_{a,k}}.$$

Thus by Proposition 3.1 we have show that $x^{\odot} \in \overline{\mathcal{D}(A^*)}$. □

Let $A^{\odot} = A^*|_{X^{\odot}}$ denotes the part of A^* in $X^{\odot}$ and $\mathcal{D}(A^{\odot}) = \{x^* \in \mathcal{D}(A^*), A^*x^* \in X^{\odot}\}$. We will write that $R^{\odot}(t) = R^*(t)|_{X^{\odot}}$

Theorem 3.4 *Let $\{R(t), t \geq 0\}$ be an exponentially bounded k-regularized resolvent family on X with the generator A, $\{R^*(t)\}_{t\geq 0}$ be its adjoint operator on X^* and A^* is the adjoint of A. If the assumption condition (H) holds and $\rho(A) \neq \Phi$, then $\{R^{\odot}(t), t \geq 0\}$ is a k-regularized resolvent family on $X^{\odot}$, and $A^{\odot}$ is its generator.*

Proof. By the known condition and Lemma 1.4, there are constants ω and μ such that for all $\lambda > \omega$, $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ and

$$||H^{(n)}(\lambda, A)|| \leq Mn!(\lambda - \omega)^{-(n+1)}, \quad n = 0, 1, 2, \cdots \tag{3.3}$$

Also, from Lemma 1.5 and Lemma 1.6 and (3.3) it follows that if $\lambda > \omega$, $\frac{1}{\hat{a}(\lambda)} \in \rho(A^*)$ and

$$||H^{(n)}(\lambda, A^*)|| \leq Mn!(\lambda - \omega)^{-(n+1)}, \quad n = 0, 1, 2, \cdots \tag{3.4}$$

Let $H(\lambda, A^{\odot})$ is the part of $H(\lambda, A^*)$ in $X^{\odot}$. Thus from (3.4) we have

$$||H^{(n)}(\lambda, A^{\odot})|| \leq Mn!(\lambda - \omega)^{-(n+1)}, \quad n = 0, 1, 2, \cdots \tag{3.5}$$

where $H(\lambda, A^{\odot}) = \hat{k}(\lambda)(I - \hat{a}(\lambda)A^{\odot})^{-1}$. Also,a direct computation show that for $\lambda, \mu > \omega$

$$\begin{aligned} & \hat{a}(\lambda)\hat{k}(\lambda)^{-1}H(\lambda, A^{\odot}) - \hat{a}(\mu)\hat{k}(\mu)^{-1}H(\mu, A^{\odot}) \\ = \; & [\hat{a}(\lambda) - \hat{a}(\mu)]\hat{k}(\lambda)^{-1}\hat{k}(\mu)^{-1}H(\lambda, A^{\odot})H(\mu, A^{\odot}) \end{aligned} \tag{3.6}$$

It follows that from Lemma 1.3

$$\lim_{\lambda \to \infty} \hat{k}(\lambda)^{-1}H(\lambda, A^{\odot})x^* = x^*, \quad \forall x^* \in X^{\odot}. \tag{3.7}$$

From (3.6), (3.7) and Corollary 2.5 we see that $H(\lambda, A^{\odot})$ is a $k-$resolvent of the closed densely defined operator $A^{\odot}$ in $X^{\odot}$. From (3.5) and Lemma 1.4 it follows that $A^{\odot}$ is the generator of a k-regularized resolvent family $\{R^{\odot}(t), t \geq 0\}$ in $X^{\odot}$. For $x \in X$ and $x^* \in X^{\odot}$ we have

$$< x^*, (I - \hat{a}(\frac{n}{t})A)^{-m-1}x > = < (I - \hat{a}(\frac{n}{t})A^{\odot})^{-m-1}x^*, x >, \tag{3.8}$$

Let $n \to \infty$ in (3.8) and by Lemma 1.7 we obtain

$$< x^*, R(t)x > = < R^{\odot}(t)x^*, x > .$$

Thus for $x^* \in X^{\odot}$, $R^*(t)x^* = R^{\odot}(t)x^*$ and $R^{\odot}(t) = R^*(t)|_{X^{\odot}}$.

In the following, we will prove that $A^{\odot} = A^*|_{X^{\odot}}$. Let $x^{\odot} \in \mathcal{D}(A^{\odot})$. Then we have

$$\lim_{t\to 0} \frac{1}{(k * a)(t)}(R^*(t)x^{\odot} - k(t)x^{\odot}) = \lim_{t\to 0} \frac{1}{(k * a)(t)}(R^{\odot}(t)x^{\odot} - k(t)x^{\odot}) = A^{\odot}x^{\odot},$$

where the limit is the strongly sense. Hence these limits exist also in the weak*-sense. By Theorem 3.2, we see that $x^{\odot} \in \mathcal{D}(A^*)$ and $A^*x^{\odot} = A^{\odot}x^{\odot} \in X^{\odot}$. Thus $A^{\odot} \subset A^*|_{X^{\odot}}$.

To prove the converse inclusion, let $x^* \in \mathcal{D}(A^*|_{X^\odot})$. Then we have $x^* \in \mathcal{D}(A^*)$ and $A^*x^* \in X^\odot$. But this implies that

$$\begin{aligned}\frac{1}{(k*a)(t)}(R^\odot(t)x^* - k(t)x^*) &= \frac{1}{(k*a)(t)}(R^\odot(t)x^* - k(t)x^*)\\ &= \frac{1}{(k*a)(t)}A^*(\omega^* - \int_0^t a(t-s)R^*(s)x^*ds)\\ &= \frac{1}{(k*a)(t)}(\omega^* - \int_0^t a(t-s)R^*(s)A^*x^*ds)\\ &= \frac{1}{(k*a)(t)}\int_0^t a(t-s)R^*(s)A^*x^*ds.\end{aligned}$$

Since $A^*x^* \in X^\odot$, letting $t \to 0$ in the above integrand of the last integral, we obtain from Theorem 3.3,

$$\lim_{t\to 0}(k*a)^{-1}(t)(R^\odot(t)x^* - k(t)x^*) = A^*x^*.$$

This show that $x^* \in \mathcal{D}(A^\odot)$ and $A^\odot x^* = A^*x^*$. Thus $A^*|_{X^\odot} \subset A^\odot$. □

By Lemma 1.5, if X is reflexive, then A^* is a closed densely defined operator in X^*. Thus we have the following corollary.

Corollary 3.5. Let X is a reflexive Banach space and Let $\{R(t), t \geq 0\}$ be an exponentially bounded k-regularized resolvent family on X with the generator A. Then $\{R^*(t)\}_{t\geq 0}$ is an exponentially bounded k-regularized resolvent family on X^* whose the generator is A^* the adjoint of A, where $\{R^*(t)\}_{t\geq 0}$ is the adjoint operator family of $\{R(t)\}_{t\geq 0}$.

From Corollary 3.5 we know that if X is a reflexive, then $X^\odot = X^*$, $A^\odot = A^*$ and $R^\odot(t) = R^*(t)$.

Remark 1. If we take that $k(t) \equiv 1$ and $a(t) \equiv 1$ or $a(t) \equiv t$ in Theorem 3.4, then we may obtain the dual theorems of the C_0-semigroup (see [9])and the strongly continuous cosine function. If $k(t) \equiv 1$ or $t^n/n!$, then we may obtain the dual theorems of the resolvent operator family or n-times integral resolvent family (see [8]).

4. The Favard class of $R(t)$ and $R^*(t)$

Definition 4.1. Let $\{R(t), t \geq 0\}$ be an exponentially bounded k-regularized resolvent family on X and $\{R^*(t), t \geq 0\}$ be its adjoint operator family X^*. Their Favard class are defined by

$$Fav(R) = \{x \in X;\ \limsup_{t\to 0}[(k*a)(t)]^{-1}\|R(t)x - k(t)x\| < \infty\}$$

and

$$Fav(R^*) = \{x^* \in X^*;\ \limsup_{t\to 0}[(k*a)(t)]^{-1}\|R^*(t)^*x^* - k(t)x^*\| < \infty\}.$$

If we introduce the norm on $Fav(R)$

$$\|x\|_{Fav(R)} = \|x\| + \overline{\lim}_{t\to 0}[(k*a)(t)]^{-1}\|R(t)x - k(t)x\|,\ \forall x \in Fav(R).$$

then $[Fav(R)] = (Fav(R), \|\cdot\|_{Fav(R)})$ become a Banach space. Similarly, we may define the space $[Fav(R^*)]$.

Starting from the k-regularized resolvent family $R^{\odot}(t)$, the duality construction can be repeated. We define $R^{\odot *}(t)$ to be the adjoint of $R^{\odot}(t)$ and write $X^{\odot\odot}$ for $(X^{\odot})^{\odot}$. X is isometrically isomorphic to a subspace $(X^{\odot})^{\odot}$. We identify it with X. If this subspace is all of $(X^{\odot})^{\odot}$, we call X $\odot$-reflexive.

Theorem 4.2. *If the assumption condition H holds, then we have the following assertions.*

(a) $Fav(R) = \{x \in X; \exists\ M' > 0$ and $x_n \in \mathcal{D}(A),$ s.t. $\|Ax_n\| \leq M'$ and $x_n \to x\}$.

(b)$Fav(R^{\odot *}) = \mathcal{D}(A^*)$. In particular, if X is reflexive, then $Fav(R) = \mathcal{D}(A)$.

(c)$Fav(R) = \mathcal{D}(A^{\odot *}) \cap X$. In particular, if X is $\odot$-reflexive, then $Fav(R) = \mathcal{D}(A^{\odot *})$.

Proof. (a) If $x \in Fav(R)$, then by Lemma 1.2, the assumption condition H and Definition 4.1 we have $x_n \in \mathcal{D}(A)$ and

$$x_n = \frac{1}{(k*a)(\frac{1}{n})}\int_0^{\frac{1}{n}} a(\frac{1}{n}-s)R(\frac{1}{n})xds \to x\ (n \to +\infty).$$

Hence

$$\begin{aligned}\|Ax_n\| &= \|A\frac{1}{(k*a)(\frac{1}{n})}\int_0^{\frac{1}{n}} a(\frac{1}{n}-s)R(\frac{1}{n})xds\| \\ &= \|\frac{1}{(k*a)(\frac{1}{n})}[R(\frac{1}{n})x - k(\frac{1}{n})x]\| \\ &= \|\frac{1}{(k*a)(\frac{1}{n})}O(\frac{1}{n})\| \\ &\leq \frac{1}{\epsilon_{a,k}\int_0^t |a(t-s)k(s)|ds}O(\frac{1}{n}) = M'.\end{aligned}$$

Conversely, if $x_n \in \mathcal{D}(A)$ and $x_n \to x$, $\|Ax_n\| \leq M'$, then for $t \to 0$

$$\begin{aligned}\|R(t)x - k(t)x\| &= \lim_{n\to\infty}\|R(t)x_n - k(t)x_n\| \\ &= \|\int_0^t a(t-s)AR(s)x_n ds\| \\ &\leq M'\int_0^t M^2e^{2\omega s}ds \\ &= O(t).\end{aligned}$$

(b) If $x^* \in \mathcal{D}(A^*)$ and $x \in X$, then we have

$$\begin{aligned}|\langle x, R(t)^* - k(t)x^*\rangle| &= |\langle (a*R)(t)x, A^*x^*\rangle| \\ &\leq M|\int_0^t a(t-s)e^{\omega t}ds|\|x\|\|A^*x^*\|.\end{aligned}$$

Therefore $\|R(t)^*x^* - k(t)x^*\| = O(t)$ as $t \to 0$, i.e. $x^* \in Fav(R^*)$.

Conversely, if $x^* \in Fav(R^*)$, then from Lemma 1.2 we have $\frac{1}{(k*a)(t)}\int_0^t a(t-s)R(t)xds \to x$ for $t \to 0^+$. Hence we obtain that $|\langle Ax, x^*\rangle| \leq M\|x\|$ for $x \in \mathcal{D}(A)$. Therefore $x^* \in \mathcal{D}(A^*)$. If X is reflexive, then we have that $A^{**} = A$ and $R^{**}(t) = R(t)$ and hence $Fav(R) = \mathcal{D}(A)$.

(c) If $x \in \mathcal{D}(A^{\odot *}) \cap X$, then if for any $y \in \overline{\mathcal{D}(A^*)}$ we have

$$\begin{aligned} |\langle x, R(t)^* y - k(t)y\rangle| &= |\langle x, R(t)^{\odot *} y - k(t)y\rangle| \\ &= \|A^{\odot *}x\| \int_0^t |a(t-s)| \|R(s)\| \|y\| ds. \end{aligned}$$

Hence

$$\|R(t)x - k(t)x\| \leq M_x | \int_0^t a(t-s)ds|,$$

i.e. $x \in Fav(R)$.

Conversely, if $x \in Fav(R)$, then from Lemma 1.2 we have

$$|\langle x, A^{\odot} y^*\rangle| \leq \|y^*\| \sup_{0<t<1} \{(k * a)(t)\}^{-1} \|R(t)x - k(t)x\|$$

for any $y^* \in \mathcal{D}(A^{\odot})$. Thus $|\langle x, A^{\odot} y^*\rangle| \leq M_x \|y^*\|$, i.e. $x \in \mathcal{D}(A^{\odot *})$. If X is reflexive, then we see from (b) that $Fav(R(t)) = \mathcal{D}(A^{\odot *})$. □

Remark 2. If we take that $k(t) \equiv 1$ and $a(t) \equiv 1$ or $a(t) \equiv t$ in Theorem 4.2, then we obtain the correlative result of the *Favard* class on C_0-semigroup (see [1, 2]) and strongly continuous cosine function.

References

[1] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Berlin, Springer-Verlag, 1967.

[2] P. Clement, O. Diekmann, M. Gyllenberg, H. J. A. M. Heimans and H. R. Thieme, *Perturbation theory for dual semigroups I:The sun-reflexive case*, Math. Ann., 277(1987), 709-725.

[3] G. Da Prato and M. Iannelli, *Linear integro-differential equation in Banach space*, Rend. Sem. Math. Padova, 62(1980), 207-219.

[4] C. Lizama, *Regularized solutions for Volterra equations*, J. Math. Anal. Appl., 243(2000), 278-292.

[5] C. Lizama, *A representation formula for strongly continuous resolvent families*, J. Integral Equs Appl., 9(1997), 278-292.

[6] C. Lizama and J. Sánchez. *On perturbantion of K-regularized resolvent families*, Taiwanese J. Math., 7(2003), 217-227.

[7] C. Lizama and P. Humberto, *On duality and spectral properties of (a, k)-regularized resolvents*, Proc. Roy. Soc. Edinburgh, 139A(2009), 505-517.

[8] M. Jung, *Duality theory for solutions to Volterra integral equation*, J. Math. Anal. Appl., 230(1999), 112-134.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[10] J. Prüss, *Evolutionary Integral Equations anD Applications*, Monographs in Mathematics, 87, Birkhaüser Verlag, 1993.