Journal of Applied Analysis and Computation
Volume 1, Number 1, February 2011 Website:http://jaac-online.com/
pp. 143–153

A STUDY ON ZOLADEK'S EXAMPLE*

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Abstract In this paper, we consider an example of third-order polynomial planar system, proposed by Zoladek who claimed that this example had eleven small-amplitude limit cycles around a center. We use focus value computation to show that for this example there may exist maximal nine small-amplitude limit cycles around the center due to Hopf bifurcation.

Keywords Hilbert's 16th problem, integrable system, limit cycle, focus value, Hopf bifurcation.

MSC(2000) 34C07, 34C23.

1. Introduction

The second part of the well-known Hilbert's 16th problem [1] is to consider the existence of limit cycles in planar polynomial vector fields. This is a very difficult problem, not completely solved even for quadratic systems after more than 100 years since Hilbert proposed the problem. If the problem is restricted to the vicinity of isolated singular points, it becomes a particular version to estimate the number of small-amplitude limit cycles (or small limit cycles) bifurcating from an elementary center or an elementary focus. This is equivalent to studying degenerate Hopf bifurcations, and the main task becomes computing the so-called *focus values* of the point and determining center conditions. In the past six decades, many researchers have considered the local problem and obtained many results. Bautin [2] first proved that quadratic systems can have maximal three small limit cycles around a center or a focus point. For cubic systems, many investigations have shown that in the vicinity of a singular point the number of small-amplitude limit cycles can be five [4], six [5], seven [3, 5, 6], eight [7, 8], and nine [8]. On the other hand, the number of limit cycles existed in multiple singular points for cubic planar polynomial systems can be ten [9], eleven [10, 11], twelve [12, 13, 14], and thirteen [15, 16, 17]. More information about the Hilbert's 16th problem may be found in the survey article [10]. It should be pointed out that the nine small-amplitude limit cycles given in [8] are obtained by perturbing an elementary center (linear center) of general cubic systems, while the nine small-amplitude limit cycles obtained in this paper are obtained by perturbing a center of an integrable system with cubic polynomials.

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*This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the National Natural Science Foundation of China (NNSFC).

In this paper, we will pay particular attention to the example of third-order planar integral system, proposed by Zoladek [18] who claimed that this system could have eleven small-amplitude limit cycles around a center. The example is related to the following equations:

$$\begin{aligned} \dot{x} &= x^3 + x\,y + \tfrac{5}{2}\,x + a, \\ \dot{y} &= -\,a\,x^3 + 6\,x^2\,y - 3\,x^2 + 4\,y^2 + 2\,y - 2\,a\,x, \end{aligned} \tag{1.1}$$

where a is a parameter. System (1.1) has a rational Darboux integral:

$$H = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4\,x^2 + 4\,y)^5}{(x^5 + 5\,x^3 + 5\,x\,y + \frac{5}{2}\,x + a)^4}, \tag{1.2}$$

and the integrating factor of system (1.1) is $M = 20\,f_1^4\,f_2^{-5}$. For $a < -\,2^{5/4}$, system (1.1) has a center at

$$\mathrm{C}_0 = \Big(-\frac{a}{2}, -\frac{a^2+2}{4}\Big), \tag{1.3}$$

and five (real or complex) fixed points at $(x,y) = (r,\ -\frac{2\,r^3+5\,r+2\,a}{2\,r})$, where r is the root of polynomial equation: $r^5 - 10\,r - 4\,a = 0$. In addition, system (1.1) has a saddle point and a non-elementary point at infinity. One example of the phase portrait of system (1.1) for $a = -\,4$ is shown in Figure 1.

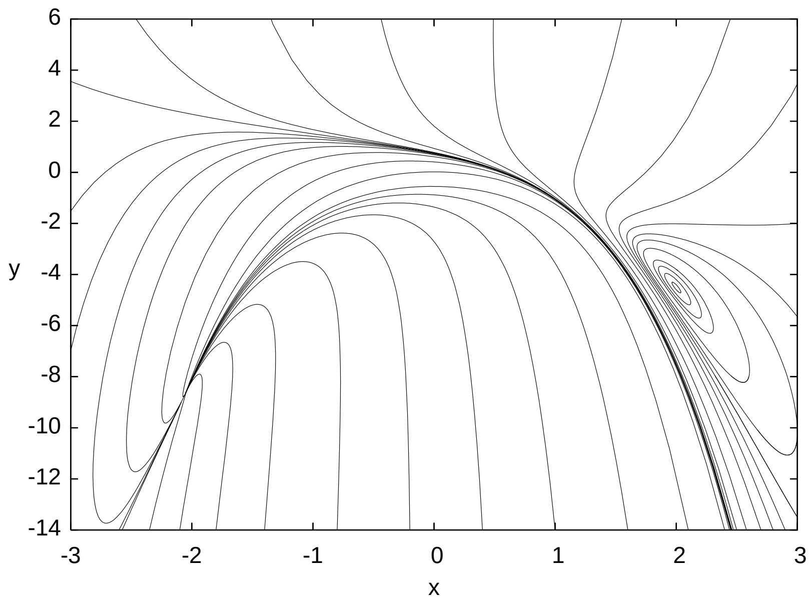

Figure 1. A phase portrait of system (1.1) when $a = -\,4$.

In [18], the author used second-order Poincaré-Pontriagin integral (or Abelian integral) to show that there exist eleven small-amplitude limit cycles around the center C_0. In this paper, we shall use focus value computation to show by perturbing system (1.1) with cubic polynomials that there may exist maximal nine small-amplitude limit cycles around C_0 due to Hopf bifurcation.

To consider perturbing system (1.1), we add cubic polynomial perturbations to system (1.1) up to ε^2, as follows:

$$\begin{aligned}\dot{x} &= x^3 + x\,y + \tfrac{5}{2}\,x + a + \varepsilon\,p(\varepsilon,x,y),\\ \dot{y} &= -\,a\,x^3 + 6\,x^2\,y - 3\,x^2 + 4\,y^2 + 2\,y - 2\,a\,x + \varepsilon\,q(\varepsilon,x,y),\end{aligned}\tag{1.4}$$

where $p(\varepsilon,x,y)$ and $q(\varepsilon,x,y)$ are given by

$$\begin{aligned}p(\varepsilon,x,y) = \;& a_{001} + a_{101}\,x + a_{011}\,y + a_{201}\,x^2 + a_{111}\,x\,y + a_{021}\,y^2\\ & + a_{301}\,x^3 + a_{211}\,x^2\,y + a_{121}\,x\,y^2 + a_{031}\,y^3\\ & + \varepsilon\Big[a_{002} + a_{102}\,x + a_{012}\,y + a_{202}\,x^2 + a_{112}\,x\,y + a_{022}\,y^2\\ & \quad + a_{302}\,x^3 + a_{212}\,x^2\,y + a_{122}\,x\,y^2 + a_{032}\,y^3\Big],\\ q(\varepsilon,x,y) = \;& b_{001} + b_{101}\,x + b_{011}\,y + b_{201}\,x^2 + b_{111}\,x\,y + b_{021}\,y^2\\ & + b_{301}\,x^3 + b_{211}\,x^2\,y + b_{121}\,x\,y^2 + b_{031}\,y^3\\ & + \varepsilon\Big[b_{002} + b_{102}\,x + b_{012}\,y + b_{202}\,x^2 + b_{112}\,x\,y + b_{022}\,y^2\\ & \quad + b_{302}\,x^3 + b_{212}\,x^2\,y + b_{122}\,x\,y^2 + b_{032}\,y^3\Big].\end{aligned}\tag{1.5}$$

It is easy to show that under the following conditions:

$$\begin{aligned}a_{00k} = \;& \tfrac{1}{64}\Big[32\,a\,a_{10k} + 16(a^2+2)\,a_{01k} - 16a^2\,a_{20k} - 8a(a^2+2)\,a_{11k}\\ & -4(a^2+2)^2 a_{02k} + \,8a^3\,a_{30k} + 4a^2(a^2+2)\,a_{21k}\\ & +2a(a^2+2)^2 a_{12k} + (a^2+2)^3 a_{03k}\Big],\\ b_{00k} = \;& \tfrac{1}{64}\Big[32\,a\,b_{10k} + 16(a^2+2)\,b_{01k} - 16a^2\,b_{20k} - 8a(a^2+2)\,b_{11k}\\ & -4(a^2+2)^2 b_{02k} + \,8a^3\,b_{30k} + 4a^2(a^2+2)\,b_{21k}\\ & +2a(a^2+2)^2 b_{12k} + (a^2+2)^3 b_{03k}\Big],\\ b_{01k} = \;& -\,a_{10k} + \tfrac{1}{16}\Big[16a\,a_{20k} + 4(a^2+2)a_{11k} - 12a^2 a_{30k} - 4(a^2+2)a_{21k}\\ & -(a^2+2)^2 a_{12k} + 8a\,b_{11k} + 8(a^2+2)b_{02k} - 4a^2 b_{21k}\\ & -4a(a^2+2)b_{12k} - 3(a^2+2)^2 b_{03k}\Big]\\ \equiv \;& a^*_{01k},\end{aligned}\tag{1.6}$$

where $k = 1,\,2$, then the perturbed system (1.4) still has the same center C_0. Now, applying a translation $x = -\frac{a}{2} + \bar{x},\ y = -\frac{a^2+2}{4} + \bar{y}$, and a linear transformation to system (1.4), we obtain the following new system:

$$\begin{aligned}\dot{\bar{x}} &= \omega_c\,\bar{y} + \tfrac{a^2-8}{\sqrt{2a^4-64}}\,\bar{x}^2 + \bar{x}\,\bar{y} + \tfrac{a^2}{\sqrt{2a^4-64}}\,\bar{x}^3 + \varepsilon\,\bar{p}(\varepsilon,\bar{x},\bar{y}),\\ \dot{\bar{y}} &= -\,\omega_c\,\bar{x} - 3\,\bar{x}^2 - \tfrac{2(a^2+28)}{\sqrt{2a^4-64}}\,\bar{x}\,\bar{y} + 4\,\bar{y}^2 - \tfrac{4a^2(a^2+5)}{a^4-32}\,\bar{x}^3 + \tfrac{6\,a^2}{\sqrt{2a^4-64}}\,\bar{x}^2\bar{y} + \varepsilon\,\bar{q}(\varepsilon,\bar{x},\bar{y}),\end{aligned}\tag{1.7}$$

in which

$$\omega_c = \frac{1}{2\sqrt{2(a^2+2)}}\Big[(a^4-32) + \omega_1\,\varepsilon + \omega_2\,\varepsilon^2 + \omega_3\,\varepsilon^3 + \cdots\Big]^{1/2}$$

and

$$\begin{aligned} \bar{p}(\varepsilon,\bar{x},\bar{y}) = \;& \bar{a}_{101}\,\bar{x} + \bar{a}_{011}\,\bar{y} + \bar{a}_{201}\,\bar{x}^2 + \bar{a}_{111}\,\bar{x}\,\bar{y} + \bar{a}_{021}\,\bar{y}^2 \\ & + \bar{a}_{301}\,\bar{x}^3 + \bar{a}_{211}\,\bar{x}^2\,\bar{y} + \bar{a}_{121}\,\bar{x}\,\bar{y}^2 + \bar{a}_{031}\,\bar{y}^3 \\ & + \varepsilon\Big[\bar{a}_{102}\,\bar{x} + \bar{a}_{012}\,\bar{y} + \bar{a}_{202}\,\bar{x}^2 + \bar{a}_{112}\,\bar{x}\,\bar{y} + \bar{a}_{022}\,\bar{y}^2 \\ & \quad + \bar{a}_{302}\,\bar{x}^3 + \bar{a}_{212}\,\bar{x}^2\,\bar{y} + \bar{a}_{122}\,\bar{x}\,\bar{y}^2 + \bar{a}_{032}\,\bar{y}^3\Big] \\ & + \varepsilon^2\big[\cdots\big] + \varepsilon^3\big[\cdots\big] + \cdots \\ \bar{q}(\varepsilon,\bar{x},\bar{y}) = \;& \bar{b}_{101}\,\bar{x} + \bar{b}_{011}\,\bar{y} + \bar{b}_{201}\,\bar{x}^2 + \bar{b}_{111}\,\bar{x}\,\bar{y} + \bar{b}_{021}\,\bar{y}^2 \\ & + \bar{b}_{301}\,\bar{x}^3 + \bar{b}_{211}\,\bar{x}^2\,\bar{y} + \bar{b}_{121}\,\bar{x}\,\bar{y}^2 + \bar{b}_{031}\,\bar{y}^3 \\ & + \varepsilon\Big[\bar{b}_{102}\,\bar{x} + \bar{b}_{012}\,\bar{y} + \bar{b}_{202}\,\bar{x}^2 + \bar{b}_{112}\,\bar{x}\,\bar{y} + \bar{b}_{022}\,\bar{y}^2 \\ & \quad + \bar{b}_{302}\,\bar{x}^3 + \bar{b}_{212}\,\bar{x}^2\,\bar{y} + \bar{b}_{122}\,\bar{x}\,\bar{y}^2 + \bar{b}_{032}\,\bar{y}^3\Big] \\ & + \varepsilon^2\big[\cdots\big] + \varepsilon^3\big[\cdots\big] + \cdots \end{aligned} \tag{1.8}$$

Here, ω_j, $\bar{a}_{ijk}$ and $\bar{b}_{ijk}$ are explicitly expressed in terms of the original coefficients a_{ijk}, b_{ijk} and a.

Next, employing a method (e.g. the perturbation method given in [19]) to compute the focus value of system (1.7), we obtain the following focus values:

$$\begin{aligned} v_0 = \;& v_{00} + v_{01}\,\varepsilon + v_{02}\,\varepsilon^2 + \cdots \equiv v_{00} + \tfrac{1}{2}\,(b_{011} - b_{011}^*)\,\varepsilon + \tfrac{1}{2}\,(b_{012} - b_{012}^*)\,\varepsilon^2 + \cdots \\ v_k = \;& v_{0k} + v_{k1}\,\varepsilon + v_{k2}\,\varepsilon^2 + \cdots, \qquad k = 1,\, 2,\, \cdots \end{aligned} \tag{1.9}$$

where $v_{0k} = 0$, $k = 1, 2, \cdots$, since C_0 is a center. Thus, when $b_{011} = b_{011}^*$, $v_{01} = 0$, and $b_{012} = b_{012}^*$ yields $v_{02} = 0$. For convenience, in the following, we call v_{k1} as ε-order focus values, and v_{k2} as ε^2-order focus values.

Remark 1. An alternative procedure to analyze system (1.1) is first to apply a translation and a linear transformation to system (1.1) and then add perturbations $p(\varepsilon, x, y)$ and $q(\varepsilon, x, y)$ to the transformed system. This procedure will generate the same result as that given by analyzing system (1.4).

In the next two sections, we shall consider the existence of small-amplitude limit cycles, based on the ε-order and ε^2-order focus values, respectively.

2. The number of limit cycles based on ε-order focus values

We first consider the ε-order focus values. Denote $\tilde{H}(3)$ for the maximal number of small-amplitude limit cycles bifurcating from the center C_0. Then, for this case, we have the following theorem.

Theorem 1. *With the ε-order focus values, $\tilde{H}(3) = 9$.*

Proof. Based on the obtained ε-order focus values, we solve the first eight equations: v_{k1}, $k = 1, 2, \cdots, 8$ for $b_{101}, b_{201}\, b_{111}, b_{021}, b_{301}, b_{211}, b_{121}$ and b_{031} to obtain

$$\begin{aligned}
b_{101} = b^*_{101} \quad = \quad & -a\,a_{101} - \tfrac{a}{2}\,a_{011} + \tfrac{1}{2}\,(a^2\!-)\,a_{201} + \tfrac{a}{4}\,(a^2\!+\!10)\,a_{111} \\
& + \tfrac{1}{24}\,(3a^4\!-\!4a^2\!+\!17)\,a_{021} - \tfrac{a}{4}\,(a^2+8)\,a_{301} + \tfrac{1}{8}\,(a^4+2a^2+5)\,a_{211} \\
& - \tfrac{a}{48}\,(3a^4+12a^2+116)\,a_{121} + \tfrac{1}{21994930176000}\,(46747280800000a^8 \\
& -687341568000a^6 - 4370117689344000 - 15886629487529168a^4 \\
& +34371511716608475a^2 - 4370117689344000)\,a_{031}, \\
b_{201} = b^*_{201} \quad = \quad & -3\,a_{101} - \tfrac{15\,a}{2}\,a_{011} + \tfrac{9\,a}{2}\,a_{201} + \tfrac{3}{4}\,(3a^2\!+\!10)\,a_{111} + \tfrac{a}{8}\,(9a^2\!-\!10)\,a_{021} \\
& - \tfrac{3}{4}\,(3a^2\!+\!4)\,a_{301} - \tfrac{3\,a}{8}\,(3a^2-8)\,a_{211} - \tfrac{1}{16}\,(9a^4-20a^2+172)\,a_{121} \\
& + \tfrac{a}{128303759360000}\,(2590088655000000a^6 + 83421692820598608a^4 \\
& -631135948827669425a^2 + 373978546560195000)\,a_{031}, \\
b_{111} = b^*_{111} \quad = \quad & -9\,a_{011} + 7\,a_{201} + 3\,a_{021} + \tfrac{9}{2}\,a_{211} + 9\,a\,a_{121} \\
& + \tfrac{3}{32075939840000}\,(19146020265571122a^4 - 81345441450987275a^2 \\
& +23437660127232000)\,a_{031}, \\
b_{021} = b^*_{021} \quad = \quad & +4\,a_{111} - 6\,a_{121} - \tfrac{9\,a}{10485760000}\,(51418022656a^4 - 311155569125)\,a_{031}, \\
b_{301} = b^*_{301} \quad = \quad & -13\,a_{011} + 5\,a_{201} + a\,a_{111} - \tfrac{1}{24}\,(3a^2+164)\,a_{021} - 2\,a\,a_{301} \\
& -\tfrac{1}{8}\,(3a^2-80)\,a_{211} - \tfrac{a}{12}\,(a^2-120)\,a_{121} \\
& +\tfrac{1}{769822556160000}\,(42335402600908032a^6 + 1032569283113924199a^4 \\
& -6419549479578590300a^2 + 2146353266294784000)\,a_{031}, \\
b_{211} = b^*_{211} \quad = \quad & 3\,a\,a_{021} + 6\,a_{301} - \tfrac{3\,a}{32075939840000}(943726387828224a^4 \\
& -10706188880760835a^2 + 11678964377872500)\,a_{031}, \\
b_{121} = b^*_{121} \quad = \quad & -\tfrac{15}{2}\,a_{021} + \tfrac{7}{2}\,a_{211} - \tfrac{3}{2566075187200}\,(187685481266381a^2 \\
& -97602502656000)\,a_{031}, \\
b_{031} = b^*_{031} \quad = \quad & \tfrac{8}{3}\,a_{121} + \tfrac{1240857537\,a}{41943040}\,a_{031}.
\end{aligned}$$

Defining the critical point

$$B^*_1 = (b^*_{101},\, b^*_{201},\, b^*_{111},\, b^*_{021},\, b^*_{301},\, b^*_{211},\, b^*_{121},\, b^*_{031}), \tag{2.1}$$

then at this critical point, $v_{k1} = 0$, $k = 1, 2, \cdots, 8$, and

$$\begin{aligned}
v_{91} = \quad & -\frac{258237837\ a_{031}}{32\,a^9(a^4-32)}, \\
v_{101} = \quad & \frac{23476167\ a_{031}\,(57697a^4 - 35728a^2 - 88704)}{64\,a^{11}(a^4-32)^2}, \\
v_{111} = \quad & -\frac{23476167\,a_{031}\,(2304313595a^8 - 1702233920a^6 - 11829269248a^4}{1024\,a^{13}(a^4-32)^3} \\
& -\frac{39211065344a^2 + 8642101248)}{1024\,a^{13}(a^4-32)^3}.
\end{aligned} \tag{2.2}$$

This clearly shows that when $a_{031} \neq 0$, we have $v_{91} \neq 0$ for $a < -2^{5/4}$. Setting $a_{031} = 0$ results in $v_{91} = v_{101} = v_{111} = \cdots 0$. Hence, at most we can have nine small-amplitude limit cycles bifurcating from the center C_0, based on the analysis of ε-order focus values.

Further, evaluating the determinant of the following Jacobian at the critical point B_1^*,

$$J_{B_1^*} = \begin{bmatrix} \frac{\partial v_{11}}{\partial b_{101}} & \frac{\partial v_{11}}{\partial b_{201}} & \frac{\partial v_{11}}{\partial b_{111}} & \frac{\partial v_{11}}{\partial b_{021}} & \frac{\partial v_{11}}{\partial b_{301}} & \frac{\partial v_{11}}{\partial b_{211}} & \frac{\partial v_{11}}{\partial b_{121}} & \frac{\partial v_{11}}{\partial b_{031}} \\ \frac{\partial v_{21}}{\partial b_{101}} & \frac{\partial v_{21}}{\partial b_{201}} & \frac{\partial v_{21}}{\partial b_{111}} & \frac{\partial v_{21}}{\partial b_{021}} & \frac{\partial v_{21}}{\partial b_{301}} & \frac{\partial v_{21}}{\partial b_{211}} & \frac{\partial v_{21}}{\partial b_{121}} & \frac{\partial v_{21}}{\partial b_{031}} \\ \frac{\partial v_{31}}{\partial b_{101}} & \frac{\partial v_{31}}{\partial b_{201}} & \frac{\partial v_{31}}{\partial b_{111}} & \frac{\partial v_{31}}{\partial b_{021}} & \frac{\partial v_{31}}{\partial b_{301}} & \frac{\partial v_{31}}{\partial b_{211}} & \frac{\partial v_{31}}{\partial b_{121}} & \frac{\partial v_{31}}{\partial b_{031}} \\ \frac{\partial v_{41}}{\partial b_{101}} & \frac{\partial v_{41}}{\partial b_{201}} & \frac{\partial v_{41}}{\partial b_{111}} & \frac{\partial v_{41}}{\partial b_{021}} & \frac{\partial v_{41}}{\partial b_{301}} & \frac{\partial v_{41}}{\partial b_{211}} & \frac{\partial v_{41}}{\partial b_{121}} & \frac{\partial v_{41}}{\partial b_{031}} \\ \frac{\partial v_{51}}{\partial b_{101}} & \frac{\partial v_{51}}{\partial b_{201}} & \frac{\partial v_{51}}{\partial b_{111}} & \frac{\partial v_{51}}{\partial b_{021}} & \frac{\partial v_{51}}{\partial b_{301}} & \frac{\partial v_{51}}{\partial b_{211}} & \frac{\partial v_{51}}{\partial b_{121}} & \frac{\partial v_{51}}{\partial b_{031}} \\ \frac{\partial v_{61}}{\partial b_{101}} & \frac{\partial v_{61}}{\partial b_{201}} & \frac{\partial v_{61}}{\partial b_{111}} & \frac{\partial v_{61}}{\partial b_{021}} & \frac{\partial v_{61}}{\partial b_{301}} & \frac{\partial v_{61}}{\partial b_{211}} & \frac{\partial v_{61}}{\partial b_{121}} & \frac{\partial v_{61}}{\partial b_{031}} \\ \frac{\partial v_{71}}{\partial b_{101}} & \frac{\partial v_{71}}{\partial b_{201}} & \frac{\partial v_{71}}{\partial b_{111}} & \frac{\partial v_{71}}{\partial b_{021}} & \frac{\partial v_{71}}{\partial b_{301}} & \frac{\partial v_{71}}{\partial b_{211}} & \frac{\partial v_{71}}{\partial b_{121}} & \frac{\partial v_{71}}{\partial b_{031}} \\ \frac{\partial v_{81}}{\partial b_{101}} & \frac{\partial v_{81}}{\partial b_{201}} & \frac{\partial v_{81}}{\partial b_{111}} & \frac{\partial v_{81}}{\partial b_{021}} & \frac{\partial v_{81}}{\partial b_{301}} & \frac{\partial v_{81}}{\partial b_{211}} & \frac{\partial v_{81}}{\partial b_{121}} & \frac{\partial v_{81}}{\partial b_{031}} \end{bmatrix}_{B_1^*},$$

yields

$$\det(J_{B_1^*}) = -\,\frac{373423834799904305184768}{5\,a^{36}(a^4-32)^8} \neq 0, \qquad \text{for} \;\; a < -\,2^{5/4}. \tag{2.3}$$

Thus, by proper perturbations, one can obtain nine small-amplitude limit cycles around the center C_0. The proof is complete. □

Remark 2. It is seen from the above proof that when the nine b_{ij1} coefficients are used for solving the ε-order focus values, only a_{031} is needed to obtain nine limit cycles, and all other a_{ij1} coefficients can be set zero for ε-order analysis.

Moreover, it is noted that in addition to the critical condition B_1^*, setting $a_{031} = 0$ results in all the ε-order focus values to be zero. Thus, we have the following result.

Theorem 2. *All the ε-order focus values become zero under the following conditions:*

$$\begin{aligned}
a_{031} =\; & 0, \\
b_{101} =\; & -\,a\,a_{101} - \tfrac{a}{2}\,a_{011} + \tfrac{1}{2}\,(a^2-)\,a_{201} + \tfrac{a}{4}\,(a^2+10)\,a_{111} + \tfrac{1}{24}\,(3a^4-4a^2+17)\,a_{021} \\
& - \tfrac{a}{4}\,(a^2+8)\,a_{301} + \tfrac{1}{8}\,(a^4+2a^2+5)\,a_{211} - \tfrac{a}{48}\,(3a^4+12a^2+116)\,a_{121}, \\
b_{201} =\; & -\,3\,a_{101} - \tfrac{15\,a}{2}\,a_{011} + \tfrac{9\,a}{2}\,a_{201} + \tfrac{3}{4}\,(3a^2+10)\,a_{111} + \tfrac{a}{8}\,(9a^2-10)\,a_{021} \\
& - \tfrac{3}{4}\,(3a^2+4)\,a_{301} - \tfrac{3\,a}{8}\,(3a^2-8)\,a_{211} - \tfrac{1}{16}\,(9a^4-20a^2+172)\,a_{121}, \\
b_{111} =\; & -\,9\,a_{011} + 7\,a_{201} + 3\,a_{021} + \tfrac{9}{2}\,a_{211} + 9\,a\,a_{121}, \\
b_{021} =\; & 4\,a_{111} - 6\,a_{121}, \\
b_{301} =\; & -\,13\,a_{011} + 5\,a_{201} + a\,a_{111} - \tfrac{1}{24}\,(3a^2+164)\,a_{021} - 2\,a\,a_{301} - \tfrac{1}{8}\,(3a^2-80)\,a_{211} \\
& - \tfrac{a}{12}\,(a^2-120)\,a_{121}, \\
b_{211} =\; & 3\,a\,a_{021} + 6\,a_{301}, \\
b_{121} =\; & -\tfrac{15}{2}\,a_{021} + \tfrac{7}{2}\,a_{211}, \\
b_{031} =\; & \tfrac{8}{3}\,a_{121}.
\end{aligned} \tag{2.4}$$

3. The number of limit cycles based on ε^2-order focus values

Now, we turn to consider the ε^2-order focus values under the conditions given in (2.4) such that all the ε-order focus values are zero. Note that except the coefficient a_{031}, we leave other unused a_{ij1} coefficients in the expressions of b_{ij1} and hope they might be used in the ε^2-order analysis. From the ε^2-order analysis, we have the following result.

Theorem 3. *With the ε^2-order focus values, $\tilde{H}(3) = 9$.*

Proof. Based on the calculated ε^2-order focus values, we solve the first eight equations: $v_{k2} = 0,\ k = 1, 2, \cdots, 8$ for $b_{102}, b_{202}\, b_{112}, b_{022}, b_{302}, b_{212}, b_{122}$ and b_{032} to obtain

$$
\begin{aligned}
b_{102} = b_{102}^* =\ & -a\, a_{102} - \tfrac{a}{2}\, a_{012} + \tfrac{1}{2}\,(a^2\!-)\, a_{202} + \tfrac{a}{4}\,(a^2\!+\!10)\, a_{112} \\
& + \tfrac{1}{24}\,(3a^4\!-\!4a^2\!+\!17)\, a_{022} - \tfrac{a}{4}\,(a^2+8)\, a_{302} + \tfrac{1}{8}\,(a^4+2a^2+5)\, a_{212} \\
& - \tfrac{a}{48}\,(3a^4+12a^2+116)\, a_{122} + \tfrac{1}{21994930176000}\,(46747280800000a^8 \\
& - 687341568000a^6 - 4370117689344000 - 15886629487529168a^4 \\
& + 34371511716608475a^2 - 4370117689344000)\, a_{032} + \tilde{b}_{102}, \\
b_{202} = b_{202}^* =\ & -3\, a_{102} - \tfrac{15\,a}{2}\, a_{012} + \tfrac{9\,a}{2}\, a_{202} + \tfrac{3}{4}\,(3a^2\!+\!10)\, a_{112} + \tfrac{a}{8}\,(9a^2\!-\!10)\, a_{022} \\
& - \tfrac{3}{4}\,(3a^2\!+\!4)\, a_{302} - \tfrac{3\,a}{8}\,(3a^2-8)\, a_{212} - \tfrac{1}{16}\,(9a^4-20a^2+172)\, a_{122} \\
& + \tfrac{a}{128303759360000}\,(2590088655000000a^6 + 834216928205986O8a^4 \\
& - 631135948827669425a^2 + 373978546560195000)\, a_{032} + \tilde{b}_{202}, \\
b_{112} = b_{112}^* =\ & -9\, a_{012} + 7\, a_{202} + 3\, a_{022} + \tfrac{9}{2}\, a_{212} + 9\, a\, a_{122} \\
& + \tfrac{3}{32075939840000}\,(19146020265571122a^4 - 81345441450987275a^2 \\
& + 23437660127232000)\, a_{032} + \tilde{b}_{112}, \\
b_{022} = b_{022}^* =\ & +4\, a_{112} - 6\, a_{122} - \tfrac{9\,a}{10485760000}\,(51418022656a^4 - 311155569125)\, a_{032} \\
& + \tilde{b}_{022}, \\
b_{302} = b_{302}^* =\ & -13\, a_{012} + 5\, a_{202} + a\, a_{112} - \tfrac{1}{24}\,(3a^2+164)\, a_{022} - 2\, a\, a_{302} \\
& - \tfrac{1}{8}\,(3a^2-80)\, a_{212} - \tfrac{a}{12}\,(a^2-120)\, a_{122} + \tfrac{1}{769822556160000} \\
& (42335402600908032a^6 + 1032569283113924199a^4 \\
& - 6419549479578590300a^2 + 2146353266294784000)\, a_{032} + \tilde{b}_{302}, \\
b_{212} = b_{212}^* =\ & 3\, a\, a_{022} + 6\, a_{302} - \tfrac{3\,a}{32075939840000}(943726387828224a^4 \\
& - 10706188880760835a^2 + 11678964377872500)\, a_{032} + \tilde{b}_{212}, \\
b_{122} = b_{122}^* =\ & -\tfrac{15}{2}\, a_{022} + \tfrac{7}{2}\, a_{212} - \tfrac{3}{2566075187200}\,(187685481266381a^2 \\
& - 97602502656000)\, a_{032} + \tilde{b}_{122}, \\
b_{032} = b_{032}^* =\ & \tfrac{8}{3}\, a_{122} + \tfrac{1240857537\,a}{41943040}\, a_{032} + \tilde{b}_{032},
\end{aligned}
$$

where the coefficients $\tilde{b}_{ij2}$ are explicitly expressed in terms of $a_{201}, a_{111}, a_{021}$, $a_{301}, a_{211}, a_{121}$, and are given in Appendix.

Similarly, defining the critical point,

$$
B_2^* = (b_{102}^*, b_{202}^*, b_{112}^*, b_{022}^*, b_{302}^*, b_{212}^*, b_{122}^*, b_{032}^*), \tag{3.1}
$$

then at this critical point, $v_{k2} = 0$, $k = 1, 2, \cdots, 8$, and

$$\begin{aligned}
v_{92} &= -\frac{28693093\,(9a_{032} - 5a_{021}\,a_{121})}{32\,a^9(a^4-32)},\\
v_{102} &= \frac{2608463\,(9a_{032} - 5a_{021}\,a_{121})}{64\,a^{11}(a^4-32)^2}\,(57697a^4 - 35728a^2 - 88704),\\
v_{112} &= -\frac{2608463\,(9a_{032} - 5a_{021}a_{121})}{1024\,a^{13}(a^4-32)^3}\\
&\quad \times(2304313595a^8 - 1702233920a^6 - 11829269248a^4 - 39211065344a^2 + 8642101248).
\end{aligned} \tag{3.2}$$

This implies that when $9\,a_{032} - 5\,a_{021}\,a_{121} \neq 0$, we have $v_{92} \neq 0$ for $a < -\,2^{5/4}$. Setting $a_{032} = \frac{5}{9}\,a_{021}\,a_{121}$ yields $v_{92} = v_{102} = v_{112} = \cdots 0$. Therefore, at most we can have nine small-amplitude limit cycles bifurcating from the center C_0, based on the analysis of ε-order focus values.

Further, evaluating the determinant of the following Jacobian at the critical point B_2^*,

$$J_{B_2^*} = \begin{bmatrix}
\frac{\partial v_{12}}{\partial b_{102}} & \frac{\partial v_{12}}{\partial b_{202}} & \frac{\partial v_{12}}{\partial b_{112}} & \frac{\partial v_{12}}{\partial b_{022}} & \frac{\partial v_{12}}{\partial b_{302}} & \frac{\partial v_{12}}{\partial b_{212}} & \frac{\partial v_{12}}{\partial b_{122}} & \frac{\partial v_{12}}{\partial b_{032}}\\
\frac{\partial v_{22}}{\partial b_{102}} & \frac{\partial v_{22}}{\partial b_{202}} & \frac{\partial v_{22}}{\partial b_{112}} & \frac{\partial v_{22}}{\partial b_{022}} & \frac{\partial v_{22}}{\partial b_{302}} & \frac{\partial v_{22}}{\partial b_{212}} & \frac{\partial v_{22}}{\partial b_{122}} & \frac{\partial v_{22}}{\partial b_{032}}\\
\frac{\partial v_{32}}{\partial b_{102}} & \frac{\partial v_{32}}{\partial b_{202}} & \frac{\partial v_{32}}{\partial b_{112}} & \frac{\partial v_{32}}{\partial b_{022}} & \frac{\partial v_{32}}{\partial b_{302}} & \frac{\partial v_{32}}{\partial b_{212}} & \frac{\partial v_{32}}{\partial b_{122}} & \frac{\partial v_{32}}{\partial b_{032}}\\
\frac{\partial v_{42}}{\partial b_{102}} & \frac{\partial v_{42}}{\partial b_{202}} & \frac{\partial v_{42}}{\partial b_{112}} & \frac{\partial v_{42}}{\partial b_{022}} & \frac{\partial v_{42}}{\partial b_{302}} & \frac{\partial v_{42}}{\partial b_{212}} & \frac{\partial v_{42}}{\partial b_{122}} & \frac{\partial v_{42}}{\partial b_{032}}\\
\frac{\partial v_{52}}{\partial b_{102}} & \frac{\partial v_{52}}{\partial b_{202}} & \frac{\partial v_{52}}{\partial b_{112}} & \frac{\partial v_{52}}{\partial b_{022}} & \frac{\partial v_{52}}{\partial b_{302}} & \frac{\partial v_{52}}{\partial b_{212}} & \frac{\partial v_{52}}{\partial b_{122}} & \frac{\partial v_{52}}{\partial b_{032}}\\
\frac{\partial v_{62}}{\partial b_{102}} & \frac{\partial v_{62}}{\partial b_{202}} & \frac{\partial v_{62}}{\partial b_{112}} & \frac{\partial v_{62}}{\partial b_{022}} & \frac{\partial v_{62}}{\partial b_{302}} & \frac{\partial v_{62}}{\partial b_{212}} & \frac{\partial v_{62}}{\partial b_{122}} & \frac{\partial v_{62}}{\partial b_{032}}\\
\frac{\partial v_{72}}{\partial b_{102}} & \frac{\partial v_{72}}{\partial b_{202}} & \frac{\partial v_{72}}{\partial b_{112}} & \frac{\partial v_{72}}{\partial b_{022}} & \frac{\partial v_{72}}{\partial b_{302}} & \frac{\partial v_{72}}{\partial b_{212}} & \frac{\partial v_{72}}{\partial b_{122}} & \frac{\partial v_{72}}{\partial b_{032}}\\
\frac{\partial v_{82}}{\partial b_{102}} & \frac{\partial v_{82}}{\partial b_{202}} & \frac{\partial v_{82}}{\partial b_{112}} & \frac{\partial v_{82}}{\partial b_{022}} & \frac{\partial v_{82}}{\partial b_{302}} & \frac{\partial v_{82}}{\partial b_{212}} & \frac{\partial v_{82}}{\partial b_{122}} & \frac{\partial v_{82}}{\partial b_{032}}
\end{bmatrix}_{B_2^*},$$

yields

$$\det(J_{B_2^*}) = -\frac{37342383479990430518476 8}{5\,a^{36}(a^4-32)^8} \neq 0, \qquad \text{for} \;\; a < -\,2^{5/4}.$$

Hence, by proper perturbations, one can obtain nine small-amplitude limit cycles around the center C_0. The proof is finished. □

Remark 3. The unused a_{ij1} coefficients remained in the system does not help to get more limit cycles. It is not surprising to see that $\det(J_{B_2^*}) = \det(J_{B_1^*})$, and the formulas for the focus values: v_{92}, v_{102}, v_{112}, are exactly the same as that for v_{91}, v_{101}, v_{111} if $a_{021} = 0$ or $a_{121} = 0$.

Remark 4. Again it is seen from the above proof that when the nine b_{ij2} coefficients are used for solving the ε^2-order focus values, only a_{032} (if setting $a_{021} = 0$ or $a_{121} = 0$) is needed to obtain nine limit cycles, and all other a_{ij2} coefficients can be set zero for ε^2-order analysis.

It is also noted that in addition to the critical condition B_2^*, setting $a_{032} = \frac{5}{9}\,a_{021}\,a_{121}$ leads to all the ε^2-order focus values being zero. Thus, we have the following result.

Theorem 4. *All the ε^2-order focus values become zero under the critical condition B_2^* with $a_{032} = \frac{5}{9}\,a_{021}\,a_{121}$.*

Remark 5. From the proofs of Theorems 1 and 3, it is easy to see that even with higher ε^n-order focus values, it is not possible to obtain more than nine small-amplitude limit cycles bifurcating from the center.

4. Conclusion

In this paper, based on ε-order and ε^2-order focus values, we have shown that the example given by Zoladek [18] can exhibit maximal nine small-amplitude limit cycles around the center. It is unlikely to have more small-amplitude limit cycles even using higher ε^n-order focus values.

Appendix

In this appendix, the coefficients $\tilde{b}_{ij2}$ given in Section 3 are listed below, where $\tilde{b}_{ij2}$ is denoted by bbij2.

```
bb102:=-119/16*a211*a1*a011-133/48*a121^2*a1^3+10247/96*a121*a021+994561814791373/2474429644800
        *a021*a1^4*a121+23/96*a021*a1^3*a211-253/18*a121^2*a1-355/96*a121*a211-7*a121*a201-45/32
        *a021*a1^2*a301+29/8*a121*a011+1/48*a111*a021+75/16*a111*a211+5*a111*a201+87/16*a211*a1
        *a201+115/48*a021*a1*a201-63/32*a211*a1^2*a301-569/384*a211*a1^4*a121-9/2*a111*a011-227/48
        *a021*a1*a011+9/2*a1*a201^2+3/2*a1*a011^2-55/64*a1^3*a211^2-1/2*a011*a301-9/16*a1^4*a121
        *a201-7/24*a021*a101-15/8*a211*a101+71/192*a1^3*a021^2+9/16*a1^4*a121*a011+4*a1*a211^2
        -2*a101*a201+2*a101*a011-9/8*a1^3*a211*a201+5/2*a111*a301*a1-9*a1*a201*a011-1/4*a111^2
        *a1^3-384434875/325582848*a121*a1^8*a021+9/8*a1^3*a211*a011+1/8*a111*a211*a1^4+a111*a101
        *a1+1/24*a021^2*a1^5-153348194797115/175959441408*a121*a021*a1^2+415/96*a121*a211*a1^2
        +63/32*a111*a211*a1^2+17/24*a301*a021-29/12*a121*a301*a1+65/96*a1*a021*a211+215/96*a111
        *a021*a1^2+113/48*a111*a121*a1^3+83/24*a121*a201*a1^2-1/4*a301^2*a1^3-41/12*a121*a101*a1
        +7/4*a011*a1^2*a301-113/48*a121*a301*a1^3-a101*a1*a301-5/2*a111^2*a1-101/192*a121^2*a1^5
        +29/24*a121*a011*a1^2+31/24*a201*a1^3*a021-31/24*a011*a1^3*a021+1/16*a111*a121*a1^5
        -1/24*a111*a021*a1^4-1/48*a121*a1^6*a021-1/24*a211*a1^5*a021-1/16*a121*a1^5*a301
        +1/24*a021*a1^4*a301+1/2*a111*a301*a1^3-7/4*a111*a011*a1^2-1/3*a101*a1^2*a021
        -7/4*a201*a1^2*a301+263/24*a111*a121*a1-1/8*a211*a1^4*a301-469/96*a1*a021^2
        +7/4*a1^2*a111*a201:
bb202:=-289987194845/176160768*a121*a1*a021-7/8*a201*a1^3*a121-43/4*a301*a121+15/2*a301*a111
        +21/8*a1*a211*a101-5/2*a1*a021*a301+85/4*a121*a1*a011+113/4*a111*a121-565/12*a121^2
        +227/32*a1^2*a021^2-749/48*a121^2*a1^2+156662310309817/57378078720*a121*a1^3*a021
        +15/32*a1^3*a211*a111+113/16*a1^2*a021*a201+7/64*a121^2*a1^6
        -83047798165451/229113856000*a121*a1^5*a021+3*a201^2-15/2*a111^2+15/2*a211^2+18*a021*a011
        -251/12*a021^2-15417194375/1374683136*a121*a1^7*a021+21/64*a1^4*a211^2-9/4*a1^2*a111^2
        +3*a111*a101+113/64*a1^4*a021^2-1/2*a121*a1^3*a011+15*a111*a1*a011+3/8*a1^2*a211^2
        -3*a301*a101-71/8*a111*a121*a1^2+245/16*a1*a021*a111-59/32*a1^2*a021*a211
        -67/32*a1^4*a021*a211+77/32*a1^3*a021*a111+7/4*a101*a121*a1^2-77/32*a1^3*a021*a301
        -83/16*a1^2*a021*a011+49/128*a121*a1^5*a211+5/12*a121^2*a1^4-6*a201*a011-41/8*a1*a021*a101
        -12*a021*a201-33/16*a211*a1^2*a011+55/32*a121*a211*a1^3-21/16*a211*a1^2*a201
        -15/32*a211*a1^3*a301-1/8*a301*a1^4*a121-153/16*a1*a211*a111+15/2*a201*a211-27/2*a211*a011
        +1/8*a111*a121*a1^4-95/32*a121*a211*a1-29/4*a021*a211+9/2*a301*a1^2*a121-15*a301*a1*a011
        -9/2*a111*a1*a201+9/2*a111*a1^2*a301+3*a1*a301*a211+9/2*a1*a301*a201
        -9/4*a1^2*a301^2-3*a011^2-7*a101*a121-a201*a1*a121:
bb112:=9/8*a211*a1*a011-456179816730741/458227712000*a021*a1^4*a121-131979/112*a121*a021
        +45/64*a211*a1^4*a121+3249631747890371/769822556160*a121*a021*a1^2-9/16*a021*a1^3*a211
        -9/16*a021*a1^2*a301+39/8*a021*a1*a011+21/16*a121*a211*a1^2+9/4*a211*a101+9/16*a211
        *a1^2*a301-9/8*a111*a121*a1^3+9/8*a021*a1*a201-9*a011*a301+9/16*a111*a021*a1^2
        -39/8*a111*a021+9*a121*a301*a1-9/16*a111*a211*a1^2-213/16*a1*a021*a211-21*a121*a201
        -9/4*a121*a201*a1^2+9/8*a121*a301*a1^3+9/2*a121*a101*a1+9*a111*a011+9/4*a121*a011*a1^2
        +437/16*a1*a021^2+9/8*a121^2*a1^3+42*a121*a011-45/8*a111*a211+3/4*a021*a101+3*a301*a021
       -9/8*a211*a1*a201-43/8*a121^2*a1-687/16*a121*a211+9/32*a1^3*a021^2+9/32*a1^3*a211^2
       +9/32*a121^2*a1^5-81/4*a111*a121*a1:
NB022:=-2845760393/16777216*a121*a1*a021+3/8*a121*a1^3*a021+3/4*a111*a121*a1^2+3/2*a201*a1*a121
        +15/2*a111*a121+200851651/8192000*a121*a1^5*a021-3/4*a301*a1^2*a121-3/16*a121^2*a1^4
        -3/8*a121*a211*a1^3+39/2*a021*a011+6*a121*a211*a1-1/4*a121^2*a1^2-3/2*a121*a1*a011
```

```
        -41/4*a021^2-9*a021*a201-21/4*a021*a211-9/2*a211*a011 -85/4*a121^2-3*a101*a121:
bb302:=-16459830161293273/21994930176000*a021*a1^4*a121-1589743/1008*a121*a021+133/48*a121^2
        *a1^3-20*a121*a201+64111487909344943/13856806010880*a121*a021*a1^2-283/48*a021*a1^3*a211
        -29/8*a211*a1*a201+25/4*a211*a101-7874888876657/257753088000*a121*a1^6*a021-33/16*a111
        *a211*a1^2+41/32*a1^3*a211^2+101/64*a211*a1^4*a121+449/48*a111*a021*a1^2+455/24*a021
        *a1*a201-347/48*a1*a021*a211-13/8*a111*a121*a1^3+33/16*a211*a1^2*a301-491/24*a021*a1*a011
        -449/48*a021*a1^2*a301-a111^2*a1+281/6*a121*a011+13/8*a121*a301*a1^3+2*a111*a301*a1
        +15/32*a121^2*a1^5+5*a301*a201+427/96*a1^3*a021^2-3*a121*a201*a1^2-135/4*a111*a121*a1
        +1247/48*a1*a021^2-7/8*a211*a1*a011+11/2*a121*a101*a1+9*a121*a211*a1^2-161/24*a121^2*a1
        +923/24*a111*a021-119/12*a021*a101-905/48*a121*a211-a1*a301^2+17/8*a121*a011*a1^2
        +9/4*a1*a211^2-26*a011*a301+26*a111*a011-205/8*a111*a211+20*a121*a301*a1
        -41/3*a301*a021-5*a111*a201+10*a301*a211:
bb212:=-305753642565281/549873254400*a121*a1^3*a021-6*a1*a021*a111+5/2*a121^2*a1^2-3*a121*a1*a011
        -30*a021*a211+200851651/4096000*a121*a1^5*a021+6*a1*a021*a301-15*a021*a201+41/2*a021^2
        -9/2*a121*a211*a1+39*a021*a011+17229676735/29360128*a121*a1*a021
        +9/8*a1^2*a021*a211+3/8*a1^2*a021^2:
bb122:=26812211609483/219949301760*a121*a021*a1^2-4*a1*a021^2-31/4*a121^2*a1-2475/28*a121*a021
        +45/4*a121*a211+15/2*a111*a021-15/2*a301*a021+15/2*a121*a011:
bb032 :=
35/6*a021^2-5/2*a021*a211+5*a121^2-413619179/25165824*a121*a1*a021:
```

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